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On teaching mathematics[1]

Mathematics is a part of physics. Physics is an experimental science, a part of natural science. Mathematics is the part of physics where experiments are cheap.

The Jacobi identity (which forces the altitudes of a triangle to meet in a point) is an experimental fact in the same way as the fact that the earth is round (that is, homeomorphic to a ball). But it can be discovered with less expense.

In the middle of the twentieth century an attempt was made to separate physics and mathematics. The consequences turned out to be catastrophic. Whole generations of mathematicians grew up without knowing half of their science and, of course, in total ignorance of any other sciences. They first began teaching their ugly scholastic pseudo-mathematics to their students, then to schoolchildren (forgetting Hardy's warning that ugly mathematics has no permanent place under the sun).

Since scholastic mathematics that is cut off from physics is fit neither for teaching nor for application in any other science, the result was a universal hatred of mathematicians, both on the part of the poor schoolchildren (some of whom in the meantime became ministers) and of the users.

The ugly building constructed by undereducated mathematicians who were exhausted by their inferiority complexes and who were unable to make themselves familiar with physics, reminds one of the rigorous axiomatic theory of odd numbers. Obviously, it is possible to create such a theory and make pupils admire the perfection and internal consistency of the resulting structure (in which, for example, the sum of an odd number of terms and the product of any number of factors are defined). From this sectarian point of view, even numbers could either be declared a heresy or, with the passage of time, be introduced into the theory supplemented with a few "ideal" objects (in order to comply with the needs of physics and the real world).

Unfortunately, it was an ugly twisted construction of mathematics like the one above which predominated in the teaching of mathematics for decades. Having originated in France, this perversity quickly spread to the teaching of foundations of mathematics, first to university students, then to school children in all specializations (first in France, then in other countries, including Russia).

To the question "what is 2 + 3" a French primary school pupil replied "3 + 2, since addition is commutative". He did not know what the sum was equal to and could not even understand what was being asked!

Another French pupil (quite rational, in my opinion) defined mathematics as follows: "there is a square, but that still has to be proved". Judging by my teaching experience in France, the university students' idea of mathematics (even those

[1]This is an extended text of an address at a discussion on the teaching of mathematics in Palais de Découverte in Paris on 7 March 1997.

taught mathematics at the Ecole Normale Supérieure: I am sorriest of all for these obviously intelligent but deformed young people) is as poor as that of this pupil.

For example, these students have never seen a paraboloid and a question about the shape of the surface given by the equation $xy = z^2$ puts the mathematicians studying at ENS into a stupour. Drawing a curve given by parametric equations (like $x = t^3 - 3t$, $y = t^4 - 2t^2$) on a plane is a totally impossible problem for them (and, probably, even for most French professors of mathematics).

Beginning with l'Hôpital's first textbook on calculus ("calculus for understanding curved lines") and continuing roughly until Goursat's textbook, the ability to solve such problems was considered (along with knowledge of the times tables) to be a necessary part of the craft of every mathematician.

Mentally challenged zealots of "abstract mathematics" removed all the geometry (whereby connections with physics and reality most often occur in mathematics) from the curriculum. The calculus textbooks by Goursat, Hermite and Picard were recently dumped by the student library of the Universities Paris 6 and 7 (Jussieu) as obsolete and, therefore, harmful (they were only rescued by my intervention).

ENS students who have sat through courses on differential and algebraic geometry (read by respected mathematicians) turned out be acquainted neither with the Riemann surface of the elliptic curve $y^2 = x^3 + ax + b$ nor even with the topological classification of surfaces (not to mention elliptic integrals of the first kind and the group property of an elliptic curve, that is, the Euler-Abel addition theorem). They were only taught Hodge structures and Jacobian varieties!

How could this happen in France, which gave the world Lagrange and Laplace, Cauchy and Poincaré, Leray and Thom? It seems to me that a reasonable explanation was given by I. G. Petrovskii, who taught me in 1966: genuine mathematicians do not gang up, but the weak need gangs in order to survive. They can unite on various grounds (it could be super-abstraction, anti-Semitism or "applied and industrial" problems), but the essence is always the solution of a social problem: survival in conditions of more literate surroundings.

By the way, I shall remind you of a warning given by Pasteur: there never have been and never will be any "applied sciences", there are only *applications of sciences* (quite useful ones!).

In those days I treated Petrovskii's words with some doubt, but now I am becoming more and more convinced of how right he was. A considerable part of super-abstract activity simply boils down to industrializing the shameless grabbing of discoveries from discoverers and then systematically assigning them to epigons–generalizers. Just as America is not named after Columbus, mathematical results are almost never called by the names of their discoverers.

In order to avoid being misquoted, I have to note that for some unknown reason my own achievements have never been expropriated in this way, although this was constantly happening both to my teachers (Kolmogorov, Petrovskii, Pontryagin, Rokhlin) and to my pupils. Professor M. Berry once formulated the following two principles.

The Arnold Principle. If a notion bears a personal name, then this name is not the name of the discoverer.

The Berry Principle. The Arnold Principle is applicable to itself.

It is, however, time to go back to the teaching of mathematics in France.

When I was a first-year student in the Faculty of Mechanics and Mathematics at Moscow State University, the lectures on calculus were read by the set-theoretic topologist L. A. Tumarkin, who conscientiously retold the old classical calculus course of French type in the Goursat version. He told us that integrals of rational functions along algebraic curves can be taken if the corresponding Riemann surface is a sphere and, generally speaking, cannot be taken if its genus is higher, and that for sphericity it is sufficient to have a sufficiently large number of double points on the curve of a given degree (which forces the curve to be unicursal: it is possible to draw its real points on the projective plane without lifting the pen from the paper).

These facts capture the imagination to the extent that (even when given without any proofs) they give a better and more correct idea of modern mathematics than all the volumes of the Bourbaki treatise. Indeed, here we discover the existence of a wonderful connection between things which seem to be completely different: on the one hand, the existence of an explicit expression for the integrals and the topology of the corresponding Riemann surface and, on the other hand, the number of double points and the genus of the corresponding Riemann surface, which also exhibits itself in the real domain in the form of unicursality.

Jacobi noted the most fascinating property of mathematics, that in it one and the same function controls both the presentations of an integer as a sum of four squares and the real movement of a pendulum.

These discoveries of connections between heterogeneous mathematical objects can be compared with the discovery of the connection between electricity and magnetism in physics or with the discovery of the similarity in the geology of the east coast of America and the west coast of Africa.

The emotional significance of such discoveries for teaching is difficult to overestimate. It is they who teach us to search and find such wonderful phenomena of harmony in the universe.

The de-geometrization of mathematical education and the divorce from physics sever these ties. For example, not only students but also modern algebraic geometers on the whole do not know the fact, observed here by Jacobi, that an elliptic integral of the first kind expresses the time of motion along an elliptic phase curve in the corresponding Hamiltonian system.

Rephrasing the famous words on the electron and the atom, it can be said that a hypocycloid is as inexhaustible as an ideal in a polynomial ring. But teaching ideals to students who have never seen a hypocycloid is as ridiculous as teaching addition of fractions to children who have never cut (at least mentally) a cake or an apple into equal parts. No wonder that children prefer to add a numerator to a numerator and a denominator to a denominator.

From my French friends I have heard that a tendency towards super-abstract generalizations is their traditional national trait. I do not entirely disagree that this might be a question of a hereditary disease, but I would like to underline the fact that I borrowed the cake-and-apple example from Poincaré.

The scheme of construction of a mathematical theory is exactly the same as that in any other natural science. First we consider some objects and make some observations in special cases. Then we try and find the limits of application of our observations by seeking counter-examples to prevent the unjustified extension of our observations to too wide a range of events (example: the number of partitions of the consecutive odd numbers 1, 3, 5, 7, 9 into an odd number of summands gives the sequence 1, 2, 4, 8, 16, but then comes 29).

As a result we formulate the empirical discovery that we have made (for example, Fermat's conjecture or Poincaré's conjecture) as clearly as possible. After this there comes the difficult period of checking the reliability of the conclusions obtained.

At this point a special technique has been developed in mathematics. This technique, when applied to the real world, is sometimes useful, but can sometimes also lead to self-deception. This technique is called modelling. When constructing a model, the following idealization is made: certain facts which are only known with a certain degree of probability or with a certain degree of accuracy, are considered to be "absolutely" correct and are accepted as "axioms". The sense of this "absoluteness" lies precisely in the fact that we allow ourselves to operate with these "facts" according to the rules of formal logic, in the process declaring as "theorems" all that we can derive from them.

It is obvious that in any real-life activity it is impossible to place total reliance on such deductions, if only because the parameters of the phenomena studied are never known absolutely exactly and a small change in parameters (for example, in the initial conditions of a process) can totally change the result. It is for this reason that a reliable long-term weather forecast is impossible and will remain impossible, no matter how much we develop computers and devices which record initial conditions.

In exactly the same way a small change in the axioms (of which we cannot be completely sure) is capable, generally speaking, of leading to completely different conclusions from those that are obtained from theorems which have been deduced from the accepted axioms. The longer and fancier the chain of deductions ("proofs"), the less reliable is the final result.

Complex models are rarely useful (except for those writing their dissertations).

The mathematical technique of modelling consists in ignoring this trouble and speaking about your deductive model as if it coincided with reality. The fact that this path, which is obviously incorrect from the point of view of natural science, often leads to useful results in physics is called "the inconceivable effectiveness of mathematics in the natural sciences" (or "the Wigner principle").

Here we can add a remark by I. M. Gel'fand: there exists another phenomenon comparable in its inconceivability with the inconceivable effectiveness of mathematics in physics noted by Wigner, and that is the equally inconceivable ineffectiveness of mathematics in biology.

"The subtle poison of mathematical education" (in F. Klein's words) for a physicist consists precisely in that the absolutized model is separated from reality and is no longer comparable with it. Here is a simple example. Mathematics teaches us that the solution of the Malthus equation $dx/dt = x$ is uniquely determined by the initial conditions (that is, that the corresponding integral curves in the (t, x)-plane do not intersect). This conclusion of the mathematical model bears little relevance to reality. A computer experiment shows that all these integral curves have common points on the negative t-semi-axis. Thus, curves with the initial conditions $x(0) = 0$ and $x(0) = 1$ practically intersect at $t = -10$, and at $t = -100$ you cannot fit in an atom between them. Properties of the space at such small distances are not described at all by Euclidean geometry. The application of the uniqueness theorem in this situation obviously exceeds the accuracy of the model. This has to be respected in practical applications of the model, otherwise one might find oneself in serious trouble.

I would like to note, however, that the same uniqueness theorem explains why the final stage of mooring a ship to a quay is carried out manually: on steering if the velocity of approach were defined as a smooth (linear) function of the distance, the process of mooring would require an infinitely long period of time. The alternative is an impact with the quay (which is damped by suitable non-ideally elastic bodies). Incidentally, this problem had to be seriously confronted when landing the first descending apparatus on the Moon and Mars, and also when docking with space stations; here the uniqueness theorem is working against us.

Unfortunately, neither such examples nor discussion of the danger of fetishising theorems are to be found in modern mathematical textbooks, even the better ones. I even get the impression that scholastic mathematicians (who have little knowledge of physics) believe in the principal difference of axiomatic mathematics from the modelling which is common in natural science and which always requires the subsequent checking of deductions by an experiment.

Not to mention the relative character of initial axioms, one cannot ignore the inevitability of logical mistakes in long arguments (say, in the form of a computer breakdown caused by cosmic rays or quantum oscillations). Every working mathematician knows that, without some form of control (best of all by examples), after some ten pages half the signs in formulae will be wrong and twos will find their way from denominators into numerators.

The technology of combatting such errors is the same external control by experiments or observations as is to be found in any experimental science, and it should be taught from the very beginning to all juniors in schools.

Attempts to create "pure" deductive-axiomatic mathematics have led to the rejection of the scheme used in physics (observation, model, investigation of the model, conclusions, testing by observations) and its replacement by the scheme definition, theorem, proof. It is impossible to understand an unmotivated definition but this does not stop the criminal algebraist-axiomatizers. For example, they would readily define the product of natural numbers by means of the long multiplication rule. Then the commutativity of multiplication becomes difficult to prove but it is still possible to deduce it as a theorem from the axioms. It is then possible to force poor students to learn this theorem and its proof (with the aim of bolstering the authority of both the science and the persons teaching it). It is obvious that such definitions and proofs can do nothing but harm to the teaching and practical work.

It is only possible to understand the commutativity of multiplication by counting and re-counting soldiers by ranks and files or by calculating the area of a rectangle in two ways. Any attempt to do without this interference by physics and reality with mathematics is sectarian and isolationist, and destroys the image of mathematics as a useful human activity in the eyes of all sensible people.

I shall reveal a few more such secrets (in the interest of poor students).

The *determinant* of a matrix is the (oriented) volume of the parallelepiped whose edges are its columns. If students are told this secret (which is carefully hidden in purified algebraic education), then the whole theory of determinants becomes a clear chapter of the theory of multilinear forms. If determinants are defined otherwise, then any sensible person will forever hate all determinants, Jacobians and the implicit function theorem.

What is a *group*? Algebraists teach that this is supposedly a set with two operations that satisfy a load of easily-forgettable axioms. This definition provokes a natural protest: why would any sensible person need such pairs of operations? "Oh, curse this maths" concludes the student (who, possibly, becomes the Minister for Science in the future).

We get a totally different situation if we start off not with the group but with the concept of a transformation (a one-to-one mapping of a set onto itself) as was done historically. A collection of transformations of a set is called a group if along with any two transformations it contains the result of their consecutive application, and along with any transformation its inverse.

This is the entire definition. The so-called "axioms" are in fact just (obvious) *properties* of groups of transformations. What axiomatizers call "abstract groups" are just groups of transformations of various sets considered up to isomorphism (a one-to-one mapping preserving the operations). As Cayley proved, there are no "more abstract" groups in the world. So why do the algebraists keep on tormenting students with the abstract definition?

By the way, in the 1960s I taught group theory to Moscow *schoolchildren*. Avoiding all the axiomatics and staying as close as possible to physics, in half a year I reached Abel's theorem on the insolubility of a general equation of degree five by radicals (having on the way taught the pupils complex numbers, Riemann surfaces, fundamental groups and monodromy groups of algebraic functions). This course was later published by one of the audience, V. Alekseev, as a book: *Abel's theorem via problems*.

What is a *smooth manifold*? In a recent American book I read that Poincaré was not acquainted with this notion (which he himself introduced) and that the "modern" definition was only given by Veblen in the late 1920s: a manifold is a topological space which satisfies a long series of axioms.

For what sins must students try and find their way through all these twists and turns? Actually, in Poincaré's *Analysis Situs* there is an absolutely clear definition of a smooth manifold which is much more useful than the "abstract" one.

A smooth k-dimensional submanifold of the Euclidean space $\mathbb{R}^N$ is a subset which in a neighbourhood of each of its points is the graph of a smooth mapping of $\mathbb{R}^k$ into $\mathbb{R}^{N-k}$ (where $\mathbb{R}^k$ and $\mathbb{R}^{N-k}$ are coordinate subspaces). This is a straightforward generalization of the commonest smooth curves on the plane (such as the circle $x^2 + y^2 = 1$) and of curves and surfaces in three-dimensional space.

Between smooth manifolds are naturally defined smooth mappings. Diffeomorphisms are mappings which, together with their inverses, are smooth.

An "abstract" smooth manifold is a smooth submanifold of a Euclidean space considered up to diffeomorphism. There are no "more abstract" finite-dimensional smooth manifolds in the world (Whitney's theorem). Why do we keep on tormenting students with the abstract definition? Would it not be better to prove for them the theorem on the explicit classification of closed two-dimensional manifolds (surfaces)?

It is this wonderful theorem (which asserts, for example, that any compact connected oriented surface is a sphere with a number of handles) that gives a correct impression of what modern mathematics is, and not the super-abstract generalizations of naïve submanifolds of a Euclidean space which in fact do not give anything new and are presented as achievements of the axiomatizers.

The theorem on the classification of surfaces is a top-class mathematical achievement, comparable with the discovery of America or X-rays. This is a genuine discovery of mathematical natural science and it is even difficult to say whether the fact itself is more attributable to physics or to mathematics. In its significance for both applications and the development of the right Weltanschauung it by far surpasses such "achievements" of mathematics as the proof of Fermat's last theorem or the proof of the fact that any sufficiently large whole number can be represented as the sum of three primes.

For the sake of publicity modern mathematicians sometimes present such sporting achievements as the last word in their science. Understandably this not only does not contribute to society's appreciation of mathematics but, on the contrary, causes a healthy distrust of the necessity of wasting energy on (rock-climbing-type) exercises with such exotic questions of interest to no-one.

The theorem on the classification of surfaces should be included in high school mathematics courses (probably, without proof), but for some reason is not even included in university mathematics courses (from which in France, by the way, all geometry has been banished in recent decades).

The return of mathematical teaching at all levels from scholastic chatter to presenting the important domain of natural science is the main current problem for France. I was astonished to learn that all the mathematical books with the best and most important methodical approach are almost unknown to students here (and apparently have not been translated into French). Among these are *Numbers and figures* by Rademacher and Toeplitz, *Geometry and the imagination* by Hilbert and Cohn-Vossen, *What is mathematics?* by Courant and Robbins, *How to solve it* and *Mathematics and plausible reasoning* by Polya, and *Development of mathematics in the nineteenth century* by F. Klein.

I remember well what a strong impression Hermite's calculus course (which does exist in Russian translation!) made on me in my school years.

Riemann surfaces appear in it, I think, in one of the first lectures (all the analysis is, of course, complex, as it should be). Asymptotics of integrals are investigated by means of path deformations on Riemann surfaces under the motion of branch points (nowadays, we would call this Picard-Lefschetz theory). Picard, by the way, was Hermite's son-in-law; mathematical abilities are often inherited by sons-in-law: the dynasty Hadamard, P. Levy, L. Schwarz, U. Frisch is another famous example in the Paris Academy of Sciences.

Hermite's "obsolete" course of one hundred years ago (now probably discarded from student libraries of French universities) was much more modern than the very boring calculus textbooks with which students are nowadays tormented.

If mathematicians do not come to their senses, then the consumers, who continue to need mathematical theory that is modern in the best sense of the word and who preserve the immunity of any sensible person to useless axiomatic chatter, will in the end turn down the services of the undereducated scholastics in both the schools and the universities.

A teacher of mathematics who has not got to grips with at least some of the volumes of the course by Landau and Lifshitz will then become a relic like the person nowadays who does not know the difference between an open and a closed set.

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