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A mathematical trivium II

V.I. Arnol'd

It seems almost a miracle that contemporary methods of teaching have not yet completely stifled holy curiosity.

A. Einstein

My appeal in "A mathematical trivium" [1] that *written* examinations should be used in mathematics provoked many replies with criticism of oral and written examinations both from Russia and from other countries in Europe and America. The authors of many letters from Russia reckoned that instructors could solve, on average, one-third of the problems of the trivium [1]. On this basis they thought that the problems of the trivium were too difficult. This should probably be set against the fact that in the USSR about 40% of heads of mathematics departments have had no mathematical education, and the position will probably deteriorate even further.

Written examinations in various mathematical subjects have been tried out in several Russian higher education institutions, so some conclusions can be drawn. The majority of the instructors taking part considered that it was easier to conduct written examinations than oral ones. Supprising as it may seem, copying is less of a danger than might have been expected, since it is easily detected (incorrect solutions are copied on the whole). The increase in the number of unsatisfactory marks (because students who have learnt nothing are shown up) can hardly be regarded as a shortcoming of a written examination.

The selection of examination problems aroused the most frequent criticism. Here strange things sometimes happened (the appearance of which, however, is also useful: the personality of the setter and local traditions clearly appear in the character and the very formulation of the questions).

For example, the official national American written examination (known by three initials, which I have forgotten) contained the following test-problem in 1992:

"Which of the following most resembles the relation between angle and a degree:

1) time and a minute,

2) milk and a quart,

3) area and a square inch, ... (and three more pairs)."

The "correct" answer is area and a square inch. Justification: the degree is the smallest unit of measurement of angle, and the square inch that of area, but the minute can be further divided, for example into seconds.

For us this answer is of course preposterous. But American students whom I tested almost always gave precisely this "correct" answer. For a long time I could not understand why this was until a well-known American physicist explained to me his (correct) answer: "The point is that I correctly imagined the degree of stupidity of the person who set these questions".

I hope that today such questions no longer threaten our examinees. But the attempts which we see to Americanize education (beginning in primary and secondary schools) may in time lead to this. Of course, I am against such Americanization and was not calling for it in [1].

The European traditions of mathematical examinations, which vary from country to country, are also instructive. In some cases university examinations degenerate into a refined system of casuistry like that which we apply (or used to apply?) in the entrance examinations for higher education institutions, recorded in regrettably well-known collections of problems (beginning with Novoselov and others).

Littlewood's *A mathematician's miscellany* shows that up to a certain time even university examinations were like this in England. It seems to me that some of the specimens of European examination questions given below are also capable of arousing sympathy for the unfortunate students compelled to go through such mathematical tortures. "What distinguishes these scholastic cultures is this — they lead the mind away from everything refined, and respect only those childish contrivances on which a whole life is wasted and which they regard as the natural occupation of professionally respectable people" (Renan).

We too are in danger of sliding into such "childish contrivances", and I urge setters of examination questions to make them rich in content, easy, beautiful, instructive, and interesting.

I was surprised by the abundance in the examinations of European universities of questions whose answers could be copied directly from the textbook. They are probably permissible when the examination is carried out under the supervision of the police (as in France), but not under our conditions. It is interesting also that in the English system the marks for a student's work not only increase as his quality improves, but also decrease with an *improvement in the quality of the work* of his fellows and rivals. This competitive character in the examination may possibly also prevent copying, but to see this here is for some reason not desirable.

A written examination for the course in ordinary differential equations in the Mechanics and Mathematics Faculty of Moscow State University was first held in the spring of 1991. Two hours were allotted to the problems given below. Correction (10–15 pieces of work per instructor) took an hour, after which the results were announced to the students. For another hour the students looked through their work and analysed their mistakes (with the aid of the instructors). This part of the examination was voluntary, but almost all the students wanted to discuss their work with the instructors.

The criteria for marking were made precise after the correction of the work, but in general were roughly as follows: satisfactory — more than one correct solution, good — more than two, excellent — more than three.

When another group takes the same examination in a few days' time, all the problems are completely changed. So that the reader can see the degree of similarity between the versions used on the same day,(1) and the degree of difference in subsequent days, versions for all the days are given below. They are compiled taking into account the fact that the students examined later know the problems from previous days (which in itself is not bad for the students, but requires additional effort from the setter). Setting the questions would be easier if the whole class could be examined simultaneously, but this could not be organized for technical reasons.

Examination in differential equations, Mechanics and Mathematics Faculty, Moscow State University, 1991

First day (one of 6 versions).

1. Find the image of the vector (1, 0) based at the point $(\pi, 0)$ under the action of the transformation at time $t = 1$ of the phase flow of the system $\dot{x} = y, \dot{y} = \sin x$.

2. In what coordinate system do the variables separate in the equation

$$\frac{dy}{dx} = xy^2 + x^3y^3\,?$$

3. Does the Cauchy problem

$$y\frac{\partial u}{\partial x} + (x^3 - x)\frac{\partial u}{\partial y} = y^2, \qquad u(0,y) = 0,$$

have a solution in the neighbourhood of the point $(x_0 = 0, y = y_0)$, and is it unique?

4. Is the solution of the system

$$\dot{x} = yz, \quad \dot{y} = -xz, \quad \dot{z} = 0$$

with initial condition (x_0, y_0, z_0) stable in the sense of Lyapunov?

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(1) In Russia it is customary for the questions for a single examination to be set in several different versions, which are then allocated to the students at random (Translator's note).

Second day. Consider the system (6 versions):

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x^2. \end{cases} \begin{cases} \dot{x} = y^3, \\ \dot{y} = -x. \end{cases} \begin{cases} \dot{x} = y, \\ \dot{y} = x^4. \end{cases} \begin{cases} \dot{x} = -y^2, \\ \dot{y} = x^2. \end{cases} \begin{cases} \dot{x} = y^4, \\ \dot{y} = -x. \end{cases} \begin{cases} \dot{x} = y^2, \\ \dot{y} = x^2. \end{cases}$$

1) Find the positions of equilibrium and investigate their stability.

2) Can all the solutions of the system be extended indefinitely?

3) How many non-zero solutions for which $y(0) = x(1) = 0$ does the system have?

4) Find the derivative with respect to a at $a = 0$ of the solution with initial condition $x(0) = y(0) = a$.

Third day. Consider the system (6 versions):

$$\begin{cases} \dot{x} = x^2 y, \\ \dot{y} = -xy^2. \end{cases} \begin{cases} \dot{x} = -x^2 y^2, \\ \dot{y} = xy^3. \end{cases} \begin{cases} \dot{x} = -x^2 y^3, \\ \dot{y} = xy^4. \end{cases} \begin{cases} \dot{x} = -xy^2, \\ \dot{y} = x^2 y. \end{cases} \begin{cases} \dot{x} = xy^3, \\ \dot{y} = -x^2 y^2. \end{cases} \begin{cases} \dot{x} = -xy^4, \\ \dot{y} = x^2 y^3. \end{cases}$$

1) and 2)—as for the previous day.

3) Find a diffeomorphism of the plane rectifying the direction field of the phase curves in the neighbourhood of the point (1, 1).

4) Find all the first integrals continuous on the whole plane that coincide with y on the y-axis.

Fourth day. Given the system (6 versions):

$$\dot{z} = iz^2. \quad \dot{z} = \bar{z}^2. \quad \dot{z} = iz^2\bar{z}. \quad \dot{z} = z\bar{z}^2. \quad \dot{z} = iz\bar{z}^2. \quad \dot{z} = i\bar{z}^2.$$

1)—as for the previous days.

2) Find all initial conditions for which the solutions can be extended forwards indefinitely.

3) Find the image of the vector (0, 1) based at the point 0 under the action of the transformation of the phase flow at time $t = 1$.

4) Find all first integrals continuous in the neighbourhood of the point $z = 1$ and equal to 1 on the real axis.

Fifth day. Consider the problem (one of 6 versions):

$$x\frac{\partial u}{\partial x} - (1 + x^4 + y^2)\frac{\partial u}{\partial y} = 2u, \quad u(0, y) = 0.$$

1) Does the problem have an unbounded solution defined on the whole plane?

2) Is the quantity u bounded on the characteristics?

3) Do all the characteristics intersect the surface $y = x + u^2$?

4) Does the equation of characteristics have a first integral whose derivative with respect to u is equal to 1 at the origin? Find the derivative of this derivative with respect to u along a characteristic vector.

Sixth day. Consider the equation (6 versions):

$$\ddot{x} + x = \sinh^3 x. \quad \ddot{x} + x = \sinh^3 x/2.$$
$$\ddot{x} + \sin x = x^3. \quad \ddot{x} + \sin x = x^3/2.$$
$$\ddot{x} + x = 2x^3. \quad \ddot{x} + x = x^3/2.$$

1) Can the solution with initial condition $x(0) = 1$, $\dot{x}(0) = 0$ be extended to the whole t-axis?

2) Does the solution with initial condition $x(0) = a$, $\dot{x}(0) = 0$ have a bounded third derivative with respect to a at $a = 0$?

3) Compute the value of this derivative for $t = 2\pi$.

4) Compute the tenth derivative with respect to a at $a = 0$ of the solution with initial condition $x(0) = a$, $\dot{x}(0) = 0$.

Seventh day. Given the equation (6 versions):

$$\dot{x} = x^2 - \sin^2 t. \quad \dot{x} = x^2 - \cos^2 t.$$
$$\dot{x} = \sin^2 t - x^2. \quad \dot{x} = \cos^2 t - x^2.$$
$$\dot{x} = \sinh^2 x - \sin^2 t. \quad \dot{x} = \sinh^2 x - \cos^2 t.$$

1) Find the third derivative at zero of the solution with initial condition $x(0) = 0$.

2) Does this equation extend to the whole of the t-axis?

3) Does the equation have unbounded solutions?

4) Find the number of asymptotically stable periodic solutions of the equation.

The examination problems for first and third year students of various European universities given below were published in [2].

University of Warwick (England)[(1)]

During the first year the students study from 10 to 15 subjects ("modules") of which 8 are compulsory. In the third year the number of subjects is between 8 and 15 (but, beginning with the second year, there are also easier programmes with fewer subjects, which qualify for another degree). Each subject corresponds to 30 hours of lectures, but teaching is mostly given by postgraduate-student instructors ("tutors"). In the first year one of the compulsory subjects is supervised work, and in the third year the only compulsory subject of specialization, "applied mathematics", consists of course work corresponding to two subjects.

[(1)] The author quotes from [2], so the original questions have been translated from English to French to Russian to English. The final version given here is unlikely to be word for word the same as that originally set (Translator's note).

The examinations are short (1½ to 2 hours) and consist of three to five independent questions. Almost all the questions contain problems from the course. If n questions are given, the mark is derived from the best $n-1$ answers. To illustrate the variety of subjects the following questions have been selected:

For first-year students: four questions in the subjects "Analysis II" (compulsory), "Linear algebra" (compulsory), "Group theory B" (compulsory for specialists in pure mathematics), and "Life in three dimensions" (compulsory for specialists in applied mathematics).

For third-year students: three questions in the subjects "Complex analysis" (the only compulsory subject for specialists in pure mathematics), "Algebraic topology", and "Catastrophe theory".

Analysis II.

Define accurately what is meant by the uniform continuity of a function $f: I \to \mathbb{R}$, and formulate the negation of this definition, that is, what is meant by a function not being uniformly continuous.

Prove in detail, verifying that your definition is satisfied, that $f(x) = x$ is uniformly continuous on $\mathbb{R}$, but $g(x) = x^2$ is not. Let $f: [a, b] \to \mathbb{R}$ be bounded and integrable on $[a, b]$, and let $F: [a, b] \to \mathbb{R}$ be defined by $F(x) = \int_0^x f(t)dt$.

Assuming as known all the properties of the integral that you need (which should be clearly stated), prove that F is uniformly continuous on $[a, b]$; if in addition f is continuous, prove that F is differentiable and that $F' = f$.

Linear algebra.

1. $T: V \to V$ is a linear transformation from a vector space V to itself. Define the linear transformations $T^2: V \to V$ and $T^3: V \to V$.

2. V is a three-dimensional vector space over $\mathbb{R}$ and $T: V \to V$ is a linear transformation such that $T^2 \neq 0$, but $T^3 = 0$. Let e_1 be a vector in V such that $T^2 e_1 \neq 0$. Let $e_2 = Te_1$ and $e_3 = T^2 e_1$. Prove that e_1, e_2, e_3 form a basis of V.

3. Find the matrix of the transformation $T: V \to V$ in the basis e_1, e_2, e_3.

4. Prove that any 3×3 matrix A over $\mathbb{R}$ for which $A^2 \neq 0$, but $A^3 = 0$, is similar to the matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Group theory.

Define homomorphism and kernel. Prove that the kernel of a homomorphism is a normal subgroup.

State the first homomorphism theorem.

Prove that the dihedral group D_{2n} has no surjective homomorphisms onto the cyclic group C_n of order n if $n > 2$.

Produce an injective homomorphism of C_n into D_n.

Life in three dimensions.

1. Let $S = \{((2+\cos x)\cos y, (2+\cos x)\sin y, \sin x) | \ 0 \leqslant x < 2\pi, 0 \leqslant y < 2\pi\}$. Find the area of S.

2. a) For the function $g : \mathbb{R}^3 \to \mathbb{R}$ and the vector field $v : \mathbb{R}^3 \to \mathbb{R}$, prove that $\operatorname{div}(gv) = \nabla g \cdot v + g \operatorname{div} v$. Deduce from this that for a bounded region $\Omega \subset \mathbb{R}^3$

$$\int_\Omega g \cdot \operatorname{div} v = \int_{\partial\Omega} gv \cdot n\, dA - \int_\Omega \nabla g \cdot v,$$

where n is the outward normal to $\partial\Omega$ and dA is the infinitesimal element of area on $\partial\Omega$.

b) Assuming in addition that $v = \nabla g$, $\operatorname{div}(\nabla g) = \operatorname{div} v = 0$, and that $g|_{\partial\Omega} = 0$, show that g (and hence v) is identically zero in Ω.

3. For $f : \mathbb{R} \to \mathbb{R}$ we define $u : \mathbb{R}^3 \to \mathbb{R}$ by $u(x, y, z) = f(x^2+y^2+z^2)$.

a) Compute ∇u in terms of x, y, z, and f'.

b) Let $v(x, y, z) = \nabla u(x, y, z)$ and $\operatorname{div} v = 0$. For $R > 1$ let $\Omega_R = \{(x, y, z) \in \mathbb{R}^3 | 1 \leqslant x^2+y^2+z^2 \leqslant R^2\}$. By applying the divergence theorem to v in Ω_R, show that $R^3 f'(R^2) = f'(1)$. Letting $t = R^2$, show that $f(t) = f(1)+2f'(1)(1-t^{-1/2})$.

Complex analysis.

1. Define the residue $\operatorname{res}(f, z_0)$ of a function f at an isolated singularity z_0. If z_0 is a simple pole, prove that $\operatorname{res}(f, z_0) = \lim_{z\to z_0}(z-z_0)f(z)$.

2. State Cauchy's residue theorem.

3. Prove that

$$\int_0^\pi \frac{dt}{1 + b\cos^2 t} = \frac{2}{ib}\int_C \frac{z\,dz}{z^4 + 2(1 + 2/b)z^2 + 1},$$

where b is real and positive and $C(t) = e^{it}$ $(-\pi \leqslant t \leqslant \pi)$.

Deduce from 1 and 2 that this integral is equal to $\pi/\sqrt{1+b}$.

Algebraic topology.

1. Define the cellular homology $H_n^c(K)$ of a cell complex K. Define an isomorphism $\varphi : H_n(K) \to H_n^c(K)$ (it is not necessary to prove that φ is well-defined).

2. Compute the homology of the cell complex $K = e_0^0 \cup e_1^1 \cup e_2^1 \cup e_3^1 \cup_f e^2$, where f corresponds to the word $a_1a_2a_3a_2^{-1}a_3^{-1}a_1$.

Describe the element of order 2 in $H_1(K)$.

Catastrophe theory.

1. Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be a plane curve traversed with unit velocity, let $k(t)$ be its curvature at $\gamma(t)$, and $n(t)$ the unit normal vector at $\gamma(t)$. The bifurcation set of the "squared distance" function $f_a(t) = \|\gamma(t)-a\|^2$ is called the *caustic curve* of γ. Show that the caustic can be parametrized by $a = \gamma(t)+n(t)/k(t)$ if the curvature does not vanish. What happens at a point at which the curvature vanishes?

Sketch the curve $y^2 = x^2(1-x^2)$ and (without calculation) try to understand what its caustic looks like. (The Serret–Frenet formulae may be used without proof.)

2. Define the curve of the centres of mass of displaced water and the metacentric curve of a (two-dimensional) vessel. Show that the curve of the centres of mass of displaced water of an elliptical vessel is also an ellipse and describe its metacentric curve. Describe briefly how the equilibrium of the ship varies when its centre of gravity moves along an axis of the ellipse (distinguishing the major and minor axes).

(You need not prove that linear transformations map the centre of gravity of a plane domain into the centre of gravity of its image.)

University of Copenhagen

There are fewer subjects here (4 for first-year students specializing in mathematics and physics). There is a specific set of courses for each speciality. Each subject is studied by lectures, accompanied by exercises. The examinations are written, as a rule, and consist of 3 or 4 independent questions, which may include problems from the course. But these are not always examinations in the classical sense of the word: two or three of the examinations for third-year students, given below, consist of exercises for which the students are given two and a half days and are on their honour to work independently.

There follow:

for first-year students—one question in analysis and one in linear algebra;

for third-year students—one question in group theory, one in differential geometry, and two in systems of differential equations.

Question 1. Let $\Omega = \{(x, y) \in \mathbb{R}^2 \mid x \neq 0\}$ and $\bar{f}(x,y) = \dfrac{\sin xy}{x}$ for $(x, y) \in \Omega$.

a) Calculate $\partial\bar{f}/\partial x = D_1\bar{f}(x, y)$ and $\partial\bar{f}/\partial y = D_2\bar{f}(x, y)$ for all $(x, y) \in \Omega$.

b) Find a power series $\sum_{n=0}^{\infty} a_n t^n$ such that

$$\bar{g}(t) = \frac{\sin t}{t} = \sum_{n=0}^{\infty} a_n t^n \quad \text{for all } t \in \mathbb{R} \setminus \{0\},$$

and hence deduce that $\bar{g} : \mathbb{R}\setminus\{0\} \to \mathbb{R}$ can be extended to a C^∞ function $g : \mathbb{R} \to \mathbb{R}$. Find $g(0)$ and $g'(0)$.

c) Prove that $\bar{f} : \Omega \to \mathbb{R}$ can be extended to a C^∞ function $f : \mathbb{R}^2 \to \mathbb{R}$. Find $f(0, y)$, $D_1f(0, y)$ and $D_2f(0, y)$ for all $y \in \mathbb{R}$.

Hint. $\bar{f}(x,y) = y\dfrac{\sin xy}{xy} = yg(xy)$ for $x \neq 0$, $y \neq 0$.

Question 2. Consider the matrix $B(c)$, $c \in \mathbb{R}$,

$$B(c) = \begin{pmatrix} c & 0 & 0 \\ 2 & c+1 & 2 \\ 1 & 1 & 1 \end{pmatrix}.$$

a) For each $c \in \mathbb{R}$, define a polynomial q_c of degree 2 such that $p_{B(c)}(t) = (c-t)q_c(t)$ for all c and $t \in \mathbb{R}$, where $p_{B(c)}$ is the characteristic polynomial of $B(c)$.

b) Prove that $q_c(c) = 0$ for $c = -1$. Show that for $c \neq 1$, $B(c)$ has three eigenvalues.

c) Find the set $D = \{c \in \mathbb{R} \mid B(c)$ is $\mathbb{R}$-diagonalizable$\}$. Prove that the set $O = \{c \in \mathbb{R} \mid B(c)$ is diagonalizable in an orthogonal basis$\}$ is empty.

Question 3. Let S be a regular surface, U an open subset of $\mathbb{R}^2$, and let $(u, v) \to X(u, v)$ be a parametrization of S, defined in U. Throughout this problem it is assumed that the coefficients E, F, G of the first fundamental form of S, calculated by means of the parametrization (X, U), have the following properties: E depends only on u, G only on v, and F is identically zero.

a) Let $w(t)$ be a vector field along a parametrized curve $X(u(t), v(t))$. Prove that the angle between $X_u(u(t), v(t))$ and $w(t)$ does not depend on t.

b) Show by means of a) (or independently) that the Gaussian curvature K is identically zero.

c) Prove that $X(u)$ is locally isometric to $\mathbb{R}^2$, using a parametrization of $\mathbb{R}^2$ of the form $(u, v) \to (\varphi(u), \psi(v))$, if desired.

Question 4. A group of order 12 has 6 conjugacy classes. State the number of classes of irreducible representations and the dimensions of these representations. Given that the orders of the classes $K_1, \ldots, K_6$ are equal to 1, 1, 2, 2, 3, 3, and that the functions φ_1, φ_2, φ_3 given below are characters, determine which of them are irreducible and construct the character table of the group.

	K_1	K_2	K_3	K_4	K_5	K_6
φ_1	1	1	1	1	−1	−1
φ_2	1	−1	−1	1	−1	1
φ_3	3	1	−2	0	1	−1

Question 5. Consider the boundary-value problem

$$\varepsilon y'' + (x^2 - 7x + 12)y' + (x_5)y = 0,$$
$$y(0) = \frac{91}{144}, \quad y(1) = \frac{3}{4}.$$

For any $\varepsilon > 0$, let $y(x, \varepsilon)$ denote the exact solution of this problem. For each ε determine an approximate solution $y^u(x, \varepsilon)$ such that

$$\max_{0 \leq x \leq 1} |y(x, \varepsilon) - y^u(x, \varepsilon)| = O(\varepsilon) \quad \text{as } \varepsilon \to 0.$$

Question 6. Consider the autonomous system in a plane region

$$\begin{cases} x' = x^2 + y^2 - 2xy - 5x + 5y + 4, \\ y' = xy - 4y, \end{cases} \qquad \Omega = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0\}.$$

a) Find the positions of equilibrium and the types of the non-degenerate positions of equilibrium.

b) Show that an orbit $(x(t), y(t))$, $t \in I$, that is in the first quadrant for $t_0 \in I$, remains in the closed first quadrant for all $t \geqslant t_0$, $t \in I$.

c) Find a closed bounded region that has the following property: each orbit that enters the region remains there subsequently.

France

For first-year students: parts I and II of the June 1991 examination in the University of Paris VI (the third part is a yes/no questionnaire on the whole of the material covered).

For third-year students: the examination on integration in the University of Paris VII in 1990.

Examination in the University of Paris VI (first year)

I

Let

$$A = \begin{pmatrix} 0 & 0 & -1 \\ 7 & 5 & -1 \\ 8 & 6 & -3 \end{pmatrix}.$$

The aim of the question is to find complex matrices B such that $B^3 = A$.

1) Find the eigenvalues and eigenspaces for A. Find a matrix P such that $D = P^{-1}AP$ has diagonal form with $a < b < c$ on the diagonal, and the first row of P consists of 1's.

2) Let E be a complex n-dimensional vector space and f an endomorphism of E with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let $v_1, v_2, \ldots, v_n$ denote non-zero eigenvectors corresponding to $\lambda_1, \lambda_2, \ldots, \lambda_n$ respectively.

Let g be an endomorphism of E such that $f \circ g = g \circ f$. Show that the vectors $v_1, v_2, \ldots, v_n$ are eigenvectors for g, and hence deduce that g can be diagonalized in the same basis as f.

3) Show that if $B^3 = A$, then $AB = BA$, and hence show that $\Delta = P^{-1}BP$ is diagonal.

4) Find all diagonal matrices Δ with complex elements for which $\Delta^3 = D$.

5) Hence find the number of matrices B with complex elements such that $B^3 = A$, and calculate explicitly those which have only real elements.

II

1) The plane is referred to an orthonormal basis (O_x, O_y). A function φ of class C^1 is given on an interval I of the axis $\mathbb{R}$. For each t in I, let M_t denote the point with coordinates $(\varphi(t), t^2+t)$, let P_t be the point $(t^2, 0)$, and Γ_φ the curve described by the point M_t as t varies in I.

Show that a necessary and sufficient condition for the tangent to Γ_φ at M_t to pass through the point P_t for each t except $t = -1/2$ is that the function φ satisfies a certain differential equation, and find this equation.

2) Let (E) be the differential equation

$$x(x+1)y' - (2x+1)y + x^2(2x+1) = 0\,.$$

a) For which points of the plane may it be stated a priori that locally one and only one integral curve passes through each of them? For which points of the plane may it be stated a priori that no integral curve passes through each of them?

b) Determine the set of points where the tangents to all the integral curves of equation (E) are horizontal.

c) Find the solutions of the differential equation (E) on the intervals

$$]-\infty, -1[, \quad]-1, 0[\quad \text{and} \quad]0, +\infty[.$$

d) Can these solutions be extended to -1? To 0? Is there a solution on $\mathbb{R}$?

3) We consider the curves Γ_φ defined in 1) for the functions φ that satisfy the equation (E) on $\mathbb{R}\setminus\{-1\}$, extended by continuity to -1. A sketch of the curves is attached.

a) The sketch suggests that the curves Γ_φ have two common points. Prove this and determine the tangents to the curves Γ_φ at these points.

b) The sketch suggests that each curve Γ_φ has a cusp R_φ and that the points R_φ lie on a straight line. Prove this and find the equation of this line.

Examination in the University of Paris VII (third year)

Parts I, II and III are fairly independent.

I

Let g be a bounded Borel function on the interval $[0, 1]$.

1) Show that for any $t \in [0, T]$ and fixed n

$$\sum_{k=1}^{\infty} ((-1)^{k-1}/k!) \int_0^T e^{kn(t-T+s)} g(s)\,ds = \int_0^T \left(1 - \exp(-e^{n(t-T+s)})\right) g(s)\,ds.$$

2) Assume henceforth that there exists a constant M such that

$$\left|\int_0^T g(s)e^{ns}\,ds\right| \le M$$

for each integer n. Show that for each $t \in [0, T]$ the left-hand side of equation 1) tends to 0 as n tends to infinity. Deduce that for all $t \leqslant T$

$$\int_{T-t}^{T} g(s)\,ds = 0.$$

3) Show that g is zero almost everywhere. If g is continuous, then it is identically zero.

4) Deduce from the foregoing that if f is a continuous function on $[1, v]$ such that

$$\left|\int_1^v x^n f(x)\,dx\right| \leq N < \infty$$

for all integers n, then f is identically zero.

5) If f is a continuous function on $[0, T]$ such that $\int_0^T t^n f(t)dt = 0$ for all n, then f is identically zero. (This can be deduced from 4), but can also be proved directly using the Stone–Weierstrass theorem and results from the present course.)

II

Let f be a locally bounded Borel function that vanishes on $]-\infty, 0]$. For fixed $T > 0$ put

$$I_n = \int_0^T e^{ns}(f * f)(s)\,ds.$$

1) By making the change of variables $u = T-v$, $s = 2T-v-w$, show that

$$I_n = \int_0^T dv \int_{T-v}^{T} e^{n(2T-v-w)} f(T-v) f(T-w)\,dw.$$

2) Henceforth we assume that $f*f$ is identically zero. Show that

$$\left(\int_0^T e^{-nv} f(T-v)\,dv\right)^2 = \iint_\Delta e^{-n(v+w)} f(T-v) f(T-w)\,dv\,dw,$$

where Δ is the region $\{(v, w) \mid 0 \leqslant v, w \leqslant T, v+w \leqslant T\}$.

3) Show, using I, that if f is continuous on $[0, 1]$, then f is identically zero.

III

Let f and g be two continuous functions that vanish on $]-\infty, 0]$ and are such that $f * g = 0$.

1) Put $f_1(t) = tf(t)$, $g_1(t) = tg(t)$. Show that

$$(f_1 * g) + (f * g_1) = 0,$$

and hence deduce that

$$(f * g) * (f_1 * g_1) + (f * g_1) * (f * g_1) = 0.$$

2) Deduce from this that for any n and t

$$\int_0^t f(t-u)u^n g(u)\,du = 0.$$

3) Show finally that f and g are identically zero.

Below there follow problems from the examinations on differential equations in 1992 in the Mechanics and Mathematics Faculty of Moscow State University.

Examination in linear theory (N.Kh. Rozov, January 1992; 3 hours)

1) Consider the linear equation

$$(\alpha t+\alpha)\frac{d^3y}{dt^3}+\pi\alpha\frac{d^2y}{dt^2}-(\alpha-1)t^2\cdot\tan t\cdot y=\ln\frac{e+t}{e-t} \tag{1}$$

with the additional conditions

$$y(a)=1,\quad y'(a)=A,\quad y''(a)=\alpha; \tag{2}$$

here α, a, A are real parameters.

a) (2 marks) Find all the values of the parameters α, a, A for which the existence and uniqueness theorem guarantees that there is a unique solution of the Cauchy problem for the third-order differential equation (1) with initial conditions (2).

b) (1 mark) What is the domain of definition of the non-extendable solution of the initial problem (1)–(2) in the case $\alpha=-1$, $a=-2$, $A=-3$?

c) (1 mark) Represent the initial problem (1)–(2) in the form of a Cauchy problem for a normal linear system of differential equations.

2. Let (3) denote the linear homogeneous equation corresponding to the non-homogeneous equation (1) with $\alpha=-1$. For this equation (3) consider the three Cauchy problems with the respective initial conditions

$$y(1)=2,\quad y'(1)=0,\quad y''(1)=0; \tag{4}$$

$$y(1)=-1,\quad y'(1)=1\quad y''(1)=-1; \tag{5}$$

$$y(1)=0,\quad y'(1)=0,\quad y''(1)=-2; \tag{6}$$

let $\varphi_1(t)$, $\varphi_2(t)$, $\varphi_3(t)$, $t\in I$, be non-extendable solutions of these problems.

a) (1 mark) Find the interval I explicitly and prove that $\{\varphi_1(t),\varphi_2(t),\varphi_3(t)\}$ is a fundamental system of solutions of equation (3) on I.

b) (1 mark) Obtain an expression for the Wronskian determinant of the solutions $\varphi_1(t)$, $\varphi_2(t)$, $\varphi_3(t)$, valid on the interval I.

c) (2 marks) Write down the solution of the Cauchy problem for equation (3) with initial conditions

$$y(1)=y_0,\quad y'(1)=y_1,\quad y''(1)=y_2$$

in terms of the functions $\varphi_1(t)$, $\varphi_2(t)$, $\varphi_3(t)$.

3. (3 marks) Does the set of all real solutions of the equation

$$y''' + y = a\cos t, \qquad a = \text{const} > 0,$$

contain a periodic function?

4. (5 marks) Calculate the fundamental matrix e^{tA} for the linear homogeneous system

$$\dot{x} = Ax, \qquad x \in \mathbb{R}^3,$$

with constant matrix

$$A = \begin{pmatrix} -2 & 1 & -2 \\ 1 & -2 & 2 \\ 3 & -3 & 5 \end{pmatrix}.$$

5. (4 marks) Let $K(t, \tau)$ be the Cauchy matrix (fundamental matrix normalized at the point τ) for the linear homogeneous system

$$\dot{x} = A(t)x, \qquad x \in \mathbb{R}^n,$$

where the matrix $A(t)$ is continuous on $\mathbb{R}$. Express the derivative $\partial K(t, \tau)/\partial\tau$ in terms of the matrices A and K.

Examination in non-linear differential equations (N.Kh. Rozov, June 1992)

1. (3 marks) It is known that the function $u(t) \in C([0, \infty])$ satisfies the double inequality

$$0 \le u(t) \le \frac{1}{\pi} + \int_0^t e^{-\pi\tau}[u(\tau)]^2\,d\tau \quad \text{for all} \quad t \in [0, \infty).$$

Find an upper bound for the quantity $\sup_{[0,\infty]} u(t)$.

2. (4 marks) Can the third-order equation

$$\dddot{x} = f(t, x, \dot{x}, \ddot{x})$$

with a continuously differentiable right-hand side $f(t, x, u, v)$ have both functions

$$x_1 = 3 + \sin t - 2\cos t, \qquad t \in \mathbb{R},$$
$$x_2 = \frac{1}{1-t}, \qquad -1 < t < \frac{1}{2},$$

among its solutions? Justify your answer.

3. For what (real) values of the parameter a is the trivial solution of the system of differential equations

$$\begin{cases} \dot{x} = ax + y + (a+1)x^2, \\ \dot{y} = x + ay \end{cases}$$

a) (1 mark) Asymptotically stable?

b) (2 marks) Stable in the sense of Lyapunov, but not asymptotically stable?

c) (1 mark) Unstable?

4. Consider the phase portrait of the equation

$$\ddot{x} + 2\delta\dot{x} - x = 0$$

on the phase plane $(x, y = \dot{x})$.

a) (1 mark) Determine the type of the phase portrait of this equation for each (real) value of the parameter δ.

b) (2 marks) Sketch the phase portraits of the given equation for $\delta = -1$ and $\delta = 1$.

c) (1 mark) Find the position of equilibrium in the phase plane for the given equation when $\delta = 0$, and explain whether it is asymptotically stable, stable in the sense of Lyapunov, or unstable (justifying your answer). How many rectilinear trajectories are there in this case in the phase portrait?

5. a) (1 mark) State the theorem on differentiability with respect to a parameter of the solution of a system of differential equations.

b) (4 marks) Calculate the derivative with respect to λ at $\lambda = 0$ of the solution $x = \varphi(t, \lambda)$ of the Cauchy problem

$$\ddot{x} + x = \lambda \sin t + \lambda x^2,$$
$$x(0) = 0, \quad \dot{x}(0) = 0 .$$

Examination in differential equations (A.F. Filippov, June 1992)

1. For what constants a and b are all solutions of the system

$$\begin{cases} \dot{x} = 2y - 4x + a, \\ \dot{y} = 2x - y + b \end{cases}$$

bounded for $t > 0$?

2. a) Give the definition of stability in the sense of Lyapunov.

b) Find a solution with period π of the system

$$\begin{cases} \dot{x} = x - y, \\ \dot{y} = 2x - y + 6\sin^2 t . \end{cases}$$

c) Is this solution stable?

3. For the equation

$$\ddot{x} + 4x - 6x^2 = 0$$

a) Find the trajectory in the phase plane that passes through the point $(1, 0)$.

b) Find the solution of the equation with initial conditions

$$x(0) = 1, \quad \dot{x}(0) = 0.$$

4. Find the derivative with respect to the parameter μ at $\mu = 0$ of the solution of the problem

$$y' = \mu x + \frac{1}{2y}, \quad y(1) = 1 - 2\mu, \quad x > 0.$$

5. a) State the theorem on the existence of a solution of the Cauchy problem for a quasilinear partial differential equation.

b) Solve the Cauchy problem

$$xy\frac{\partial z}{\partial x} + xz\frac{\partial z}{\partial y} = yz, \qquad \text{curve } L : x = 1, z = 1 + y^2.$$

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