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Piecewise smooth vector fields: index of singularities and some results about the existence of limit cycles

Campos vetoriais suaves por partes: índice de singularidades e alguns resultados sobre a existência de ciclos limites

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Tese apresentada ao Instituto de Matemática, Estatística e Computação Científica da Universidade Estadual de Campinas como parte dos requisitos exigidos para a obtenção do título de Doutora em Matemática.

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"We ourselves feel that what
we are doing is just a drop in the ocean.
But the ocean would be less because of
that missing drop."
Mother Teresa.

Resumo

Os campos vetoriais de Filippov são objeto de estudo de grande relevância, tanto em termos de seus aspectos teóricos quanto aplicados. Além disso, a análise de conjuntos minimais desempenha um papel fundamental na compreensão do comportamento qualitativo global de sistemas dinâmicos. Assim, a determinação da existência ou não desses conjuntos é um tema crucial e amplamente explorado nesta área de pesquisa. Nesta tese, investigamos se determinadas classes de campos vetoriais de Filippov apresentam ciclos limite após pequenas perturbações. A característica de Euler de uma variedade compacta bidimensional e o comportamento local dos campos vetoriais suaves definidos nela estão interligados pelo Teorema de Poincaré-Hopf. Até então, tal resultado não havia sido estabelecido para campos vetoriais de Filippov, e demonstramos a sua validade neste contexto. Enquanto, nos casos suaves, as singularidades correspondem aos pontos onde o campo vetorial se anula, no âmbito dos campos vetoriais de Filippov, a noção de singularidade abrange novos tipos de pontos, a saber, pontos de pseudo-equilíbrio e tangência. Neste contexto, a definição clássica de índice para singularidades em campos vetoriais suaves é estendida para abranger as singularidades dos campos vetoriais de Filippov.

Palavras-chave: Ciclo limite. Campos vetoriais de Filippov. Índice de singularidades. Sistemas dinâmicos. Sistemas Hamiltonianos.

Abstract

Filippov vector fields are the subject of highly relevant study, both in terms of their theoretical and applied aspects. Additionally, the analysis of minimal sets plays a fundamental role in understanding the global qualitative behavior of dynamical systems. Therefore, determining the existence or non-existence of these sets is a crucial and extensively explored topic in this research area. In this thesis, we investigate whether certain classes of Filippov vector fields exhibit limit cycles after small perturbations. The Euler characteristic of a compact two-dimensional manifold and the local behavior of smooth vector fields defined on it are interconnected through the Poincaré-Hopf Theorem. Until now, such a result had not been established for Filippov vector fields, and we demonstrate its validity in this context. While, in smooth cases, singularities consist of points where the vector field vanishes, in the context of Filippov vector fields, the notion of singularity also includes new types of points, namely, pseudo-equilibrium points and tangency points. In this context, the classical definition of the index for singularities in smooth vector fields is extended to encompass the singularities of Filippov vector fields.

Keywords: Limit cycle. Filippov vector fields. Singularity index. Dynamical systems. Hamiltonian systems.

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Introduction

Understanding singularity indices and the Poincaré-Hopf theorem holds paramount importance in the realm of dynamical systems and topology. Singularity indices provide a profound insight into the nature of critical points within vector fields, offering a means to classify and comprehend the behavior of complex dynamical systems. These indices serve as key indicators of stability, helping researchers distinguish between different types of singularities and predict the overall behavior of a system. The Poincaré-Hopf theorem, on the other hand, establishes a deep connection between the topology of a manifold and the distribution of singularities on that manifold (see, for instance [54]). It serves as a fundamental tool for characterizing the global behavior of vector fields and understanding the topological features of the underlying space.

Filippov dynamical systems constitute a distinct class of nonlinear dynamic systems characterized by nonsmooth differential equations. They derive their name from the Russian mathematician Alexei Filippov, who pioneered the theory underlying these systems. Considering the significance of Filippov systems and their extensive practical applications (see, for instance, [27, 45]), and recognizing the absence of a result akin to the Poincaré-Hopf theorem in this particular context, we have formulated an index (see, for instance, [7]). This index serves as a generalization of the continuous vector fields index, with the added capability of being computable for Filippov discontinuous fields.

Theorem A. *Let $\mathcal{Z}$ be a Filippov vector field (defined on a 2-dimensional compact manifold M). Denote the set of the singularities of $\mathcal{Z}$ by $\mathcal{S}$ and assume that they are all isolated. Then,*

$$\sum_{p\in\mathcal{S}} I_p(\mathcal{Z}) = \chi(M),$$

where $\chi(M)$ is the Euler Characteristic of M and $I_p(\mathcal{Z})$ is the index of singularity p in the Filippov vector field $\mathcal{Z}$.

Theorem A becomes feasible due to the introduction of a new definition of singularity index, based on the following regularization invariance principle:

Theorem B. *Let Z be the Filippov vector field given by* (1.3), Z_ε *its ST-regularization* (1.8), *D an open set and $B \subset D$, a closed ball such that ∂B does not contain any singularities of Z. Then, for $\varepsilon > 0$ sufficiently small, ∂B does not contain any singularities of Z_ε and $I_{\partial B}(Z) = I_{\partial B}(Z_\varepsilon)$.*

Limit cycles hold profound significance in the study of dynamical systems, providing crucial insights into their behaviour, stability, and applications across various fields. In the context of general dynamical systems, the presence of limit cycles serves as a key indicator of sustained oscillatory behavior, offering a comprehensive understanding of periodic orbits and system dynamics. These cycles play a pivotal role in stability analysis, control design, and the identification of bifurcation scenarios, influencing fields ranging from engineering to ecology.

In the first result in which we provide an upper bound for the number of limit cycles, we work with a piecewise continuous vector field, separated by a circle, in which one of the fields has a linear center and the other field has a quadratic center, (see, for instance, [5]). In [10], we have that a quadratic differential system that has centers at the origin can be written as $\dot{x} = -y - bx^2 - cxy - dy^2$, $\dot{y} = x + ax^2 + exy - ay^2$, if at least one of the four following conditions hold (i) $e - 2b = c + 2a = 0$ or (ii) $b + d = 0$ or (iii) $c + 2a = e + 3b + 5d = a^2 + bd + 2d^2 = 0$ or (iv) $c = a = 0$. This result can also be found in the original works of Kapteyn and Bautin [2, 28, 29].

Theorem C. *Consider a differential system formed by a linear differential center and a quadratic differential center and separated by a circle. Consider the change of variables $x = kX + \alpha, \quad y = MY + \beta$, with $k, M \neq 0$. Then the following statements hold.*

(a) There are no continuous piecewise differential systems (3.1) *with quadratic differential center of type (i), (ii), (iii), with $d \neq 0$. When $d = 0$ system (iii) becomes system (iv).*

(b) *The continuous piecewise differential system* (3.1) *with a quadratic differential center of type (iv) has at most three limit cycles.*

In Filippov systems, where non-smoothness introduces additional complexity, the study of limit cycles becomes especially pertinent. Limit cycles in Filippov systems contribute to our understanding of robust periodic behavior in the presence of discontinuities, shedding light on the intricate dynamics of systems with sliding motions. The exploration of limit cycles in Filippov systems is essential for unraveling the unique challenges posed by non-smooth dynamical behavior, making it a crucial area of research with applications in control, robotics, and other domains where discontinuous dynamics are prevalent. Overall, the study of limit cycles in both general and Filippov dynamical systems plays a pivotal role in advancing our understanding of complex, real-world phenomena and enhancing the predictability and control of dynamic processes.

Theorem D. *Given a discontinuous piecewise differential system, separated by a straight line, defined by two arbitrary Hamiltonians $H_1(x, y)$ and $H_2(x, y)$*

(a) *if both Hamiltonians are of degree* 2*, then system has no limit cycles.*

(b) *if both Hamiltonians are of degree* 3*, then system has at most one limit cycle.*

(c) *if both Hamiltonians are of degree* 4*, then system has at most three limit cycles.*

Moreover, there are differential systems formed by two convenient Hamiltonians $H_1(x, y)$ and $H_2(x, y)$ of the corresponding degree realizing the upper bounds on the number of limit cycles of statements (a) and (b).

Finally, we consider three lines of discontinuity, the circle, the parabola and the hyperbola, where the vector fields are Hamiltonians of degree two.

Theorem E. *The maximum number of limit cycles of piecewise differential systems formed by two hamiltonians systems of degree two, intersecting the discontinuity line $f(x, y) = 0$ in two points the is*

(i) 3*, for $f(x, y) = x^2 + y^2 - 1$,*

(ii) 3, *for* $f(x,y) = x^2 - y$,

(iii) 2, *for* $f(x,y) = 1 - xy$.

These upper bounds are reached.

This thesis is organized as follows:

In chapter 1, we extend the classical index definition for singularities of smooth vector fields to encompass singularities of Filippov vector fields. This extension is grounded on an invariance property under a regularization process. Introducing this novel index definition enables us to formulate a version of the Poincaré–Hopf Theorem specifically tailored for Filippov vector fields. As a consequential result, we establish a Hairy Ball Theorem within this context, asserting that "any Filippov vector field defined on a sphere must possess at least one singularity (in the Filippov sense)."

In chapter 2, we classify the index of singularities of low codimension. In the context of continuous vector fields, well-established knowledge dictates that the index of a saddle is consistently minus one, while the index of a node is consistently one, irrespective of its stability. This means that by merely identifying the singularity, we can gain insight into its index without the need to calculate the integral that defines it. In this chapter, our goal is to adopt a parallel approach, extending this principle to classify Filippov singularities of codimension zero and one.

In chapter 3, we delve into the analysis of continuous piecewise differential systems delimited by a circle, comprising a linear differential center and a quadratic differential center. As it is typical in planar differential systems, one of the main difficulties for understanding their dynamics consists in controlling their limit cycles. Our research is centered on discerning the maximum number of limit cycles that can be exhibited by such a continuous piecewise differential system.

In chapter 4, our focus shifts to the examination of the maximum number of limit cycles in discontinuous piecewise differential systems, composed of two Hamiltonian systems separated by a straight line. We investigate three distinct scenarios: when both Hamiltonian systems on each side of the discontinuity line simultaneously have degrees one, two, or three. Our findings reveal that in these three cases, the maximum number of limit cycles is zero, one, and three, respectively. Additionally, we provide evidence that there exist discontinuous piecewise differential systems that achieve these maximum numbers of limit cycles.

In chapter 5, we studied the maximum number of limit cycles of discontinuous piecewise differential systems, formed by two Hamiltonians systems of degree one with three distinct lines of discontinuity either a circle, or a parabola, or a hyperbola. Our analysis reveals that the maximum number of limit cycles for piecewise differential systems intersecting the line of discontinuity at two points is three for both the circle and the parabola, and two for the hyperbola.

1 Poincaré-Hopf Theorem for Filippov vector fields on 2-dimensional compact manifolds

The content within this chapter corresponds to paper [7].

1.1 Introduction

The Poincaré–Hopf Theorem is a classical result that relates the Euler characteristic of a compact manifold with the indices of the singularities of smooth vector fields defined on it (see, for instance, [18, 47]). A well known application of such a theorem is the *Hairy Ball Theorem* which asserts that any smooth vector field defined on a sphere has at least one singularity (see, for instance, [46]). This result, although it may seem purely theoretical, is also useful in applied areas of science (see, for instance, [53, 57]).

On the other hand, Filippov vector fields constitute an important class of dynamical systems, mainly because of their wide range of applications in many areas of science (see, for instance, [27, 45]). Roughly speaking, Filippov vector fields are piecewise smooth vector fields for which the local trajectories at points of non-smoothness are provided by the Filippov's convention. The concept of singularity for Filippov vector fields encompasses the usual one (for smooth vector fields), but also comprehend some new kinds of points over the non-smoothness set, namely, pseudo-equilibria and tangency points (see, for instance [22]). The formal definition of Filippov vector fields and their singularities will be provided in Section 1.2.

So far, a version of the Poincaré–Hopf Theorem for Filippov vector fields is not known. This is mainly because of the lack of a nice index definition for singularities in this context. As expected, a version of the Hairy Ball Theorem for Filippov vector fields is not known either. In other words, the following question is open: “Is there any Filippov vector field defined on a sphere without singularities?”

In this paper, we are firstly concerned in extending the classical index definition to singularities of Filippov vector fields. Such an extension is provided by definitions 3, 4, and 5, which are based on an invariance property under a regularization process established by Theorem 3. With this new index definition, we are able to state and prove our main result, the Poincaré–Hopf Theorem for Filippov vector fields (Theorem 14). Consequently, we also get a Hairy Ball Theorem in this context, i.e. “any Filippov vector field defined on a sphere must have at least one singularity (in the Filippov sense)”.

This chapter is structured as follows. Section 1.2 is devoted to discuss the basic notions and definition of Filippov vector fields. The definition of index for singularities of Filippov vector fields is provided in Section 1.3 and some of their properties are established in Section 1.4. Our main result, the Poincaré–Hopf Theorem for Filippov vector fields, is then stated and proven in Section 1.5. Section 1.6 is dedicated to discuss the invariance property of this new index definition under a regularization process and to presenting a proof for Theorem 3. An Appendix is also provided with some concepts and properties of the classical index for singularities of smooth vector fields.

1.2 Basic notions on Filippov vector fields

In this section, we introduce Filippov’s convention for piecewise smooth vector fields defined on 2-dimensional compact manifolds. We also introduce the concept of singularities of Filippov vector fields.

First, let M be a 2-dimensional compact manifold and $N \subset M$ be a 1-dimensional compact submanifold of M. Denote by C_i, $i \in \{1, 2, \ldots, k\}$, the connected components of $M \backslash N$ (which is finite in number because of the compactness of M and N). Let $\mathcal{X}_i : M \to TM$, for $i \in \{1, 2, \ldots, k\}$, be smooth vector fields on M, i.e. $\mathcal{X}_i(p) \in T_pM$ for every $p \in M$. Accordingly, we consider a piecewise smooth vector field on M given by

$$\mathcal{Z}(p) = \mathcal{X}_i(p) \text{ if } p \in C_i, \text{ for } i \in \{1, 2, \ldots, k\}, \tag{1.1}$$

for which N is called *non-smoothness manifold*.

The trajectories of (1.1), for points in N, can be locally described by the Filippov's convention (see [13]). To do so, we have to obtain a description of (1.1) in local coordinates around points of the non-smoothness manifold N. Since N is a 1-dimensional compact submanifold of M, we can find, for each $q \in N$, a chart (U, Φ) of M around q (i.e. $\Phi : U \to \mathbb{R}^2$ is a local coordinate system and $U \subset M$ is a neighborhood of q) and a function $H : U \to \mathbb{R}$, having 0 as a regular value, such that

- $S := N \cap U = H^{-1}(0)$, and
- $U \backslash S$ is composed by two disjoint open sets, $S^+ = \{p \in U : H(p) \geqslant 0\}$ and $S^- = \{p \in U : H(p) \leqslant 0\}$, such that $\mathcal{Z}^+ = \mathcal{Z}|_{S^+}$ and $\mathcal{Z}^- = \mathcal{Z}|_{S^-}$ are smooth vector fields.

Let $D = \Phi(U)$ and consider the following smooth vector fields defined on D

$$F^+ := \Phi_* \mathcal{Z}^+ : \Sigma^+ \to \mathbb{R}^2 \text{ and } F^- := \Phi_* \mathcal{Z}^- : \Sigma^- \to \mathbb{R}^2$$

(pushforward of $\mathcal{Z}^+$ and $\mathcal{Z}^-$ by Φ, respectively). The local coordinate system Φ can be chosen in such a way that the non-smoothness manifold S is transformed into a straight segment, i.e. $f(\mathbf{x}) := H \circ \Phi^{-1}(\mathbf{x})$, $\mathbf{x} = (x, y) \in D$, is such that

$$\Sigma := f^{-1}(0) = \{(x, y) \in D : y = 0\} = \Phi(S). \tag{1.2}$$

In addition,

$$\Sigma^+ := \{\mathbf{x} \in D : f(\mathbf{x}) \geqslant 0\} = \Phi(S^+) \text{ and } \Sigma^- := \{\mathbf{x} \in D : f(\mathbf{x}) \leqslant 0\} = \Phi(S^-).$$

Thus, the piecewise smooth vector field (1.1) can be locally described around $q \in N$ by the following piecewise smooth vector field on D (see Figure 1),

$$Z(\mathbf{x}) = \Phi_*(\mathcal{Z}|_U) := \begin{cases} F^+(\mathbf{x}), & \text{if } f(\mathbf{x}) \geqslant 0, \\ F^-(\mathbf{x}), & \text{if } f(\mathbf{x}) \leqslant 0. \end{cases} \tag{1.3}$$

Usually, the Filippov vector field (1.3) is concisely denoted by $Z = (F^+, F^-)_f$.

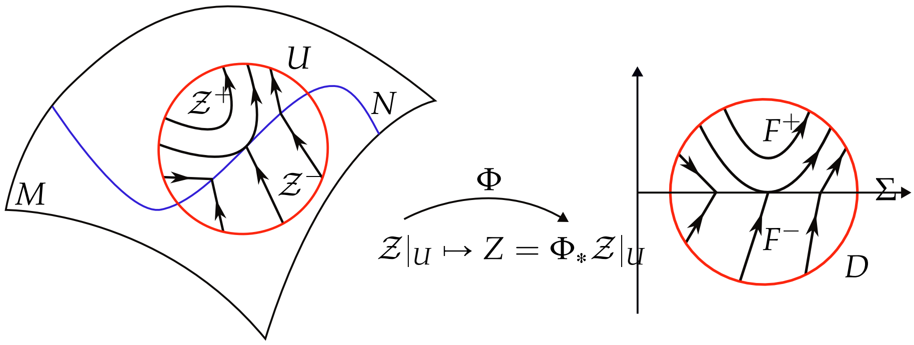


Figure 1 – Local description of the piecewise smooth vector field (1.1) in M by using local coordinates around points at the non-smoothness manifold N.

Remark 1. *We shall see that the Filippov's convention for trajectories of* (1.3) *only depends on the zero set* Σ *of the function* f*. Thus, in the local description* (1.3) *of* (1.1)*, it would be sufficient to consider* $f(x, y) = y$ *(see expression* (1.2)*). However, in the next section, we are going to introduce a regularization process for Filippov vector fields, which will be a key tool for defining a index in this context. For such a process, the expression of the function* f *plays some role (see expression* (1.8)*) and that is why we must carry* f *in the Filippov vector field* (1.3) *instead of just* Σ*. It is important to anticipate that the index definition will not depend on* f *(see Remark 4).*

In [13], Filippov conventioned that the trajectories of (1.3) correspond to the solutions of the differential inclusion

$$\dot{\mathbf{x}} \in \mathcal{F}_Z(\mathbf{x}), \tag{1.4}$$

where $\mathcal{F}_Z$ is the following set-valued function

$$\mathcal{F}_Z(\mathbf{x}) = \frac{F^+(\mathbf{x}) + F^-(\mathbf{x})}{2} + \operatorname{sign}(f(\mathbf{x}))\frac{F^+(\mathbf{x}) - F^-(\mathbf{x})}{2},$$

with

$$\operatorname{sign}(u) = \begin{cases} -1 & \text{if } u < 0, \\ [-1, 1] & \text{if } u = 0, \\ 1 & \text{if } u > 0. \end{cases}$$

The piecewise smooth vector field (1.1) is called *Filippov vector field* when its local trajectories (i.e. trajectories of (1.3) for each $q \in N$) are ruled by the Filippov's convention.

The solutions of the differential inclusion (1.4) have an easy geometrical interpretation which is fairly discussed in the research literature. In order to establish this geometrical interpretation, some regions on Σ must be distinguished. First, denote by Fh the first Lie derivative of h in the direction of the vector field F, i.e. $Ff(\mathbf{x}) = \langle \nabla f(\mathbf{x}), F(\mathbf{x}) \rangle$.

The *crossing region*, denoted by Σ^c, consists of the points $\mathbf{x} \in \Sigma$ such that $F^+f(\mathbf{x})F^-f(\mathbf{x}) > 0$. Notice that at a point $\mathbf{x} \in \Sigma^c$, the solutions either side of the non-smoothness manifold Σ, reaching $\mathbf{x}$, can be joined continuously, forming a solution that crosses Σ (see Figure 2).

The *sliding region* (resp. *escaping region*), denoted by Σ^s (resp. Σ^e), consists of the points $\mathbf{x} \in \Sigma$ such that $F^+f(\mathbf{x}) < 0$ and $F^-f(\mathbf{x}) > 0$ (resp. $F^+f(\mathbf{x}) > 0$ and $F^-f(\mathbf{x}) < 0$). Notice that, at a point $\mathbf{x} \in \Sigma^s$ (resp. $\mathbf{x} \in \Sigma^e$), both vector $F^+(\mathbf{x})$ and $F^-(\mathbf{x})$ point inward (resp. outward) Σ in such a way that the solutions on either side of Σ, reaching $\mathbf{x}$, cannot be concatenate. Alternatively, for $\mathbf{x} \in \Sigma^{s,e} = \Sigma^s \cup \Sigma^e \subset N$, the solutions on either side of Σ, reaching $\mathbf{x}$, can be joined continuously to solutions that slide on $\Sigma^{s,e}$ following the so-called *sliding vector field* (see Figure 2):

$$Z^s(\mathbf{x}) = \frac{F^-f(\mathbf{x})F^+(\mathbf{x}) - F^+f(\mathbf{x})F^-(\mathbf{x})}{F^-f(\mathbf{x}) - F^+f(\mathbf{x})}, \text{ for } \mathbf{x} \in \Sigma. \tag{1.5}$$

The sliding vector field (1.5) associated with the Filippov vector field (1.3) defined on D naturally induces a sliding vector field on $U \cap N$ associated with the Filippov vector field (1.1) defined on the manifold M.

In what follows, we introduce the concept of singularities of Filippov vector fields (see, for instance, [22]).

Definition 1. *Consider the Filippov vector field Z given by* (1.3). *We say that $\mathbf{x}_0 \in D$ is a singularity of Z if one of the following conditions hold:*

(a) $\mathbf{x}_0 \in \Sigma^+$ *(resp.* $\mathbf{x}_0 \in \Sigma^-$*) and* $F^+(\mathbf{x}_0) = 0$ *(resp.* $F^-(\mathbf{x}_0) = 0$*).*

(b) $\mathbf{x}_0 \in \Sigma^s \cup \Sigma^e$ *and* $Z^s(\mathbf{x}_0) = 0$.

(c) $\mathbf{x}_0 \in \Sigma$ *and* $F^+ f(\mathbf{x}_0) = 0$ *or* $F^- f(\mathbf{x}_0) = 0$.

In case (a), if $\mathbf{x}_0 \notin \Sigma$*, then* $\mathbf{x}_0$ *is just a singularity of one of the smooth vector fields,* F^+ *or* F^-*. Otherwise, it is called* **boundary equilibrium**. *In case (b),* $\mathbf{x}_0$ *is a called* **pseudo-equilibrium**. *In case (c),* $\mathbf{x}_0$ *is called* **tangential singularity**.

Any point that does not satisfies the definition above is called **regular**.

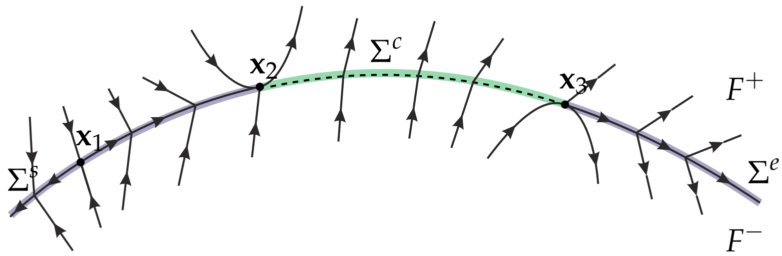


Figure 2 – Illustration of Filippov's convention. The point $\mathbf{x}_1 \in \Sigma^s$ represents a pseudo-equilibrium and the points $\mathbf{x}_2, \mathbf{x}_3 \in \partial\Sigma^c$ represent tangential singularities.

Definition 1 can be naturally extended to the Filippov vector field (1.1) defined on the manifold M as follows.

Definition 2. *Consider the Filippov vector field* $\mathcal{Z}$ *given by* (1.1). *We say that* $p_0 \in M$ *is a singularity of* $\mathcal{Z}$ *if there exists a chart* (U, Φ) *of* M *around* p_0 *such that* $\Phi(p_0)$ *satisfies Definition 1.*

1.3 Index of Filippov vector fields

This section is devoted to provide the definition for index of singularities of Filippov vector fields defined on 2-dimensional compact manifolds.

First, for $A_1, A_2 : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ vector fields and $\mathbf{x} \in U$, satisfying $\det(A_2(\mathbf{x})|A_1(\mathbf{x})) \neq 0$, we define the following auxiliary function,

$$H_{(A_1,A_2)}(\mathbf{x}) := \frac{\|A_1(\mathbf{x})\|^2 - \langle A_1(\mathbf{x}), A_2(\mathbf{x})\rangle}{\det\left(A_2(\mathbf{x})|A_1(\mathbf{x})\right)}. \tag{1.6}$$

We start by defining the index of the Filippov vector field (1.3) on a circle ∂B, where B is the closed ball $B = B_r(\mathbf{x}_0) = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x} - \mathbf{x}_0\| \leqslant r\}$.

Definition 3. *Let Z be the Filippov vector field given by (1.3) and $B \subset D$ a closed ball such that ∂B does not contain any singularities of Z. The index of Z on B is defined by*

$$I_{\partial B}(Z) := \frac{1}{2\pi}\left(J(Z) + \int_{\Gamma^+} \omega_W + \int_{\Gamma^-} \omega_W\right), \tag{1.7}$$

where ω_W is the following usual differential 1-form

$$\omega_W := \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy,$$

$\Gamma^{\pm} = \{Z(\mathbf{x}), \mathbf{x} \in \partial B_r(\mathbf{x}_0) \cap \Sigma^{\pm}\}$*, and* $J(Z) = J^+(Z) + J^-(Z)$ *with*

$$J^{\pm}(Z) = \begin{cases} \operatorname{tg}^{-1}\left(H_{(F^+,F^-)}(\pm r, 0)\right) - \operatorname{tg}^{-1}\left(H_{(F^-,F^+)}(\pm r, 0)\right), & D(\pm r) \neq 0, \\ 0, & D(\pm r) = 0, \end{cases}$$

where $D(\pm r) = \det(F^+(\pm r, 0)|F^-(\pm r, 0))$.

The above definition is based on the fact that the index given by (1.7) is invariant under Sotomayor-Texeira regularization (ST-regularization) [56], see Theorem 3 below. Roughly speaking, a regularization of a piecewise smooth vector field Z is a 1-parameter family Z_ε of C^r, $r \geqslant 0$, vector fields such that Z_ε converges to Z when $\varepsilon \to 0$. The ST-regularization is defined by

$$Z_\varepsilon(\mathbf{x}) = \frac{1 + \phi_\varepsilon \circ f(\mathbf{x})}{2} F^+(\mathbf{x}) + \frac{1 - \phi_\varepsilon \circ f(\mathbf{x})}{2} F^-(\mathbf{x}), \text{ being } \phi_\varepsilon(s) := \phi(s/\varepsilon), \tag{1.8}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is a $\mathcal{C}^r$, $r \geqslant 0$, function which is $\mathcal{C}^1$ for $s \in (-1, 1)$ satisfying $\phi(s) = \operatorname{sign}(s)$ for $|s| \geqslant 1$ and $\phi'(s) > 0$ for $s \in (-1, 1)$. We call ϕ a *monotonic transition function*. In this paper we will always consider smooth monotonic transition functions, i.e. $r \geqslant 1$.

Remark 2. *Originally, the ST-regularization was introduced for Filippov vector fields defined on Euclidean spaces. In [50], the ST-regularization was extended to Filippov vector fields defined on a smooth manifold M with a 1-dimensional non-smoothness* *manifold $N \subset M$, by considering oriented 1-foliations on $M \backslash N$ (see [50, Section 3.1]). We shall call such an extension,* **global ST-regularization**. *Roughly speaking, a global ST-regularization of the Filippov vector field $\mathcal{Z}$ given by* (1.1) *is a 1-parameter family $\mathcal{Z}_\varepsilon$ of smooth vector fields defined on M for which there exists an atlas $\mathcal{A} = \{(U_\alpha, \Phi_\alpha) : \alpha\}$ satisfying that if $U_\alpha \cap N \neq \emptyset$, then the pushforward $(\Phi_\alpha)_* \mathcal{Z}_\varepsilon$ can be written as in* (1.8).

In the sequel, we state the main result of this section that provides the invariance of Definition 3 under ST- regularization.

Theorem 3. *Let Z be the Filippov vector field given by* (1.3), *Z_ε its ST-regularization as in* (1.8), *and $B \subset D$ a closed ball such that ∂B does not contain any singularities of Z. Then, for $\varepsilon > 0$ sufficiently small, ∂B does not contain any singularities of Z_ε and $I_{\partial B}(Z) = I_{\partial B}(Z_\varepsilon)$.*

Theorem 3 is proven in Section 1.6.

Remark 4. *Theorem 3 asserts that the index of Z_ε, the ST-regularization of Z along ∂B is invariant under the choice of the function f that describes Σ as in* (1.2) *and the transition function ϕ.*

Now, we define the index of an isolated singularity of a Filippov vector field (see Definition 1).

Definition 4. *Let Z be the Filippov vector field given by* (1.3), *$\mathbf{x}_0$ an isolated singularity of Z, and $r > 0$ such that $\mathbf{x}_0$ is the unique singularity in $B = B_r(\mathbf{x}_0) \subset D$. The index of Z at $\mathbf{x}_0$ is defined as $I_{\mathbf{x}_0}(Z) := I_{\partial B}(Z)$.*

The following lemma, together with Theorem 3, will prove that the above index of an isolated singularity, of a Filippov vector field is well defined.

Lemma 5 ([55, Proposition 6])**.** *Let Z be a Filippov vector field given by* (1.3), *Z_ε its ST-regularization as in* (1.8), *and $\mathbf{x}_0$ a regular point of Z. Then, there exist a neighborhood $V_{\mathbf{x}_0}$ of $\mathbf{x}_0$ and $\varepsilon_{\mathbf{x}_0} > 0$ such that $0 \notin Z_\varepsilon(V_{\mathbf{x}_0})$ for every $\varepsilon \in (0, \varepsilon_{\mathbf{x}_0})$.*

Lemma 5 will also be a key result in Section 1.4 to establish some properties of the index of Filippov vector fields.

The next result shows that the index $I_{\mathbf{x}_0}(Z)$ does not depend on the radius r of the ball $B_r(\mathbf{x}_0) \subset D$, as long as $\mathbf{x}_0$ is the unique singularity in $B_r(\mathbf{x}_0)$. This ensures that the index of a Filippov vector field at a singularity is well defined.

Proposition 6. *Let Z be a Filippov vector field given by (1.3), $\mathbf{x}_0$ an isolated singularity of Z, and $r_1 > r_0 > 0$ such that x_0 is the unique singularity of Z inside $B_1 = B_{r_1}(\mathbf{x}_0) \subset D$. Then, $I_{\partial B_0}(Z) = I_{\partial B_1}(Z)$, where $B_0 = B_{r_0}(\mathbf{x}_0)$.*

Proof. From Theorem 3, there exists $\bar{\varepsilon} > 0$ such that $I_{\partial B_0}(Z) = I_{\partial B_0}(Z_\varepsilon)$ and $I_{\partial B_1}(Z) = I_{\partial B_1}(Z_\varepsilon)$ for every $\varepsilon \in (0, \bar{\varepsilon})$. We claim that there exists $\varepsilon^* \in (0, \bar{\varepsilon})$ and a continuous deformation $B_s \subset D$, $s \in [0,1]$, of B_0 into B_1 such that $0 \notin Z_{\varepsilon^*}(\partial B_s)$ for every $s \in [0,1]$. Indeed, let $K = \{\mathbf{x} \in \mathbb{R}^2 : r_0 \leqslant \|\mathbf{x} - \mathbf{x}_0\| \leqslant r_1\} \subset D$. Notice that B_s can be chosen in such a way that $\partial B_s \subset K$ for every $s \in [0,1]$. By taking into account that K is compact and that $Z\big|_K$ has only regular points, Lemma 5 provides $\varepsilon^* \in (0, \bar{\varepsilon})$ such that $0 \notin Z_{\varepsilon^*}(K)$ and the claim follows. Hence, since Z_{ε^*} is a smooth vector field, it follows that $I_{\partial B_0}(Z_{\varepsilon^*}) = I_{\partial B_1}(Z_{\varepsilon^*})$ (see Proposition 18 in the Appendix), which implies that $I_{\partial B_1}(Z) = I_{\partial B_0}(Z)$. □

In what follows, we extend Definition 4 to singularities of Filippov vector fields defined on a 2-dimensional manifold compact M (see Definition 2).

Definition 5. *Let $\mathcal{Z}$ be a Filippov vector field (defined on a 2-dimensional compact manifold M) given by (1.1). Let $p_0 \in M$ be an isolated singularity of $\mathcal{Z}$ and (U, Φ) a chart of M around p_0 such that $\Phi_* \mathcal{Z}$ is given by (1.3). The index of $\mathcal{Z}$ at p_0 is defined as $I_{p_0}(\mathcal{Z}) := I_{\Phi(p_0)}(\Phi_* \mathcal{Z})$.*

The next result shows that the index $I_{p_0}(\mathcal{Z})$ does not depend on the chart (U, Φ). This insures that the index of a Filippov vector field (defined on a 2-dimensional compact manifold M) at a singularity is well defined.

Proposition 7. *Let $\mathcal{Z}$ be a Filippov vector field given by (1.1). Let $p_0 \in M$ be an isolated singularity of $\mathcal{Z}$ and let (U, Φ) and (V, Ψ) be charts of M around p_0 such that*

$\Phi(p_0) = \Psi(p_0) = \mathbf{x}_0$ and the pushforward vector fields $\Phi_* \mathcal{Z}$ and $\Psi_* \mathcal{Z}$ as in (1.3). Then, $I_{\mathbf{x}_0}(\Phi_* \mathcal{Z}) = I_{\mathbf{x}_0}(\Psi_* \mathcal{Z})$.

Proof. Denote $D_1 = \Phi(U \cap V)$ and $D_2 = \Psi \circ \Phi^{-1}(D_1)$ and consider Filippov vector fields

$$Z = (F^+, F^-)_f =: \Phi_* \mathcal{Z} : D_1 \to \mathbb{R}^2 \text{ and } W = (G^+, G^-)_g := (\Psi_* \mathcal{Z})|_{D_2} : D_2 \to \mathbb{R}^2.$$

Define the diffeomorphism $\alpha := \Psi \circ \Phi^{-1} : D_1 \to D_2$ and notice that $W = \alpha_* Z$, $G^+ = \alpha_* F^+$ and $G^- = \alpha_* F^-$. It will be convenient to take $g = f \circ \alpha^{-1}$ which, from Remark 1, can be done without loss of generality.

Also, let $r_0 > \bar{r} > r_1 > 0$ be such that $\mathbf{x}_0$ is the unique singularity of Z and W inside $B_0 = B_{r_0}(\mathbf{x}_0) \subset D_1$, $\alpha(B) \subset B_0$ where $B = B_{\bar{r}}(\mathbf{x}_0)$, and $B_1 = B_{r_1}(\mathbf{x}_0) \subset B$. By Definition 4,

$$I_{\mathbf{x}_0}(\Phi_* \mathcal{Z}) = I_{\partial B}(Z) \text{ and } I_{\mathbf{x}_0}(\Psi_* \mathcal{Z}) = I_{\partial B}(W). \tag{1.9}$$

Now, let Z_ε and W_ε be ST-regularizations, of Z and W, respectively, given by (1.8). From Theorem 3, and taking Remark 4 into account, there exists $\bar{\varepsilon} > 0$ such that Z_ε and W_ε do not vanish on ∂B and

$$I_{\partial B}(Z) = I_{\partial B}(Z_\varepsilon) \text{ and } I_{\partial B}(W) = I_{\partial B}(W_\varepsilon) \tag{1.10}$$

for every $\varepsilon \in (0, \bar{\varepsilon}]$. We claim that $W_\varepsilon = \alpha_* Z_\varepsilon$. Indeed, for $\mathbf{y} \in D_2$

$$\begin{aligned}
\alpha_* Z_\varepsilon(\mathbf{y}) =& \frac{1 + \phi_\varepsilon \circ f(\alpha^{-1}(\mathbf{y}))}{2} \alpha_* F^+(\mathbf{y}) + \frac{1 - \phi_\varepsilon \circ f(\alpha^{-1}(\mathbf{y}))}{2} \alpha_* F^-(\mathbf{y}) \\
=& \frac{1 + \phi_\varepsilon \circ g(\mathbf{y})}{2} G^+(\mathbf{y}) + \frac{1 - \phi_\varepsilon \circ g(\mathbf{y})}{2} G^-(\mathbf{y}) \\
=& W_\varepsilon(\mathbf{y}).
\end{aligned}$$

Finally, since Z_ε and W_ε are smooth vector fields, we have that $I_{\partial B}(Z_\varepsilon) = I_{\alpha(\partial B)}(W_\varepsilon)$, for $\varepsilon \in (0, \bar{\varepsilon}]$ (see Proposition 17 in the Appendix). From Lemma 5 and taking into account the compactness of $K = \{\mathbf{x} \in B : r_1 \leqslant ||\mathbf{x} - \mathbf{x}_0|| \leqslant r_0\}$, we can choose $\varepsilon^* \in (0, \bar{\varepsilon})$ for which Z_{ε^*} and W_{ε^*} do not vanish on K. From choice of $r_0, \bar{r}$, and r_1, we have that $\partial B \subset K$ and $\alpha(\partial B) \subset K$, thus they can be continuously deformed into each other without passing through a singularity. Therefore, $I_{\alpha(\partial B)}(W_\varepsilon) = I_{\partial B}(W_\varepsilon)$ (see Proposition 18 in the Appendix), which implies that $I_{\partial B}(Z_\varepsilon) = I_{\partial B}(W_\varepsilon)$. Hence, from (1.9) and (1.10), it follows that $I_{\mathbf{x}_0}(\Phi_* \mathcal{Z}) = I_{\mathbf{x}_0}(\Psi_* \mathcal{Z})$. □

1.4 Properties of the index for Filippov vector fields

In this section, we apply Theorem 3 together with Lemma 5 to extend the index properties of smooth vector fields (see the Appendix) to the index for Filippov vector fields established in Section 1.3.

Proposition 8. *Let Z be a Filippov vector field given by* (1.3) *and $B \subset D$ a closed ball. If Z has no singularities on ∂B, then $I_{\partial B}(Z) \in \mathbb{Z}$.*

Proof. Let Z_ε be a ST-regularization (1.8) of Z. Theorem 3 implies that there exists $\bar{\varepsilon} > 0$ such that $I_{\partial B}(Z) = I_{\partial B}(Z_\varepsilon)$ for every $\varepsilon \in (0, \bar{\varepsilon})$. Since Z_ε is a smooth vector field, it follows that $I_{\partial B}(Z_\varepsilon) \in \mathbb{Z}$ (see the Appendix), which implies that $I_{\partial B}(Z) \in \mathbb{Z}$. □

Proposition 9. *Let Z be a Filippov vector field given by* (1.3) *and $B \subset D$ a closed ball. If Z has no singularities on B, then $I_{\partial B}(Z) = 0$.*

Proof. Let Z_ε be the ST-regularization (1.8) of Z. Theorem 3 implies that there exists $\bar{\varepsilon} > 0$ such that $I_{\partial B}(Z) = I_{\partial B}(Z_\varepsilon)$ for every $\varepsilon \in (0, \bar{\varepsilon})$. By taking into account the compactness of B and that $Z\big|_B$ has only regular points, Lemma 5 implies that there exists $\varepsilon^* \in (0, \bar{\varepsilon})$ such that $Z_{\varepsilon^*}\big|_B$ has only regular points. Since Z_{ε^*} is a smooth vector field, it follows that $I_{\partial B}(Z_{\varepsilon^*}) = 0$ (see Proposition 20 in the Appendix), which implies that $I_{\partial B}(Z) = 0$. □

Proposition 10. *Let Z be a Filippov vector field given by* (1.3) *and $B \subset D$ a closed ball. Assume that Z has no singularities on ∂B. Then,*

$$\min_{\lambda \in [0,1]} \|(1-\lambda)F^+(\mathbf{x}) + \lambda F^-(\mathbf{x})\| > 0, \text{ for every } \mathbf{x} \in \partial B \cap \Sigma.$$

In addition, assume that the Filippov vector field $\widetilde{Z} = (\widetilde{F}^+, \widetilde{F}^-)$ (defined on D and given as in (1.3)*), satisfies*

- $\|Z(\mathbf{x}) - \widetilde{Z}(\mathbf{x})\| < \|Z(\mathbf{x})\|$, *for every* $\mathbf{x} \in \partial B \backslash \Sigma$,
- $|F^{\pm} f(\mathbf{x}) - \widetilde{F}^{\pm} f(\mathbf{x})| < |F^{\pm} f(\mathbf{x})|$, *for every* $\mathbf{x} \in \partial B \cap \Sigma$,
- $|Z^s(\mathbf{x}) - \widetilde{Z}^s(\mathbf{x})| < |Z^s(\mathbf{x})|$, *for every* $\mathbf{x} \in \partial B \cap \Sigma^s$, *and*
- $\|F^{\pm}(\mathbf{x}) - \widetilde{F}^{\pm}(\mathbf{x})\| < \dfrac{1}{2} \min\limits_{\lambda \in [0,1]} \|(1-\lambda)F^+(\mathbf{x}) + \lambda F^-(\mathbf{x})\|$, *for every* $\mathbf{x} \in \partial B \cap \Sigma$.

Then, $\widetilde{Z}$ *has no singularities on* ∂B *and* $I_{\partial B}(Z) = I_{\partial B}(\widetilde{Z})$.

Proof. First, let us verify that

$$\min_{\lambda \in [0,1]} \|(1-\lambda)F^+(\mathbf{x}) + \lambda F^-(\mathbf{x})\| > 0, \text{ for every } \mathbf{x} \in \partial B \cap \Sigma.$$

Suppose that there exist $\bar{\mathbf{x}} \in \partial B \cap \Sigma$ and $\bar{\lambda} \in [0,1]$ such that $\|(1-\bar{\lambda})F^+(\bar{\mathbf{x}}) + \bar{\lambda} F^-(\bar{\mathbf{x}})\| = 0$, i.e. $(1-\bar{\lambda})F^+(\bar{\mathbf{x}}) = -\bar{\lambda} F^-(\bar{\mathbf{x}})$. Since $F^{\pm}(\bar{\mathbf{x}}) \neq 0$, then $\bar{\lambda} \notin \{0,1\}$. Therefore,

$$F^-(\bar{\mathbf{x}}) = -\left(\frac{1-\bar{\lambda}}{\bar{\lambda}}\right) F^+(\bar{\mathbf{x}}) \quad \Rightarrow \quad \frac{F^- f(\bar{\mathbf{x}})}{F^+ f(\bar{\mathbf{x}})} = -\left(\frac{1-\bar{\lambda}}{\bar{\lambda}}\right) < 0,$$

This implies that $\bar{\mathbf{x}} \in \Sigma^s$ and, consequently, $\bar{\mathbf{x}}$ is a pseudo-equilibrium, i.e. $Z^s(\bar{\mathbf{x}}) = 0$ which contradicts the hypothesis.

Now notice that, by hypothesis, $\|\widetilde{Z}(\mathbf{x})\| > 0$ for $\mathbf{x} \in \partial B \backslash \Sigma$, $|\widetilde{F}^{\pm} f(\mathbf{x})| > 0$ for $\mathbf{x} \in \partial B \cap \Sigma$, and $|\widetilde{Z}^s(\mathbf{x})| > 0$ for $\mathbf{x} \in \partial B \cap \Sigma^s$. This implies that $\widetilde{Z}$ has no singularities on ∂B.

Finally, let Z_ε and $\widetilde{Z}_\varepsilon$ be ST-regularizations of Z and $\widetilde{Z}$, respectively. Theorem 3 implies that there exists $\bar{\varepsilon} > 0$ such that $I_{\partial B}(Z) = I_{\partial B}(Z_\varepsilon)$ and $I_{\partial B}(\widetilde{Z}) = I_{\partial B}(\widetilde{Z}_\varepsilon)$ for every $\varepsilon \in (0, \bar{\varepsilon})$. In addition, from the hypothesis and taking into account that $\phi(\mathbb{R}) \subset [-1,1]$, we get

$$\begin{aligned} \|Z_\varepsilon(\mathbf{x}) - \widetilde{Z}_\varepsilon(\mathbf{x})\| &= \left\| \frac{1+\phi_\varepsilon(f(\mathbf{x}))}{2}\left(F^+(\mathbf{x}) - \widetilde{F}^+(\mathbf{x})\right) \right. \\ & \left. + \frac{1-\phi_\varepsilon(f(\mathbf{x}))}{2}\left(F^-(\mathbf{x}) - \widetilde{F}^-(\mathbf{x})\right) \right\| \\ &\leqslant \|F^+(\mathbf{x}) - \widetilde{F}^+(\mathbf{x})\| + \|F^-(\mathbf{x}) - \widetilde{F}^-(\mathbf{x})\| \\ &< \min_{\lambda\in[0,1]} \|(1-\lambda)F^+(\mathbf{x}) + \lambda F^-(\mathbf{x})\| < \|Z_\varepsilon(\mathbf{x})\|, \end{aligned}$$

for every $\mathbf{x} \in \partial B$ and $\varepsilon \in (0, \bar{\varepsilon}]$. Thus, since Z_ε and $\widetilde{Z}_\varepsilon$ are smooth vector fields, it follows that $I_{\partial B}(\widetilde{Z}_\varepsilon) = I_{\partial B}(Z_\varepsilon)$ for $\varepsilon \in (0, \bar{\varepsilon}]$ (see Proposition 21 in the Appendix), which implies that $I_{\partial B}(\widetilde{Z}) = I_{\partial B}(Z)$. □

Proposition 11. *Let Z be a Filippov vector field given by (1.3), $B \subset D$ a closed ball, and $Z_1(\cdot;\delta) = (F_1^+(\cdot;\delta), F_1^-(\cdot;\delta))_f$ be a continuous 1-parameter family of Filippov vector fields (defined on D and given as in (1.3)) such that $Z_1(\cdot;0)$ vanishes, is identically zero, that is $F_1^\pm(\cdot;0) = (0,0)$. Consider the Filippov vector field $\widetilde{Z}(\cdot;\delta) = Z + Z_1(\cdot;\delta)$. Then, there exists $\bar{\delta} > 0$ such that $I_{\partial B}(Z) = I_{\partial B}(\widetilde{Z}(\cdot;\delta))$ for every $\delta \in (0, \bar{\delta})$.*

Proof. First, notice that $\widetilde{Z}(\cdot,\delta) = (\widetilde{F}^+(\cdot,\delta), \widetilde{F}^-(\cdot,\delta))_f$, where $\widetilde{F}^\pm(\cdot,\delta) = F^\pm + F_1^\pm(\cdot,\delta)$. Since, $F_1^\pm(\mathbf{x};\delta) \to (0,0)$, uniformly for $\mathbf{x} \in B$, as $\delta \to 0$ we obtain that: $\|Z(\mathbf{x}) - \widetilde{Z}(\mathbf{x};\delta)\| \to 0$, uniformly for $\mathbf{x} \in \partial B \backslash \Sigma$, as $\delta \to 0$; $|F^\pm f(\mathbf{x}) - \widetilde{F}^\pm f(\mathbf{x};\delta)| \to 0$ and $\|F^\pm(\mathbf{x}) - \widetilde{F}^\pm(\mathbf{x};\delta)\| \to 0$, for $\mathbf{x} \in \partial B \cap \Sigma$, as $\delta \to 0$; and $|Z^s(\mathbf{x}) - \widetilde{Z}^s(\mathbf{x};\delta)| \to 0$, for $\mathbf{x} \in \partial B \cap \Sigma^s$, as $\delta \to 0$. Thus, taking into account that Z has no singularities on ∂B, we conclude that there exists $\bar{\delta} > 0$ for which the hypotheses of Proposition 10 hold for $\widetilde{Z}(\cdot;\delta)$, for $\delta \in (0, \bar{\delta})$, which implies that $I_{\partial B}(Z) = I_{\partial B}(\widetilde{Z}(\cdot;\delta))$ for every $\delta \in (0, \bar{\delta})$. □

Proposition 12. *Let $Z(\cdot;\lambda)$, $\lambda \in [0,1]$, be a continuous homotopy between Filippov vector fields given as (1.3) and $B \subset D$ a closed ball such that $Z(\cdot;\lambda)$ has no singularities on ∂B, for every $\lambda \in [0,1]$. Then, $I_{\partial B}(Z(\cdot;\lambda))$ is constant on $\lambda \in [0,1]$.*

Proof. For each $\lambda_0 \in [0,1]$, by taking $Z = Z(\cdot;\lambda_0)$ and $Z_1(\cdot;\delta) = Z(\cdot;\lambda_0+\delta) - Z(\cdot;\lambda_0)$, Proposition 11 provides a neighborhood $J_{\lambda_0} \subset [0,1]$ of λ_0 such that

$I_{\partial B}(Z(\cdot;\lambda)) = I_{\partial B}(Z(\cdot;\lambda_0))$ for every $\lambda \in J_{\lambda_0}$, which implies that the map $\lambda \in [0,1] \mapsto I_{\partial B}(Z(\cdot,\lambda))$ is continuous. Since, from Proposition 8, it is an integer-valued map, we conclude that $I_{\partial B}(Z(\cdot,\lambda))$ is constant on $\lambda \in [0,1]$. □

Proposition 13. *Let Z be the Filippov vector field given by (1.3) and $B \subset D$ a closed ball such that Z has no singularities on ∂B. Assume that Z has finitely many singularities inside B, $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$. Then,*

$$I_{\partial B}(Z) = \sum_{i=1}^{n} I_{\mathbf{x}_i}(Z).$$

Proof. Since each singularity $\mathbf{x}_i$, $i \in \{1,\dots,n\}$, is isolated, there exist r_i, $i \in \{1,\dots,n\}$, small enough, such that $\mathbf{x}_i$ is the unique singularity inside $B_i = B_{r_i}(\mathbf{x}_i)$ for $i \in \{1,\dots,n\}$, and $B_j \cap B_i = \emptyset$, for every $i \neq j$. From Theorem 3 and Definition 4, there exists $\bar{\varepsilon} > 0$ such that

$$I_{\partial B}(Z) = I_{\partial B}(Z_\varepsilon) \text{ and } I_{\mathbf{x}_i}(Z) = I_{\partial B_i}(Z) = I_{\partial B_i}(Z_\varepsilon) \text{ for } i \in \{1,\dots,n\} \text{ and } \varepsilon \in (0,\bar{\varepsilon}].$$

Since Z_ε is a smooth vector field and $B = \overline{B\setminus \cup_i B_i} \cup (\cup_i B_i)$, Proposition 19 in the Appendix provides

$$I_{\partial B}(Z) = I_{\partial B}(Z_\varepsilon) = I_{\partial(B\setminus\cup_i B_i)}(Z_\varepsilon) + \sum_{i=1}^{n} I_{\partial B_i}(Z_\varepsilon) = I_{\partial(B\setminus\cup_i B_i)}(Z_\varepsilon) + \sum_{i=1}^{n} I_{\mathbf{x}_i}(Z),$$

for $\varepsilon \in (0,\bar{\varepsilon}]$. Now, Lemma 5 asserts the existence of $\varepsilon^* \in (0,\bar{\varepsilon}]$ such that $Z_{\varepsilon *}$ has no singularities in $\overline{B\setminus \cup_i B_i}$. Thus, from Proposition 20 in the Appendix, we have that $I_{\partial(B\setminus\cup_i B_i)}(Z^*_\varepsilon) = 0$ which implies that

$$I_{\partial B}(Z) = \sum_{i=1}^{n} I_{\mathbf{x}_i}(Z).$$

□

1.5 Poincaré–Hopf Theorem for Filippov vector fields

In this section, we state and prove the main result of this chapter. Namely, we show that, by considering the generalization of index for singularities of Filippov vector fields provided in Section 1.3, the Poincaré–Hopf Theorem remains true for Filippov vector fields defined on 2-dimensional compact manifolds.

Theorem 14. *Let $\mathcal{Z}$ be the Filippov vector field (defined on a 2-dimensional compact manifold M) given by (1.1). Denote the set of the singularities of $\mathcal{Z}$ by $\mathcal{S}$ and assume that they are all isolated. Then,*

$$\sum_{p\in\mathcal{S}} I_p(\mathcal{Z}) = \chi(M),$$

where $\chi(M)$ is the Euler Characteristic of M.

Proof. Let $p_i \in M$, $i \in \{1,\ldots,n\}$, be the singularities of the Filippov vector field $\mathcal{Z}$. Consider an atlas $\mathcal{A} = \{(U_\alpha, \Phi_\alpha) : \alpha\}$ of M satisfying:

1. for each $i \in \{1,\ldots,n\}$, there exists α_i such that p_i is the unique singularity of $\mathcal{Z}$ inside U_{α_i}, and
2. $U_{\alpha_i} \cap U_{\alpha_j} = \emptyset$, for $i \neq j$.

Denote $U^i = U_{\alpha_i}$ and $\Phi^i = \Phi_{\alpha_i}$. From Definition 5, $I_{p_i}(\mathcal{Z}) = I_{\Phi^i(p_i)}(\Phi^i_*\mathcal{Z})$. Take $r_i > 0$, for $i \in \{1,\ldots,n\}$, such that $B_i = B_{r_i}(\Phi^i(p_i)) \subset D^i := \Phi^i(U^i)$. Thus, from Definition 4, we have that $I_{p_i}(\mathcal{Z}) = I_{\partial B_i}(\Phi^i_*\mathcal{Z})$.

Now, let $\mathcal{Z}_\varepsilon$ be a global ST-regularization of $\mathcal{Z}$. Taking Remark 2 into account, that atlas $\mathcal{A}$ can be chosen in such a way that $\Phi^i_*\mathcal{Z}_\varepsilon : D^i \to \mathbb{R}^2$ as in (1.8). From Theorem 3, there exists $\bar{\varepsilon} > 0$ such that $I_{\partial B_i}(\Phi^i_*\mathcal{Z}) = I_{\partial B_i}(\Phi^i_*\mathcal{Z}_\varepsilon)$, for $i \in \{1,\ldots,n\}$ and $\varepsilon \in (0,\bar{\varepsilon}]$. So far, we have obtained that

$$I_{p_i}(\mathcal{Z}) = I_{\partial B_i}(\Phi^i_*\mathcal{Z}_\varepsilon), \tag{1.11}$$

for $i \in \{1,\ldots,n\}$ and $\varepsilon \in (0,\bar{\varepsilon}]$.

Now, fix $\varepsilon^* \in (0,\bar{\varepsilon}]$. Recall that the set $\mathcal{G} \subset C^r(M)$ of the vector fields defined on M having only hyperbolic singularities is open and dense in $C^r(M)$ (see [49, Theorem 3.4]). Since $\mathcal{Z}_{\varepsilon^*} \in C^r(M)$, there exists a perturbation $\mathcal{X} \in \mathcal{G}$ of $\mathcal{Z}_{\varepsilon^*}$ (as close to $\mathcal{Z}_{\varepsilon^*}$ as we want) satisfying that $\mathcal{X}$ has finitely many singularities and none of them are contained in $M \setminus \bigcup_{i=1}^{n} (\Phi^i)^{-1}(B_i)$. Thus, for each $i \in \{1, \ldots, n\}$, let p_i^j, $j \in \{1, \ldots, m_i\}$, be the singularities of $\mathcal{X}$ in $(\Phi^i)^{-1}(B_i)$. Hence,

$$\sum_{j=1}^{m_i} I_{p_i^j}(\mathcal{X}) = \sum_{j=1}^{m_i} I_{\Phi^i(p_i^j)}(\Phi^i_* \mathcal{X}) = I_{\partial B_i}(\Phi^i_* \mathcal{X}). \tag{1.12}$$

Notice that $\mathcal{X}$ can be taken sufficiently close to $\mathcal{Z}$ in order that $||\Phi^i_* \mathcal{X}(\mathbf{x}) - \Phi^i_* \mathcal{Z}_{\varepsilon^*}(\mathbf{x})|| < ||\Phi^i_* \mathcal{Z}_{\varepsilon^*}(\mathbf{x})||$ for every $\mathbf{x} \in \partial B_i$ and $i \in \{1, \ldots, n\}$. Thus, Proposition 21 from Appendix implies that

$$I_{\partial B_i}(\Phi^i_* \mathcal{X}) = I_{\partial B_i}(\Phi^i_* \mathcal{Z}_{\varepsilon^*}),$$

which, together with the equations in (1.11) and (1.12), provide

$$\sum_{j=1}^{m_i} I_{p_i^j}(\mathcal{X}) = I_{p_i}(\mathcal{Z}). \tag{1.13}$$

Finally, applying the Poincaré-Hopf Theorem to the smooth vector field $\mathcal{X}$ (see Theorem 22 from the Appendix), it follows that

$$\sum_{i=1}^{n} \sum_{j=1}^{m_i} I_{p_i^j}(\mathcal{X}) = \chi(M). \tag{1.14}$$

Therefore, from (1.13) and (1.14), we conclude that

$$\sum_{i=1}^{n} I_{p_i}(Z) = \sum_{i=1}^{n} \sum_{j=1}^{m_i} I_{p_i^j}(\mathcal{X}) = \chi(M).$$

□

Taking into account that the Euler Characteristic of a sphere is 2, we obtain, as a direct consequence of Theorem 14, the following version of the Hairy Ball Theorem for Filippov vector fields.

Corollary 15. *Assume that M is a smooth sphere and let $\mathcal{Z}$ be the Filippov vector field given by (1.1) defined on M. Then, $\mathcal{Z}$ has at least one singularity.*

1.6 Invariance under regularization process: proof of Theorem 3

This section is devoted to the proof of Theorem 3. Consider the Filippov vector field Z given by (1.3) and let Z_ε be its ST-regularization given by (1.8). Notice that $Z_\varepsilon(\mathbf{x}) = (X_\varepsilon(\mathbf{x}), Y_\varepsilon(\mathbf{x}))$, where

$$\begin{aligned} X_\varepsilon(\mathbf{x}) &= \frac{F_1^+(\mathbf{x}) + F_1^-(\mathbf{x}) + \phi_\varepsilon(y)(F_1^+(\mathbf{x}) - F_1^-(\mathbf{x}))}{2}, \\ Y_\varepsilon(\mathbf{x}) &= \frac{F_2^+(\mathbf{x}) + F_2^-(\mathbf{x}) + \phi_\varepsilon(y)(F_2^+(\mathbf{x}) - F_2^-(\mathbf{x}))}{2}. \end{aligned} \tag{1.15}$$

Let $B = B_r(\mathbf{x}_0) \subset D$ and consider the following parametrization of its boundary ∂B,

$$\sigma(t) = (u(t), v(t)) = \big(r\cos(t+\pi/2), r\sin(t+\pi/2)\big), \;\; t \in [0, 2\pi]$$

Lemma 5 implies that Z_ε does not vanish on ∂B for $\varepsilon > 0$ sufficiently small. Since Z_ε is a smooth vector field, we know that

$$I_{\partial B}(Z_\varepsilon) = \frac{1}{2\pi}\int_{\Gamma_\varepsilon} \omega_W = \frac{1}{2\pi}\int_0^{2\pi} \Big(p_{Z_\varepsilon}(\sigma(t)) + q_{Z_\varepsilon}(\sigma(t))\Big)\mathrm{d}t, \tag{1.16}$$

where $\Gamma_\varepsilon = \{Z_\varepsilon \circ \sigma(t),\ t \in [0, 2\pi]\}$ and, for a vector field $A(\mathbf{x})$, the functions p_A and q_A are given by expressions in (1.34) of the Appendix.

Since 0 is a regular value of f, one can apply the Implicit Function Theorem to ensure that, for $\varepsilon > 0$ sufficiently small, the curve ∂B intersects the boundaries of the regularization band, $L_\varepsilon^+ = \{(x,y) \in D : f(x,y) = \varepsilon\}$ and $L_\varepsilon^- = \{(x,y) \in D : f(x,y) = -\varepsilon\}$, at points $\sigma(w_i^\varepsilon)$, for $i \in \{1, \ldots, 4\}$ in such a way that

$$w_1^\varepsilon, w_2^\varepsilon \to \pi/2 \;\text{ and }\; w_3^\varepsilon, w_4^\varepsilon \to 3\pi/2 \;\text{ as }\; \varepsilon \to 0. \tag{1.17}$$

Also, let $\partial B = \partial B_\varepsilon^1 \cup \partial B_\varepsilon^2 \cup \partial B_\varepsilon^3 \cup \partial B_\varepsilon^4 \cup \partial B_\varepsilon^5$, where, for each $i \in \{1,2,3,4,5\}$, $\partial B_\varepsilon^i = \{\sigma(t) : t \in (w_{i-1}^\varepsilon, w_i^\varepsilon)\}$ (see Figure 3).

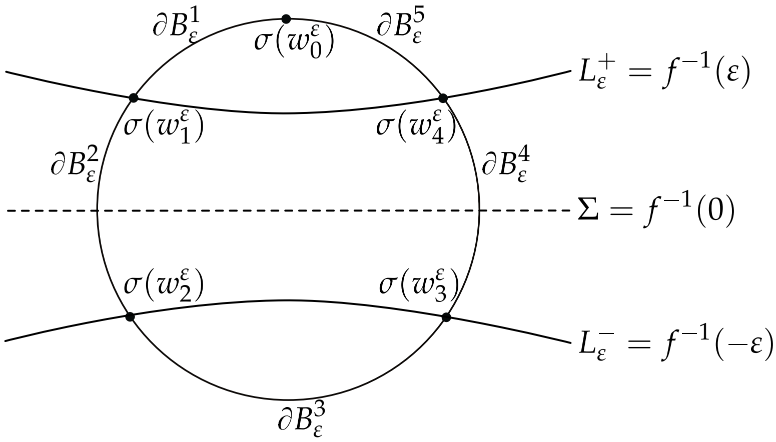


Figure 3 – Illustration of the curve ∂B and its intersection with the boundaries of the regularization band, L_ε^+ and L_ε^-.

The index (1.16) can be split into several integrals as follows

$$I_{\partial B}(Z_\varepsilon) = \frac{1}{2\pi}\left(\int_{\Gamma_\varepsilon^1} \omega_W + \int_{\Gamma_\varepsilon^2} \omega_W + \int_{\Gamma_\varepsilon^3} \omega_W + \int_{\Gamma_\varepsilon^4} \omega_W + \int_{\Gamma_\varepsilon^5} \omega_W\right), \tag{1.18}$$

where $\Gamma_\varepsilon^i = \{Z_\varepsilon(\mathbf{x}), \mathbf{x} \in \partial B_\varepsilon^i\}$ for $i \in \{1,2,3,4,5\}$. Notice that, for $\mathbf{x} \in \Gamma_\varepsilon^1 \cup \Gamma_\varepsilon^5$, $Z_\varepsilon(\mathbf{x}) = F^+(\mathbf{x})$; for $\mathbf{x} \in \Gamma_\varepsilon^3$, $Z_\varepsilon(\mathbf{x}) = F^-(\mathbf{x})$; and for $\mathbf{x} \in \Gamma_\varepsilon^2 \cup \Gamma_\varepsilon^4$, $Z_\varepsilon(\mathbf{x}) = (X_\varepsilon(\mathbf{x}), Y_\varepsilon(\mathbf{x}))$ is given by (1.15).

In order to prove that $I_{\partial B}(Z) = I_{\partial B}(Z_\varepsilon)$ for $\varepsilon > 0$ small enough, it is sufficient to show that

$$\lim_{\varepsilon \to 0} I_{\partial B}(Z_\varepsilon) = I_{\partial B}(Z), \tag{1.19}$$

because, since the index is a discrete function, we know that $\lim\limits_{\varepsilon \to 0} I_{\partial B}(Z_\varepsilon) = I_{\partial B}(Z_\varepsilon)$ for $\varepsilon > 0$ sufficiently small.

First, along Γ_ε^1, Γ_ε^3, and Γ_ε^5, we are outside the regularization band, where the integrand does not depend on ε. In this cases, taking into account the limits in (1.17), we get

$$\begin{aligned}
\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon^1}\omega_W &= \lim_{\varepsilon\to 0}\int_{w_0^\varepsilon}^{w_1^\varepsilon}\Big(p_{F^+}(\sigma(t))+q_{F^+}(\sigma(t))\Big)\mathrm{d}t\\
&= \int_0^{\pi/2}\Big(p_{F^+}(\sigma(t))+q_{F^+}(\sigma(t))\Big)\mathrm{d}t,\\
\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon^3}\omega_W &= \lim_{\varepsilon\to 0}\int_{w_2^\varepsilon}^{w_3^\varepsilon}\Big(p_{F^-}(\sigma(t))+q_{F^-}(\sigma(t))\Big)\mathrm{d}t\\
&= \int_{\pi/2}^{3\pi/2}\Big(p_{F^-}(\sigma(t))+q_{F^-}(\sigma(t))\Big)\mathrm{d}t,\\
\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon^5}\omega_W &= \lim_{\varepsilon\to 0}\int_{w_4^\varepsilon}^{w_5^\varepsilon}\Big(p_{F^+}(\sigma(t))+q_{F^+}(\sigma(t))\Big)\mathrm{d}t\\
&= \int_{3\pi/2}^{2\pi}\Big(p_{F^+}(\sigma(t))+q_{F^+}(\sigma(t))\Big)\mathrm{d}t,
\end{aligned}$$

Therefore, adding up the integrals above, we obtain

$$\lim_{\varepsilon\to 0}\left(\int_{\Gamma_\varepsilon^1}\omega_W+\int_{\Gamma_\varepsilon^3}\omega_W+\int_{\Gamma_\varepsilon^5}\omega_W\right)=\int_{\Gamma^+}\omega_W+\int_{\Gamma^-}\omega_W,$$

where $\Gamma^\pm=\{Z(\mathbf{x}),\,\mathbf{x}\in\partial B\cap\Sigma^\pm\}$.

Now, for the integral along Γ_ε^2 in (1.18), we proceed with the following change of integration variable

$$t=\tau_1^\varepsilon(s)=(1-s)w_1^\varepsilon+s\,w_2^\varepsilon,$$

which provides

$$\begin{aligned}
\int_{\Gamma_\varepsilon^2}\omega_W &= \int_{w_1^\varepsilon}^{w_2^\varepsilon}\Big(p_{Z_\varepsilon}(\sigma(t))+q_{Z_\varepsilon}(\sigma(t))\Big)dt\\
&= \int_0^1\Big(p_{Z_\varepsilon}(\sigma\circ\tau_1^\varepsilon(s))+q_{Z_\varepsilon}(\sigma\circ\tau_1^\varepsilon(s))\Big)\frac{d\tau_1^\varepsilon}{ds}(s)ds.
\end{aligned}\tag{1.20}$$

Notice that the integrand of the equation (1.20) is uniformly convergent as ε goes to 0. Then, by switching the limit with the integral and taking into account the limits in (1.17), we get

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon^2}\omega_W &= \int_0^1 \lim_{\varepsilon\to 0}\left(p_{Z_\varepsilon}(\sigma\circ\tau_1^\varepsilon(s))+q_{Z_\varepsilon}(\sigma\circ\tau_1^\varepsilon(s))\right)\frac{d\tau_1^\varepsilon}{ds}(s)ds\\ &= \int_0^1 \frac{4\phi'(1-2s)(F_1^+(-r,0)F_2^-(-r,0)-F_2^+(-r,0)F_1^-(-r,0))}{G_1(s)}ds,\end{aligned} \tag{1.21}$$

where

$$\begin{aligned}G_1(s) = & -2\left(\phi(1-2s)^2-1\right)F_1^+(-r,0)F_1^-(-r,0)+(\phi(1-2s)+1)^2F_1^+(-r,0)^2\\ & -2\left(\phi(1-2s)^2-1\right)F_2^+(-r,0)F_2^-(-r,0)+(\phi(1-2s)+1)^2F_2^+(-r,0)^2\\ & +(\phi(1-2s)-1)^2\left(F_1^-(-r,0)^2+F_2^-(-r,0)^2\right).\end{aligned}$$

By taking $u=\phi(1-2s)-1$, i.e. $s=\sigma_1(u):=(1-\phi^{-1}(u+1))/2$, the integral (1.21) becomes

$$\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon^2}\omega_W = \int_0^{-2}\frac{\alpha_1}{\beta_1(u+2)^2-2\psi_1(u+2)u+\eta_1 u^2}du, \tag{1.22}$$

where $\alpha_1=\alpha(-r,0)$, $\beta_1=\beta(-r,0)$, $\psi_1=\psi(-r,0)$, and $\eta_1=\eta(-r,0)$ with

$$\begin{aligned}\alpha(\mathbf{x}) &:= 2F_2^+(\mathbf{x})F_1^-\mathbf{x}-2F_1^+(\mathbf{x})F_2^-(\mathbf{x}),\\ \beta(\mathbf{x}) &:= F_1^+(\mathbf{x})^2+F_2^+(\mathbf{x})^2,\\ \psi(\mathbf{x}) &:= F_1^+(\mathbf{x})F_1^-(\mathbf{x})+F_2^+(\mathbf{x})F_2^-(\mathbf{x}),\\ \eta(\mathbf{x}) &:= F_1^-(\mathbf{x})^2+F_2^-(\mathbf{x})^2.\end{aligned} \tag{1.23}$$

Notice that the discriminant of the denominator of (1.22), given by $\psi_1^2-\beta_1\eta_1$, is less than or equal to zero. One can see that the discriminant is zero if and only if $\det(F^+(-r,0)|F^-(-r,0))=0$. In this cases, $\alpha_1=0$ which implies that the integral is zero. Now assume that $\psi_1^2-\beta_1\eta_1<0$. Hence,

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon^2}\omega_W &= \int_0^{-2}\frac{\alpha_1}{\beta_1(u+2)^2-2\psi_1(u+2)u+\eta_1 u^2}du\\ &= \frac{\alpha_1\left(\mathrm{tg}^{-1}\left(\frac{\psi_1-\beta_1}{\sqrt{\beta_1\eta_1-\psi_1^2}}\right)+\mathrm{tg}^{-1}\left(\frac{\psi_1-\eta_1}{\sqrt{\beta_1\eta_1-\psi_1^2}}\right)\right)}{2\sqrt{\beta_1\eta_1-\psi_1^2}}.\end{aligned} \tag{1.24}$$

Substituting (1.23) into (1.24), we get

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon^2}\omega_W =&\mathrm{tg}^{-1}\left(\frac{F_1^-(-r,0)^2+F_2^-(-r,0)^2-F_1^+(-r,0)F_1^-(-r,0)}{F_1^+(-r,0)F_2^-(-r,0)-F_2^+(-r,0)F_1^-(-r,0)}\right.\\&\left.-\frac{F_2^+(-r,0)F_2^-(-r,0)}{F_1^+(-r,0)F_2^-(-r,0)-F_2^+(-r,0)F_1^-(-r,0)}\right)\\&+\mathrm{tg}^{-1}\left(\frac{F_1^+(-r,0)^2+F_2^+(-r,0)^2-F_1^+(-r,0)F_1^-(-r,0)}{F_1^+(-r,0)F_2^-(-r,0)-F_2^+(-r,0)F_1^-(-r,0)}\right.\\&\left.-\frac{F_2^+(-r,0)F_2^-(-r,0)}{F_1^+(-r,0)F_2^-(-r,0)-F_2^+(-r,0)F_1^-(-r,0)}\right).\end{aligned}\tag{1.25}$$

Notice that the argument of the arctangent in (1.25) is of the form

$$\frac{A_1(\mathbf{x})^2+A_2(\mathbf{x})^2-A_1(\mathbf{x})B_1\mathbf{x}-A_2(\mathbf{x})B_2(\mathbf{x})}{B_1(\mathbf{x})A_2\mathbf{x}-B_2(\mathbf{x})A_1(\mathbf{x})},\tag{1.26}$$

where $A(\mathbf{x})=(A_1(\mathbf{x}),A_2(\mathbf{x}))$ and $B(\mathbf{x})=(B_1(\mathbf{x}),B_2(\mathbf{x}))$ are general vector fields. Note that (1.26) becomes $H_{(A,B)}(\mathbf{x})$, defined in (1.6), that is

$$\frac{\|A(\mathbf{x})\|^2-\langle A(\mathbf{x}),B(\mathbf{x})\rangle}{\det\big(B(\mathbf{x})|A(\mathbf{x})\big)}=H_{(A,B)}(\mathbf{x}).$$

Then,

$$\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon^2}\omega_W=\mathrm{tg}^{-1}\left(H_{(F^-,F^+)}(-r,0)\right)-\mathrm{tg}^{-1}\left(H_{(F^+,F^-)}(-r,0)\right).$$

Hence, we have obtained that

$$\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon^2}\omega_W=\begin{cases}\mathrm{tg}^{-1}\left(H_{(F^-,F^+)}(-r,0)\right)-\mathrm{tg}^{-1}\left(H_{(F^+,F^-)}(-r,0)\right), & D(-r)\neq 0,\\ 0, & D(-r)=0,\end{cases}$$

with $D(-r)=\det(F^+(-r,0)|F^-(-r,0))$, which coincides with $-J^-(Z)$.

Analogously, for the integral along Γ_ε^4 in (1.18) we proceed with the following change of integration variable

$$\tau_2^\varepsilon(s)=(1-s)w_4^\varepsilon+sw_3^\varepsilon,$$

which provides

$$\begin{aligned}\int_{\Gamma_\varepsilon^4}\omega_W &= \int_{w_3^\varepsilon}^{w_4^\varepsilon}\Big(p_{Z_\varepsilon}(\sigma(t))+q_{Z_\varepsilon}(\sigma(t))\Big)dt\\ &= \int_0^1\Big(p_{Z_\varepsilon}(\sigma\circ\tau_2^\varepsilon(s))+q_{Z_\varepsilon}(\sigma\circ\tau_2^\varepsilon(s))\Big)\frac{d\tau_2^\varepsilon}{ds}(s)ds,\end{aligned} \tag{1.27}$$

Again, as the integrand of the equation (1.27) is uniformly convergent as ε goes to 0, by switching the limit with the integral and taking into account the limits in (1.17), we get

$$\begin{aligned}\lim_{\varepsilon\to0}\int_{\Gamma_\varepsilon^4}\omega_W &= \int_0^1\lim_{\varepsilon\to0}\Big(p_{Z_\varepsilon}(\sigma\circ\tau_2^\varepsilon(s))+q_{Z_\varepsilon}(\sigma\circ\tau_2^\varepsilon(s))\Big)\frac{d\tau_2^\varepsilon}{ds}(s)ds\\ &= \int_0^1\frac{4\phi'(2s-1)(F_2^+(r,0)F_1^-(r,0)-F_1^+(r,0)F_2^-(r,0))}{G_2(s)}ds,\end{aligned} \tag{1.28}$$

where

$$\begin{aligned}G_2(s) &= \phi(2s-1)^2\Big((F_1^+(r,0)-F_1^-(r,0))^2+(F_2^+(r,0)-F_2^-(r,0))^2\Big)\\ &\quad+2\phi(2s-1)\Big(F_1^+(r,0)^2+F_2^+(r,0)^2-F_1^-(r,0)^2-F_2^-(r,0)^2\Big)\\ &\quad+(F_1^+(r,0)+F_1^-(r,0))^2+(F_2^+(r,0)+F_2^-(r,0))^2.\end{aligned}$$

By taking $u=-1+\phi(-1+2s)$, i.e. $s=\sigma_2(u):=(\phi^{(-1)}(u+1)+1)/2$, the integral (1.28) becomes

$$\lim_{\varepsilon\to0}\int_{\Gamma_\varepsilon^4}\omega_W=\int_{-2}^0\frac{\alpha_2}{\beta_2(u+2)^2-2\psi_2(u+2)u+\eta_2u^2}du, \tag{1.29}$$

where $\alpha_2=\alpha(r,0)$, $\beta_2=\beta(r,0)$, $\psi_2=\psi(r,0)$, and $\eta_2=\eta(r,0)$ are given in (1.23). Notice that the discriminant of the denominator of (1.29), given by $\psi_2^2-\beta_2\eta_2$, is less than or equal to zero. One can see that the discriminant is zero if and only if $\det(F^+(r,0)|F^-(r,0))=0$. In this cases, $\alpha_2=0$ which implies that the integral is zero. Now, assume that $\psi_1^2 - \beta_1\eta_1 < 0$. Hence,

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon^4}\omega_W &= \int_{-2}^{0}\frac{\alpha_2}{\beta_2(u+2)^2-2\psi_2(u+2)u+\eta_2 u^2}du \\ &= -\frac{\alpha_2\left(\text{tg}^{-1}\left(\frac{\psi_2-\beta_2}{\sqrt{\beta_2\eta_2-\psi_2^2}}\right)+\text{tg}^{-1}\left(\frac{\psi_2-\eta_2}{\sqrt{\beta_2\eta_2-\psi^2}}\right)\right)}{2\sqrt{\beta_2\eta_2-\psi_2^2}}.\end{aligned} \tag{1.30}$$

Substituting (1.23) into (1.30), we get

$$\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon^4}\omega_W = \text{tg}^{-1}\left(H_{(F^+,F^-)}(r,0)\right) - \text{tg}^{-1}\left(H_{(F^-,F^+)}(r,0)\right).$$

Accordingly, we have concluded that

$$\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon^4}\omega_W = \begin{cases}\text{tg}^{-1}\left(H_{(F^+,F^-)}(r,0)\right) - \text{tg}^{-1}\left(H_{(F^-,F^+)}(r,0)\right), & D(r)\neq 0,\\ 0, & D(r)=0,\end{cases}$$

with $D(r) = \det(F^+(r,0)|F^-(r,0))$, which coincides with $J^+(Z)$.

Therefore, for $\varepsilon > 0$ small enough, we get that

$$I_{\partial B}(Z_\varepsilon) = \lim_{\varepsilon\to 0} I_{\partial B}(Z_\varepsilon) = J(Z) + \frac{1}{2\pi}\left(\int_{\Gamma^+}\omega_W + \int_{\Gamma^-}\omega_W\right) = I_{\partial B}(Z),$$

which concludes the proof of Theorem 3.

1.7 Appendix

This appendix provides some concepts and results from the index theory for smooth vector fields. We are following the references [11, 18, 58].

Let ω be a *differential* 1*-form* defined on $\mathbb{R}^2$, i.e. $\omega = p(\mathbf{x})dx + q(\mathbf{x})dy$ where p and q are smooth real functions on $\mathbb{R}^2$. Let γ be a curve on $\mathbb{R}^2$ with smooth parametrization $\alpha : [a,b] \to \mathbb{R}^2$, $\alpha(t) = (u(t), v(t))$. The integral of ω along γ is defined as

$$\int_\gamma \omega := \int_a^b \Big(p(\alpha(t))u'(t) + q(\alpha(t))v'(t)\Big)dt. \tag{1.31}$$

The above definition does not depend on the parametrization α. Accordingly, the *winding number of* γ *around the origin* $W(\gamma)$ is an integer defined by

$$W(\gamma) = \frac{1}{2\pi} \int_{\gamma} \omega_W,$$

where ω_W is the following differential 1-form

$$\omega_W := \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy. \tag{1.32}$$

The next result is a useful tool for computing the winding number.

Proposition 16 ([18, Corollary 3.8])**.** *If two smooth oriented closed curves, γ and δ, are homotopic in $\mathbb{R}^2\backslash\{(0,0)\}$, then $W(\gamma) = W(\delta)$.*

Now, consider a smooth vector field $A(\mathbf{x})$ defined on an open subset $D \subset \mathbb{R}^2$. Let $\gamma \subset D$ be a smooth oriented closed curve. Assume that $A(\mathbf{x})$ is nonsingular on γ. When a point $\mathbf{x}$ moves one cycle around γ in the counterclockwise direction, the vector $A(\mathbf{x})$ winds around the origin an integral number of revolutions. The total number of revolutions $I_\gamma(A)$ is called the *rotation number of the vector field $A(\mathbf{x})$ around γ* [58]. Following the notation above,

$$I_\gamma(A) := W(\Gamma) = \frac{1}{2\pi} \int_{\Gamma} \omega_W, \tag{1.33}$$

where $\Gamma := \{A(\mathbf{x}), \mathbf{x} \in \gamma\}$. Define the auxiliary functions $p_A, q_A : D \subset \mathbb{R}^2 \to \mathbb{R}$ such that

$$\begin{aligned} p_A(\alpha(t)) &:= \frac{-A_2(\alpha(t))}{A_1^2(\alpha(t)) + A_2^2(\alpha(t))} \nabla A_1(\alpha(t)) \cdot \alpha'(t), \\ q_A(\alpha(t)) &:= \frac{A_1(\alpha(t))}{A_1^2(\alpha(t)) + A_2^2(\alpha(t))} \nabla A_2(\alpha(t)) \cdot \alpha'(t). \end{aligned} \tag{1.34}$$

Then (1.33) becomes

$$\begin{aligned} I_\gamma(A) &= \frac{1}{2\pi}\left(\int_a^b \frac{-A_2(\alpha(t))}{A_1^2(\alpha(t))+A_2^2(\alpha(t))}\nabla A_1(\alpha(t))\cdot\alpha'(t)dt \right. \\ &\left. + \int_a^b \frac{A_1(\alpha(t))}{A_1^2(\alpha(t))+A_2^2(\alpha(t))}\nabla A_2(\alpha(t))\cdot\alpha'(t)dt\right) \\ &= \frac{1}{2\pi}\int_a^b p_A(\alpha(t))dt + q_A(\alpha(t))dt. \end{aligned} \tag{1.35}$$

The next result provides the invariance of $I_\gamma(A)$ under change of coordinates by diffeomorphism. Although very intuitive, we were not able to find any reference for a proof of such a result, thus we shall provide it.

Proposition 17. *Let $A : D \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field, $\gamma \subset D$ a smooth oriented closed curve, and $\alpha : D \to D^*$ a diffeomorphism. Assume that A does not vanish on γ. Then,*

$$I_{\alpha(\gamma)}(\alpha_* A) = I_\gamma(A).$$

Proof. From (1.33), $I_\gamma(A) = W(\Gamma)$ and $I_{\alpha(\gamma)}(\alpha_* A) = W(\Gamma^*)$, where $\Gamma = \{A(\mathbf{x}) : x \in \gamma\}$ and

$$\Gamma^* := \{\alpha_* A(\alpha(\mathbf{x})) : x \in \gamma\} = \{d\alpha(\mathbf{x})A(\mathbf{x}) : x \in \gamma\}.$$

We claim that Γ^* is homotopic in $D\backslash\{(0,0)\}$ either to Γ or $-\Gamma$. In what follows we shall construct such a homotopy.

Let $\theta : [0,1] \to D$ be a parametrization of γ. Thus Γ and Γ^* are parametrized, respectively, by $\Theta(t) = A(\theta(t))$ and $\Theta^*(t) = d\alpha(\theta(t))A(\theta(t))$. We know that the set of invertible 2×2 real matrices has two path-connected components, one of them containing the identity matrix I_2 and the other one containing $-I_2$. Thus, since $d\alpha(\theta(0))$ is invertible, there exists a path $M(s)$ of invertible matrices satisfying $M(0) = [d\alpha(\theta(0))]^{-1}$ and either (a) $M(1) = I_2$ or (b) $M(1) = -I_2$. In both cases, consider the homotopy $H(s,t) = M(s)d\alpha(\theta(s \cdot t))A(\theta(t))$. Notice that $H(0,t) = A(\theta(t)) = \Theta(t)$ and, in case (a) $H(1,t) =$

$d\alpha(\theta(s \cdot t))A(\theta(t)) = \Theta^*(t)$; and in case (b) $H(1,t) = -d\alpha(\theta(s \cdot t))A(\theta(t)) = -\Theta^*(t)$. Furthermore, since A does not vanish on γ and $M(s)d\alpha(\theta(s \cdot t))$ is invertible for every $(s,t) \in [0,1] \times [0,1]$, we conclude that $H(s,t) \neq (0,0)$ for every $(s,t) \in [0,1] \times [0,1]$.

Hence, Proposition 16 implies that either $W(\Gamma^*) = W(\Gamma)$ or $W(\Gamma^*) = W(-\Gamma)$. Taking into account (1.31) and (1.32), we see that $W(-\Gamma) = W(\Gamma)$. Therefore, $I_{\alpha(\gamma)}(\alpha_* A) = W(\Gamma^*) = W(\Gamma) = I_\gamma(A)$. □

If $\mathbf{x}_0 \in D$ is an isolated critical point of $A(\mathbf{x})$, then there exists $r > 0$ such that the closed ball $B = B_r(\mathbf{x}_0) \subset D$ does not contain any critical point other than x_0. Accordingly, the *index $I_{\mathbf{x}_0}(A)$ of the vector field A at the critical point* $\mathbf{x}_0$ is defined as

$$I_{\mathbf{x}_0}(A) := I_{\partial B}(A) = \frac{1}{2\pi}\int_\Gamma \omega_W,$$

where now $\Gamma = \{A(\mathbf{x}),\ \mathbf{x} \in \partial B\}$. It is well known that if γ is a closed curve on D enclosing a finite number of isolated singularities, $x_1, x_2, \ldots, x_n$, then

$$I_\gamma(A) = \sum_{i=1}^{n} I_{x_i}(A).$$

The next results, taken from [11] and [58] are classical properties of the index for smooth vector fields that have been used throughout the paper.

Proposition 18 ([11, Lemma 6.19]). *Suppose that $\gamma \subset D$ can be continuously deformed into γ' without passing through a singularity. Then*

$$I_\gamma(A) = I_{\gamma'}(A).$$

Proposition 19 ([58, Property 1]). *Let $C_1, C_2 \subset D$ be two closed connected regions. Suppose that the intersection of the interiors of C_1 and C_2 is empty and let $C = C_1 \cup C_2$. Then*

$$I_{\partial C}(A) = I_{\partial C_1}(A) + I_{\partial C_2}(A).$$

Proposition 20 ([58, Property 2]). *Assume that A has no singularities in a bounded closed connected region $C \subset D$. Then, $I_{\partial C}(A) = 0$.*

Proposition 21 ([58, Theorem 1.3]). *Let $C \subset D$ be a closed bounded region. Suppose that A_0 and A_1 are smooth vector fields on D such that A_0 is nonsingular on ∂C and $||A_1(\mathbf{x}) - A_0(\mathbf{x})|| < ||A_0(\mathbf{x})||$ for every $\mathbf{x} \in \partial C$. Then, $A_1(\mathbf{x})$ is nonsingular on ∂D and $I_{\partial D}(A_0) = I_{\partial D}(A_1)$.*

Finally, let $\mathcal{F} : M \to TM$ be a smooth vector field defined on a two dimensional smooth manifold M. Assume that $p_0 \in M$ is an isolated singularity of $\mathcal{F}$. Let (U, Φ) be a chart of M around p_0. Then, the index of $\mathcal{Z}$ at p_0 is defined as $I_{p_0}(\mathcal{F}) := I_{\Phi(p_0)}(\Phi_* \mathcal{F})$. The next result is the famous Poincaré–Euler Theorem (see [23]) which relates the indices of the singularities of a vector field $\mathcal{F}$ defined on a compact manifold M with the Euler characteristic of M, $\chi(M)$. Details about the Euler characteristic of a compact manifold can be found in [18].

Theorem 22 (Poincaré-Hopf Theorem). *Let $\mathcal{F}$ be a smooth vector field defined on a 2-dimensional compact manifold M. Denote the set of the singularities of $\mathcal{F}$ by $\mathcal{S}$ and assume that they are all isolated. Then,*

$$\sum_{p \in \mathcal{S}} I_p(\mathcal{F}) = \chi(M),$$

where $\chi(M)$ is the Euler Characteristic of M.

2 Classification of singularities with low codimension by the index for Filippov vector fields

2.1 Introduction

In [7], a generalization of the Poincaré index for singularities in continuous vector fields was conducted, extending this concept to Filippov systems. It was demonstrated that the essential properties for a robust index definition remain valid, culminating in the validation of the Poincaré-Hopf Theorem for Filippov vector fields.

Understanding the singularity index, without the need for complex integral calculations, is of utmost importance to deepen our comprehension of the system's dynamics. In many cases, the direct calculation of the integral defining the index can be challenging and, in some instances, even impractical.

In Filippov systems, in addition to singularities representing critical points of the vector field, we encounter the so-called pseudo-singularities, where the sliding vector field vanishes. Examples of these singularities in Filippov vector fields include pseudo-saddles, pseudo-nodes, tangency points, fold-fold, and cusps.

The main objective of this chapter is to classify the indices of low codimension Filippov singularities, specifically those of codimension zero and one. Thus, by obtaining information about the existence of these singularities or even about parameters of the vector field, we can determine the values of the indices associated with these singularities. Furthermore, the application of the Poincaré-Hopf Theorem will provide additional insights into the system's dynamics and the existence of singularities, utilizing the topological information of the manifold in which the vector field is defined. This study contributes significantly to the comprehensive understanding of dynamic and topological properties in Filippov systems, providing a deeper insight into the behavior of these complex systems.

2.2 Indices of low codimension singularities

In the realm of continuous vector fields, we possess established knowledge; for instance, the index of a saddle is consistently minus one, while the index of a node is consistently one, irrespective of its stability. Thus, merely by identifying the singularity, we can provide insights into its index without the need for calculating the integral (1.7) that defines it. In this chapter, we aim to achieve a parallel approach by classifying the Filippov singularities of codimension zero and one.

2.2.1 Indices of generic Σ-singularities

Recall that the generic singularities of a planar Filippov vector field are the regular-fold, the hyperbolic pseudo-node, and the hyperbolic pseudo-saddle [22]. In what follows, we define these singularities.

Definition 6. *A point $p \in \Sigma$ is called a pseudo-equilibrium of Z if $Z^s(p) = 0$. If we have $d(\det(F^+, F^-)|_\Sigma)(p) \neq 0$, then p is called a hyperbolic pseudo-equilibrium of Z. Let $p \in \Sigma$ be a hyperbolic pseudo-equilibrium of Z.*

- *The point p is called a pseudo-saddle if $p \in \Sigma^s$ and $d(\det(F^+, F^-)|_\Sigma)(p) > 0$ or $p \in \Sigma^e$ and $d(\det(F^+, F^-)|_\Sigma)(p) < 0$.*
- *The point p is called a pseudo-node if $p \in \Sigma^s$ and $d(\det(F^+, F^-)|_\Sigma)(p) < 0$ or if $p \in \Sigma^e$ and $d(\det(F^+, F^-)|_\Sigma)(p) > 0$.*

Definition 7. *A regular-fold point is a point $p \in \Sigma$ satisfying one of the following properties:*

- *$F^+ f(p) = 0$ and $(F^+)^2 f(p) \neq 0$ and $F^- f(p) \neq 0$. In this case, we say that the regular-fold point is visible if $(F^+)^2 f(p) > 0$ and invisible if $(F^+)^2 f(p) < 0$.*

- $F^- f(p) = 0$ *and* $(F^-)^2 f(p) \neq 0$ *and* $F^+ f(p) \neq 0$*. In this case, it is visible provided* $(F^-)^2 f(p) < 0$ *and invisible provided* $(F^-)^2 f(p) > 0$*.*

Proposition 23 ([55, Proposition 8])**.** *Let p be a hyperbolic pseudo-equilibrium of Z as given in* (1.3)*. There exists a neighbourhood V of p and $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leqslant \varepsilon_0$, Z_ε has a unique critical point p_ε in V. If p is a pseudo-saddle than p_ε is a saddle. If p is a pseudo-node than p_ε is a node.*

Proposition 24 ([55, Proposition 9])**.** *Let Z be a Filippov vector field, as given in* (1.3)*, Z_ε its regularization, and p a regular-fold point of Z. Then, there exist a neighborhood V_p of p and $\varepsilon_p > 0$ such that $0 \notin Z_\varepsilon(V_p)$ for every $\varepsilon \in (0, \varepsilon_p)$.*

Theorem 25. *Let p be a generic $\Sigma-$singularity of a Filippov vector field Z. Then $I_p(Z) = 0$ provided p is a regular-fold; $I_p(Z) = 1$ provided p is a hyperbolic pseudo-node and $I_p(Z) = -1$ provided p is a hyperbolic pseudo-saddle.*

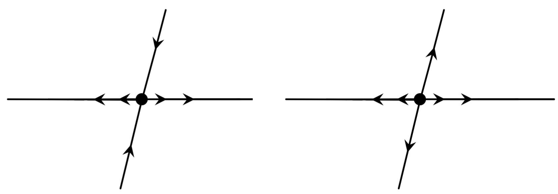
Proof. Let $p \in \Sigma$ be a regular-fold. Then by Proposition 24, there is a neighborhood V of p and $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leqslant \varepsilon_0$, Z_ε has no critical points in V. Take a neighborhood $B \subset V$ of p, by Theorem 3, $I_{\partial B}(Z) = I_{\partial B}(Z_\varepsilon) = 0$ since Z_ε has no singularities. Then $I_{\partial B}(Z) = I_p(Z) = 0$, see Figure 5.

Now let $p \in \Sigma$ be a hyperbolic pseudo-equilibrium. By Proposition 23, there exists a neighborhood V of p and $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leqslant \varepsilon_0$, Z_ε has a unique critical point p_ε near p which is a hyperbolic saddle if p is a pseudo-saddle for Z^s, or a hyperbolic node if p is a pseudo-node for Z^s. By Theorem 3, if p is a pseudo-node, then the index is 1 whereas if p is a pseudo-saddle the index is -1, see Figure 4.

□

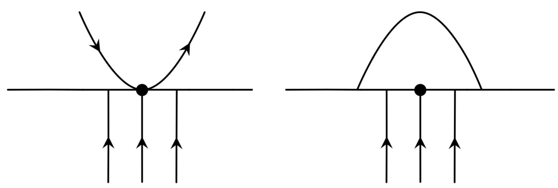
2.2.2 Indices of codimension-1 Σ-singularities

Recall that the codimension-1 Σ-singularities are the fold-fold, regular-cusp, pseudo-equilibrium, saddle-node and the boundary equilibrium.



(a) Index equal to -1 (b) Index equal to 1

Figure 4 – Indices of generic singularities, saddle and node.



(a) Indices equal to 0 (b) Index equal to 0

Figure 5 – Indices of generic fold singularities. The index does not depend on the direction of the flow.

Proposition 26. *Let $p \in \Sigma$ be a fold-fold of a Filippov vector field $Z = (F^+, F^-)$ and denote*

$$k = \operatorname{sign}(F_1^-(0,0))\frac{1}{|F_1^+(0,0)|}\frac{\partial}{\partial x}F_2^+(0,0) - \operatorname{sign}(F_1^+(0,0))\frac{1}{|F_1^-(0,0)|}\frac{\partial}{\partial x}F_2^-(0,0).$$

Then, the following statements hold:

- *If $F_1^+(0,0)F_1^-(0,0) > 0$ or $k = 0$, then $I_p(Z) = 0$.*
- *If $F_1^+(0,0)F_1^-(0,0) < 0$ and $k > 0$, then $I_p(Z) = 1$.*
- *If $F_1^+(0,0)F_1^-(0,0) < 0$ and $k < 0$, then $I_p(Z) = -1$.*

Proof. The simplified form of the fold-fold, which is obtained after rescaling the time variable (see [48]), is given by

$$Z(x,y) = \begin{cases} F^+(x,y) = (\delta^+, a^+x + x^2 f^+(x) + y g^+(x,y)), \\ F^-(x,y) = (\delta^-, a^-x + x^2 f^-(x) + y g^-(x,y)), \end{cases} \tag{2.1}$$

where

$$\delta^\pm = \text{sign}(F_1^\pm(0,0)) \quad \text{and} \quad a^\pm = \frac{1}{|F_1^\pm(0,0)|} \frac{\partial}{\partial x} F_2^\pm(0,0).$$

Notice that $p = (0,0)$ is a fold-fold of (2.1). Let $B = B_r(p)$ such that ∂B does not contain any singularities of Z. Table 1 represents the behavior of the fold-fold p for different combinations of parameters.

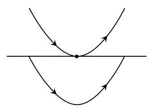
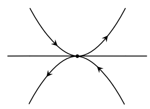
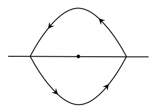
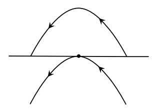
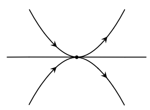
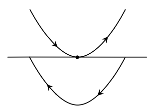
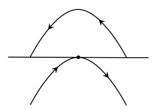
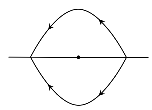
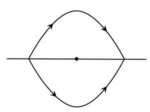
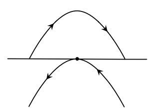
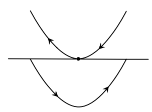
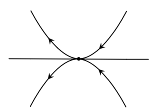
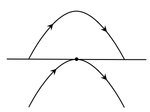
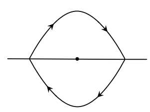
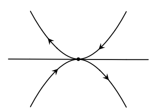
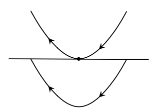
	δ^+	1		-1		
a^+	a^- \ δ^-	1	-1	1	-1	a^+a^-
> 0	> 0					> 0
	< 0					< 0
< 0	> 0					< 0
	< 0					> 0
	$\delta^+\delta^-$	1	-1	-1	1	

Table 1 – All possible behaviours near a fold-fold point of the vector field (2.1).

The approach to the demonstration is based on perturbing the vector field (2.1) to unfold the singularity. Applying Proposition 11 the index within a ball remains invariant under small perturbations. Our goal is to analyse the index of the singularities that arise within the ball B after the perturbation. Just to illustrate, Figure 6 shows the first row of Table 1 after the perturbation of the

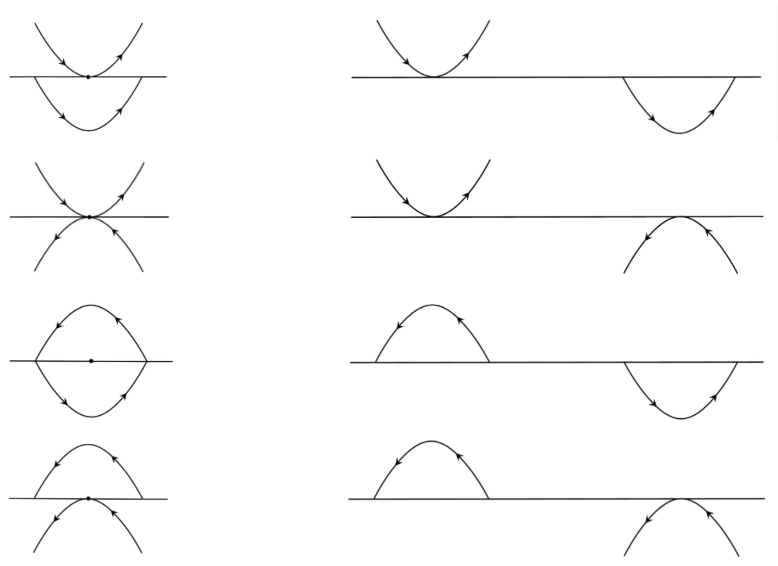


Figure 6 – First row of Table 1 after the perturbation of the vector field (2.1).

vector field. On the right, we have the representation of the perturbed field, and by analyzing each of them, we can determine in which of these four situations there is a singularity in the perturbed vector field. In Figure 6, for the second and third fold-fold, there is a singularity and in the first and fourth, there is no singularity in the perturbed field, so the index is zero.

Now, consider the following 1-parameter family of Filippov vector fields

$$Z(x,y;\lambda) = \begin{cases} F^{+}(x,y), \\ F^{-}(x-\lambda,y). \end{cases} \tag{2.2}$$

for $\lambda < r$, small enough. Notice that for $\lambda = 0$ we have $Z(x,y;0) = Z(x,y)$. Working out the crossing, sliding and escaping regions of the vector field (2.2), with $\lambda \neq 0$, and performing the change of variables $x = \lambda u$, we obtain

$$\frac{(F^{+}h)(\lambda u,0) \cdot (F^{-}h)(\lambda u,0)}{\lambda^{2}} = a^{-}a^{+}(-1+u)u + O(\lambda).$$

To distinguish between the escaping and sliding region we must take into account that

$$\frac{F^{+}h(\lambda u,0)}{\lambda u} = a^{+} + O(\lambda). \tag{2.3}$$

a^+	a^-	$u < 0$	$0 < u < 1$	$1 < u$
+	+	crossing	escaping	crossing
−	−	crossing	sliding	crossing
+	−	sliding	crossing	escaping
−	+	escaping	crossing	sliding

Table 2 – Crossing, escaping and sliding regions of the vector field (2.2) for $\lambda \neq 0$.

The outcome of this analysis is summarized on Table 2.

We need to determine the existence of singularities of the sliding vector field associated to (2.2). Let $A(\lambda, u)$ be an auxiliary function given by the numerator of the sliding vector field Z^s of (2.2) considering the change of coordinates $x = \lambda u$.

$$A(\lambda,u) = \text{sign}(\Delta(\lambda,u)) \quad \Big(a^-(-1+u)\delta^+ + (-1+u)^2\delta^+\lambda f^-((-1+u)\lambda) \\ -u\delta^- \left(a^+ + u\lambda f^+(u\lambda)\right)\Big), \tag{2.4}$$

where $\Delta(\lambda, u) = (F^+h)(\lambda u, 0) - (F^-h)(\lambda u, 0)$. Notice that $\Delta(\lambda, u) > 0$ in the sliding region, $\Delta(\lambda, u) < 0$ in the escaping region. We apply the implicit function theorem to $A(\lambda, u)$. Notice that

$$A(0,u) = 0 \Leftrightarrow u^* = \frac{a^-\delta^+}{-a^+\delta^- + a^-\delta^+} = \frac{1}{1 + \frac{-a^+\delta^-}{a^-\delta^+}}, \tag{2.5}$$

that is, $A(0, u^*) = 0$. Moreover,

$$D_u A(0,u^*) = \text{sign}(\Delta(0,u^*))(-a^+\delta^- + a^-\delta^+) = -\text{sign}(\Delta(0,u^*))k, \tag{2.6}$$

where $k := a^+\delta^- - a^-\delta^+$.

When the derivative of the sliding vector field at $\lambda = 0$ is non-zero, by the Implicit Function Theorem, there exists neighbourhoods U of 0, V of u^* and a smooth function $\eta : U \to V$ such that

$$\eta(0) = u^* = \frac{a^-\delta^+}{-a^+\delta^- + a^-\delta^+}, \tag{2.7}$$

and $A(\lambda, \eta(\lambda)) = 0$ for every $\lambda \in U$. In addition, $D_\lambda A(\lambda, \eta(\lambda)) \neq 0$. Since the zeros of the sliding vector field Z^s are the same as those of A, then we conclude that we will have only one singularity $\eta(\lambda)$ bifurcating from a fold-fold p, and the index is

$$I_p(Z) = I_{\partial B}(Z) = I_{\eta(\lambda)}(Z). \tag{2.8}$$

Now we will determine what are the conditions for the existence of a pseudo-singularity in the perturbed vector field and what type of singularity appears for each condition. The existence of pseudo-singularities only makes sense in the escape and sliding region, see Table 2. If there is a singularity, for $a^+a^- > 0$, we know that it will be between zero and one, and for $a^+a^- < 0$, the singularity will be less than zero, or greater than one. Therefore, suppose that $a^+a^- > 0$:

$$0 < \frac{a^-\delta^+}{-a^+\delta^- + a^-\delta^+} < 1 \Leftrightarrow \frac{-a^+\delta^-}{a^-\delta^+} > 0 \Leftrightarrow a^+a^-\delta^+\delta^- < 0 \tag{2.9}$$

inequality (2.9) only makes sense when $\delta^+\delta^- = -1$. Now, assume that $a^+a^- < 0$. We have to study the two possible cases:

$$u^* < 0 \Leftrightarrow \frac{1}{1 + \frac{-a^+\delta^-}{a^-\delta^+}} < 0 \Leftrightarrow \frac{-a^+\delta^-}{a^-\delta^+} < -1, \tag{2.10}$$

$$u^* > 1 \Leftrightarrow \frac{1}{1 + \frac{-a^+\delta^-}{a^-\delta^+}} > 1 \Leftrightarrow -1 < \frac{-a^+\delta^-}{a^-\delta^+} < 0. \tag{2.11}$$

Note that inequalities (2.10) and (2.11) only make sense when $\delta^+\delta^- = -1$. Thus, singularities only exist for $\delta^+\delta^- = -1$. Hence, for $\delta^+\delta^- = 1$ or $k = 0$ there is no pseudo-equilibrium inside B. By Proposition 9, $I_p(Z) = I_{\partial B}(Z) = 0$. We assume now that $\delta^+\delta^- = -1$. For the sliding region

$$D_u A(0, u^*) = -k \neq 0,$$

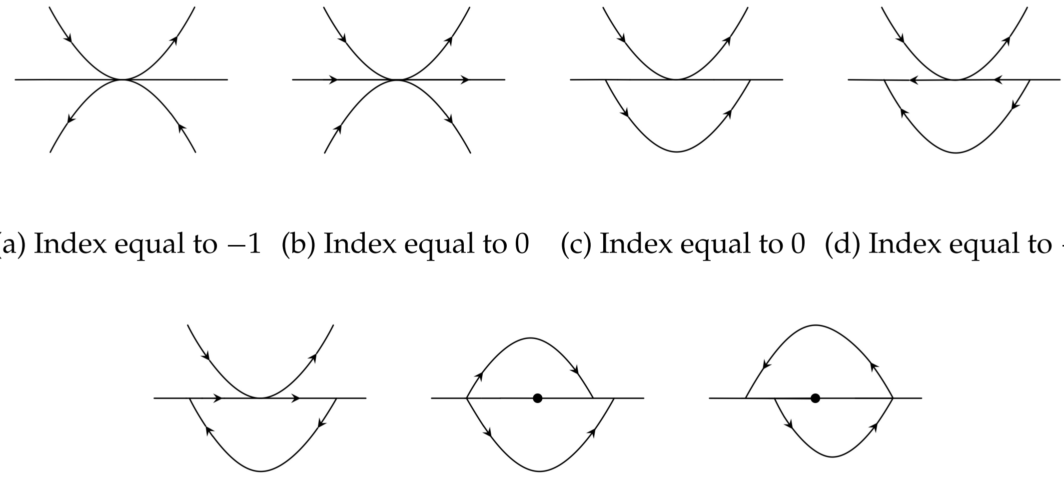
and for the escape region

$$D_u A(0, u^*) = k \neq 0,$$

by Definition 6, we obtained that $\eta(\lambda)$ given in (2.7) is a hyperbolic pseudo-equilibrium. If $k > 0$, $\eta(\lambda)$ is a pseudo-node then by (2.8) and Proposition 25 $I_p(Z) = 1$. And $\eta(\lambda)$ is a pseudo-saddle if $k < 0$ then $I_p(Z) = -1$.

From all possible cases, it follows that $I_p(Z) = 0$ if $F_1^+(0,0)F_1^-(0,0) > 0$ or $k = 0$, $I_p(Z) = 1$ if $F_1^+(0,0)F_1^-(0,0) < 0$ and $k > 0$ and $I_p(Z) = -1$ if $F_1^+(0,0)F_1^-(0,0) < 0$ and $k < 0$.

Topologically, we have the following possibilities represented in Figure 7.



(a) Index equal to -1 (b) Index equal to 0 (c) Index equal to 0 (d) Index equal to -1

(e) Index equal to 1 (f) Index equal to 0 (g) Index equal to 1

Figure 7 – Fold-fold

□

Proposition 27. *If $p \in \Sigma$ is a regular-cusp or a pseudo-saddle-node of the Filippov vector field $Z = (F^+, F^-)$, then $I_p(Z) = 0$.*

Proof. Let p a pseudo-saddle-node of the vector field Z. Unfolding the vector field, it results in a saddle pseudo-equilibrium and a node pseudo-equilibrium, that we will denote by p_1 and p_2, respectively. By Proposition 13, we have that $I_p(Z) = I_{p_1}(Z) + I_{p_2}(Z)$, and by Proposition 25, we have $I_{p_1}(Z) = -1$ and $I_{p_2}(Z) = 1$ then $I_p(Z) = 0$.

Now let p a regular-cusp. We have two options to perturb the vector field: one with two fold points and the other with no equilibrium point. In both

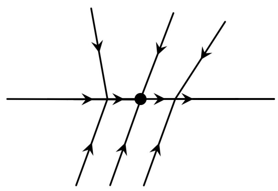


Figure 8 – Pseudo-saddle-node.

cases we have $I_p(Z) = 0$, because the index of a regular-fold is zero and if there is no equilibrium point, by Proposition 13, the index is zero.

□

The next result gives the indices of the boundary equilibrium points. Bifurcations of the boundary equilibrium were studied in [20].

Proposition 28. *Let $p \in \Sigma$ be a boundary equilibrium point of a Filippov vector field*

$$Z(x,y) = \begin{cases} F^+(x,y) & = (ax+by, cx+dy) + O(|(x,y|)^2, \\ F^-(x,y) & = (u_1,u_2) + O(|(x,y)|), \end{cases} \tag{2.12}$$

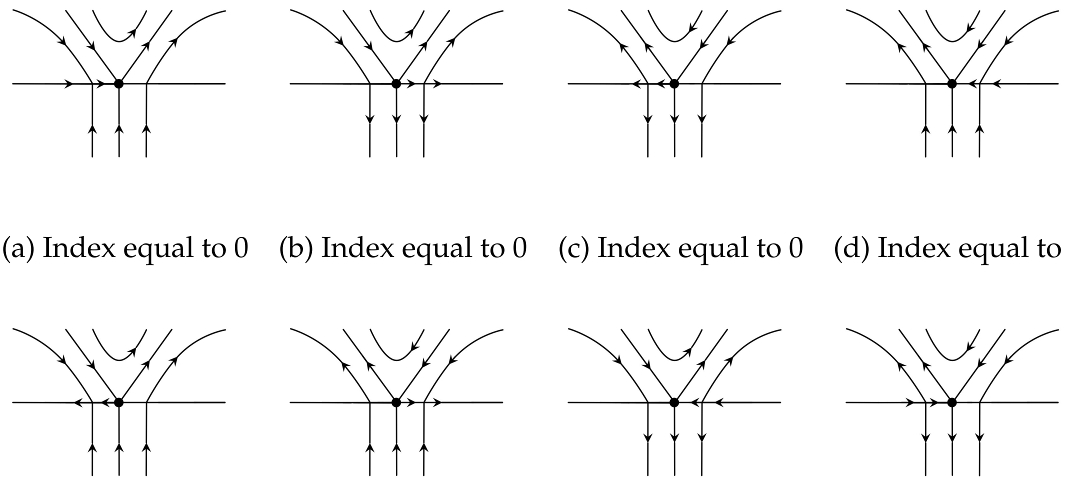
satisfying $F^+(0,0) = 0$, $F^-(0,0) = (u_1,u_2)$, $u_2 \neq 0$. The index of p is given by

- $I_p(Z) = 1$ *if* $\det(J) > 0$ *and* $-cu_1 + au_2 < 0$;
- $I_p(Z) = -1$ *if* $\det(J) < 0$ *and* $-cu_1 + au_2 < 0$;
- $I_p(Z) = 0$ *if* $\det(J)(cu_1 - au_2) < 0$;

where, J is the Jacobian matrix of F^+ of the vector field Z.

Proof. Consider the perturbed Fillipov vector field, where the perturbation is just the translation of the vector field $F^+(x,y)$, that is

$$Z(x,y;\lambda) = \begin{cases} F^+(x,y) & = (ax+b(y+\lambda), cx+d(y+\lambda)) + O(|(x,y,\lambda)|)^2, \\ F^-(x,y) & = (u_1,u_2) + O(|(x,y,\lambda)|), \end{cases} \tag{2.13}$$



(a) Index equal to 0 (b) Index equal to 0 (c) Index equal to 0 (d) Index equal to 0

(e) Index equal to −1. (f) Index equal to −1. (g) Index equal to −1. (h) Index equal to −1.

Figure 9 – Boundary equilibria saddle and their respective indices.

for $\lambda > 0$. Note that $u_2 \neq 0$ and $c \neq 0$, otherwise the codimension of the singularity would increase. For $\lambda = 0$ in (2.13) we have the unperturbed vector field. Let p a boundary equilibrium of Z. The product of the Lie derivative over the discontinuity manifold is given by

$$F^{-}f(x,0) \cdot F^{+}f(x,0) = u_2(cx + d\lambda) + O(|(x,\lambda)|).$$

We have that the point $\bar{p} = -d/c$ is a fold point. Since the zeros of the sliding vector field vary with λ, we have a sliding or escaping region in the following cases:

$$\begin{aligned} u_2 c > 0 \quad &\text{and} \quad x < \frac{-d}{c}, \\ u_2 c < 0 \quad &\text{and} \quad x > \frac{-d}{c}. \end{aligned} \tag{2.14}$$

The sliding vector field of Z is given by

$$Z^s(x,0) = \frac{-au_2x - b\lambda u_2 + cu_1x + d\lambda u_1}{cx + d\lambda - u_2}.$$

Note that $Z^s(x,0) = 0$ if and only if $x = 0$. And the derivative of Z^s is

$$(Z^s)'(x,0) = \frac{au_2 - cu_1}{u_2},$$

where $au_2 - cu_1 \neq 0$ because the singularity has codimension 1. When the derivative of the sliding vector field at $\lambda = 0$ is non zero, by the Implicit Function Theorem, there exists $\eta(\lambda)$ such that $\eta(0) = 0$,

$$\eta'(0) = \frac{du_1 - bu_2}{-cu_1 + au_2}. \tag{2.15}$$

By (2.14) we have that $\eta(\lambda)$ is a pseudo-singularity just in these two cases

$$\begin{aligned} u_2 c > 0 \quad &\text{and} \quad \frac{du_1 - bu_2}{-cu_1 + au_2} < \frac{-d}{c}, \\ u_2 c < 0 \quad &\text{and} \quad \frac{du_1 - bu_2}{-cu_1 + au_2} > \frac{-d}{c}. \end{aligned} \tag{2.16}$$

Since for this following conditions

$$\begin{aligned} u_2 c > 0 \quad &\text{and} \quad \frac{du_1 - bu_2}{-cu_1 + au_2} > \frac{-d}{c}, \\ u_2 c < 0 \quad &\text{and} \quad \frac{du_1 - bu_2}{-cu_1 + au_2} < \frac{-d}{c}, \end{aligned} \tag{2.17}$$

there are no sliding or escaping region, and thus $\eta(\lambda)$ is not a pseudo-singularity.

Then by (2.17) if

$$\frac{(bc - ad)u_2^2}{cu_1 - au_2} > 0,$$

we do not have a pseudo-singularity, then by Proposition 25, the index is zero. Now under the conditions of (2.16) we have that if

$$\frac{(bc - ad)u_2^2}{cu_1 - au_2} < 0 \quad \text{and} \quad -cu_1 + au_2 < 0,$$

then $\eta(\lambda)$ is a pseudo-node. Equivalently, if $ad - bc > 0$ and $-cu_1 + au_2 < 0$ then $\eta(\lambda)$ is also a pseudo-node, then by Proposition 25 the index of the boundary equilibrium p is equal to 1. On the other hand, if

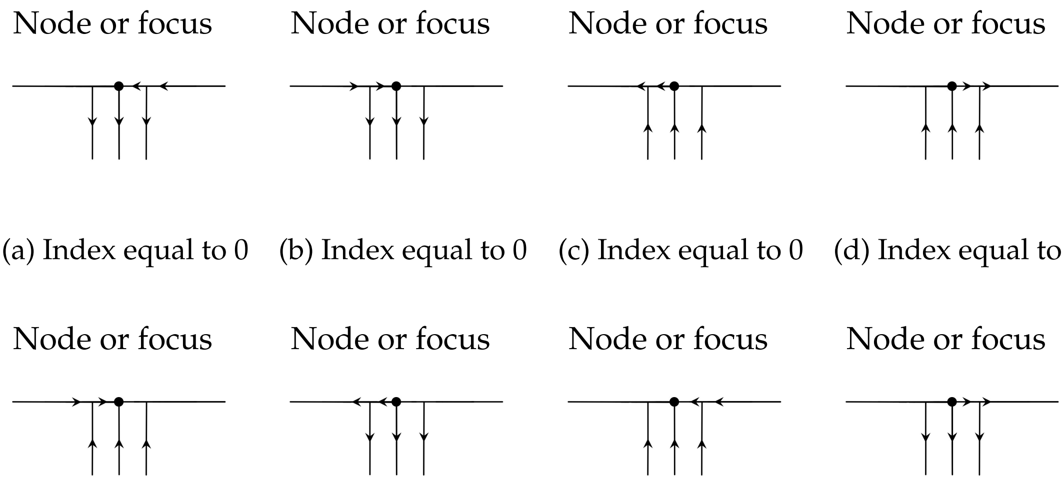
$$\frac{(bc - ad)u_2^2}{cu_1 - au_2} < 0 \quad \text{and} \quad -cu_1 + au_2 > 0,$$

then $\eta(\lambda)$ is also a pseudo-saddle. Equivalently if $ad - bc < 0$ and $-cu_1 + au_2 > 0$ then $\eta(\lambda)$ is also a pseudo-saddle then by Proposition 25 the index is equal to -1. Table 3 relates the conditions of the boundary-equilibrium p with its indices and its figures.

$u_2 c$	$\frac{du_1 - bu_2}{-cu_1 + au_2}$	u_2	$\frac{au_2 - cu_1}{u_2}$	EP	Index	Figure
> 0	$< -\frac{d}{c}$	> 0	< 0	stable node	1	Fig. 10(e)
		< 0	> 0	unstable node	1	Fig. 10(f)
		> 0	> 0	saddle	-1	Fig. 9(e)
		< 0	< 0	saddle	-1	Fig. 9(h)
< 0	$> -\frac{d}{c}$	> 0	< 0	stable node	1	Fig. 10(g)
		< 0	> 0	unstable node	1	Fig. 10(h)
		> 0	> 0	saddle	-1	Fig. 9(f)
		< 0	< 0	saddle	-1	Fig. 9(g)

Table 3 – Conditions for each index of the boundaries equilibrium.

Notice that, for J the Jacobian matrix of F^+ of the vector field Z, $\det(J) = -bc + ad$. Then $I_p(Z) = 1$ if $\det(J) > 0$ and $-cu_1 + au_2 < 0$, $I_p(Z) = -1$ if $\det(J) < 0$ and $-cu_1 + au_2 < 0$. Finally $I_p(Z) = 0$ if $\det(J)(cu_1 - au_2) < 0$. □



(a) Index equal to 0 (b) Index equal to 0 (c) Index equal to 0 (d) Index equal to 0

(e) Index equal to 1 (f) Index equal to 1 (g) Index equal to 1 (h) Index equal to 1

Figure 10 – Boundary equilibria of the type node or focus and their respective indices.

2.3 A vector field on $\mathbb{S}^2$ with only tangential singularities

In this application we will calculate the index of a Filippov vector field defined in the sphere given in [4]. Here we will not put the calculations already done in [4].

We will consider the following vector field defined in the sphere, for which there are no equilibrium points, only Fillippov singularities see [4].

We will apply Theorem 14 to this example. Consider the planar vector field

$$Z(x,y) = \begin{cases} F^+(x,y) & = (2, y(x^3 - 3x)), \\ F^-(x,y) & = (-1, 0). \end{cases} \tag{2.18}$$

Project the trajectories of (2.18) in the sphere, by the stereographic projection fixed at point $(1,0,0)$ for the vector field $F^+(x,y)$ and in the point $(-1,0,0)$ for the vector field $F^-(x,y)$. We have the vector field defined in the sphere (see [4]). After a coordinate change, we have

$$Z_1(u,v) = \begin{cases} F^+(u,v) & = (F_1^+(u,v), F_2^+(u,v)), \\ F^-(u,v) & = (F_1^-(u,v), F_2^-(u,v)), \end{cases} \tag{2.19}$$

where, $F_1^+(u,v) = -1/c$, for $c \in \mathbb{R}$,

$$F_2^+(u,v) = \frac{u}{c(v + \sqrt{u^2+v^2+1})},$$

$$F_1^-(u,v) = (\sqrt{u^2+v^2+1} - v)\left(2 - \frac{u^4}{(\sqrt{u^2+v^2+1}-v)^4} + \frac{3u^2}{(\sqrt{u^2+v^2+1}-v)^2}\right),$$

$$\begin{aligned} F_2^-(u,v) = &\ \frac{u(-1+2u^4+v^2+4v^4+v\sqrt{u^2+v^2+1}-4v^3\sqrt{u^2+v^2+1})}{v-\sqrt{u^2+v^2+1}} \\ + &\ \frac{u^2(2+8v2-6v\sqrt{u^2+v^2+1})}{v-\sqrt{u^2+v^2+1}}. \end{aligned}$$

As shown in the paper, the vector field only has Fillipov singularities, so the sum of the indices outside the region of discontinuity is zero. In the discontinuity manifold we have the symmetric vector field (2.19), as shown in figure 11.

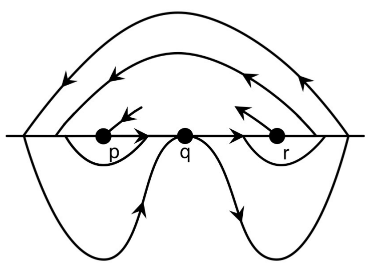


Figure 11 – Vector field (2.19).

By Proposition 25 the indices of the regular folds, p and r, see Figure 11, are zero. The index of fold-fold point q, by Proposition 26, is 1. Since the vector field is symmetric, on the other side of the sphere we have the same singularities. So the sum of the Poincaré-Hopf indices of the discontinuous vector field is 2.

3 Limit cycles of continuous piecewise smooth differential systems

The content within this chapter corresponds to paper [5].

3.1 Introduction and statement of the main results

Around the 1920's, the interest for studying piecewise differential systems started mainly in the works of Andronov, Vitt and Khaikin, see the book [51]. Nowadays this interest is increasing due to the fact that piecewise differential systems model many processes appearing in mechanics, electronics, economy, etc. For more details see the books of Simpson [52], di Bernardo et al. [9] and, the survey of Makarenkov and Lamb [44], and the vast of references which appear there in.

The easiest continuous piecewise differential systems are the ones separated by a straight line in the plane $\mathbb{R}^2$ and formed by two linear differential systems. Lum and Chua in 1990 conjectured in [42, 43] that such piecewise differential systems have at most one limit cycle. We recall that a limit cycle is an isolated periodic orbit in the set of all periodic orbits of a differential system. The previous conjecture was proved in 1990 by Freire et al. [14]. Later on a distinct and shorter proof was given in 2013 by Llibre, Ordóñez and E. Ponce [35], and more recently in 2021 a new proof has been given by Carmona, Fernández-Sánchez and Novaes [3].

In the paper [33] the authors studied the discontinuous piecewise differential systems separated by a circle and formed by two linear differential systems, and proved that those systems can have at most 3 limit cycles, and that there are systems of this type having 3 limit cycles. But the same kind of piecewise differential systems being continuous on the circle has no limit cycles.

In this chapter before studying the limit cycles of the discontinuous piecewise differential systems separated by a circle and formed by one linear differential system and a quadratic differential system, we shall study the limit cycles of the easier continuous piecewise differential systems separated by a circle and formed by one linear differential system and a quadratic differential system.

In [32] it was proved, (see Theorem 1.1) that a continuous piecewise differential system separated by a parabola, and formed by a linear differential center and a quadratic differential center, has at most one limit cycle, and that there exist such kind of piecewise differential systems with one limit cycle.

In this chapter we study the continuous piecewise differential systems separated by the circle $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ and given by

$$Z = \begin{cases} Z_1(x, y), & \text{if } x^2 + y^2 \leqslant 1, \\ Z_2(x, y), & \text{if } x^2 + y^2 \geqslant 1, \end{cases} \tag{3.1}$$

where Z_1, Z_2 are centers, one is linear and the other is quadratic. To ensure that we have a linear and a quadratic center we shall use the following two propositions.

Proposition 29 ([39, Lemma 1]). *A linear differential system having a center can be written as*

$$\dot{x} = -Bx - \frac{4B^2 + \omega^2}{4A} y + D, \quad \dot{y} = Ax + By + C, \tag{3.2}$$

with $A > 0$ and $\omega > 0$.

Proposition 30 is taken from book [10], but can also be found in the original works of Kapteyn and Bautin [2, 28, 29].

Proposition 30 ([10, Theorem 8.15]). *A quadratic differential system that has a center at the origin can be written in the form*

$$\dot{x} = -y - bx^2 - cxy - dy^2, \quad \dot{y} = x + ax^2 + exy - ay^2. \tag{3.3}$$

This system has a center at the origin if and only if at least one of the four following conditions hold:

(i) $e - 2b = c + 2a = 0$,

(ii) $b + d = 0$,

(iii) $c + 2a = e + 3b + 5d = a^2 + bd + 2d^2 = 0$,

(iv) $c = a = 0$.

The main result of this chapter is the following theorem.

Theorem 31. *Consider the differential system* (3.1) *formed by a linear differential center* (3.2) *and a quadratic differential center* (3.3) *after the change of variables* $x = kX + \alpha$ *and* $y = MY + \beta$, *with* $k, M \neq 0$. *Then the following statements hold.*

(a) There are no continuous piecewise differential systems (3.1) *with quadratic differential center of type (i), (ii), (iii), with* $d \neq 0$. *When* $d = 0$ *system (iii) becomes system (iv).*

(b) The continuous piecewise differential system (3.1) *with a quadratic differential center of type (iv) has at most three limit cycles.*

Theorem 31 is proved in Section 3.3.

The reason for considering in the statement of Theorem 31 quadratic differential centers with the mentioned change of variables, is that we increase the classes of quadratic differential centers described in Proposition 30 with four additional parameters. Unfortunately at this moment we cannot increases that class of quadratic differential centers by doing a general affine transformation which will increase in six the number of parameters, but the computations necessary for studying this class cannot be done for now.

In summary, for the class of continuous piecewise differentials systems studied here in, we provide the upper bound of three for their maximum number of limit cycles. So we arrive at have solved the extension of the 16th Hilbert problem for this class of piecewise differential systems. For the moment is unknown if this upper bound is reached. The only known examples of piecewise differential systems have one limit cycle.

In order to simplify the terminology, in what follows instead of linear differential center and quadratic differential center we shall write linear center and quadratic center, respectively.

3.2 Preliminaries

Let $I \subset \mathbb{R}$ be an open interval and let $f_0, f_1, ..., f_n : I \to \mathbb{R}$. We say that $f_0, f_1, ..., f_n$ are linearly independent functions if and only if when

$$\sum_{i=0}^{n} \lambda_i f_i(x) = 0, \ \forall\ x \in I \implies \lambda_0 = ... = \lambda_n = 0. \tag{3.4}$$

The following result, which can be found in [37], will be used in the proof of statement (b) of Theorem 31.

Proposition 32. *Let $f_0, f_1, ..., f_n : I \to \mathbb{R}$ be analytic functions. If $f_0, f_1, ..., f_n$ are linearly independent then there exist $s_1, ..., s_n \in I$ and $\lambda_0, ..., \lambda_n \in \mathbb{R}$ such that for every $j \in \{1, ..., n\}$ we have $\sum_{i=0}^{n} \lambda_i f_i(s_j) = 0$.*

Let I be an open interval and $f_0, ..., f_n$ functions defined on I. We say that $(f_0, ..., f_n)$ forms an Extended Chebyshev system (ET-system) on I, if and only if, any non-trivial linear combination of these functions has at most n zeros counting their multiplicities and this number is reached. The functions $(f_0, ..., f_n)$ are an Extended Complete Chebyshev system (ECT-system) on I if and only if for any $j \in \{0, 1, ..., n\}$, $(f_0, ..., f_j)$ form an ET-system.

The next result can be found in [30].

Proposition 33. *Let $f_0, ..., f_n$ be analytic functions defined on an open interval $I \subset \mathbb{R}$. Then $(f_0, ..., f_n)$ is an ECT-system on I if and only if for each $j \in \{0, 1, ..., n\}$ and all $y \in I$ the Wronskian*

$$W(f_0, ..., f_j)(y) = \begin{bmatrix} f_0(y) & f_1(y) & \cdots & f_j(y) \\ f_0'(y) & f_1'(y) & \cdots & f_j'(y) \\ \vdots & \vdots & \ddots & \vdots \\ f_0^{(j)}(y) & f_1^{(j)}(y) & \cdots & f_j^{(j)}(y) \end{bmatrix} \tag{3.5}$$

is different from zero.

3.3 Proof of Theorem 31

Proof of statement (a) of Theorem 31. Under the assumptions of statement (a) of Theorem 31 there are no continuous piecewise differential systems (3.1) such that one of the differential systems (3.3) satisfying (i), (ii) and (iii) with $d \neq 0$ has a quadratic center. We will prove statement (a) for the quadratic centers (i),(ii) and (iii) for $d \neq 0$, separately.

Indeed, in case (i) we have that $e - 2b = c + 2a = 0$, then system (3.3) after the change of variables $x = kX + \alpha$, $y = MY + \beta$, can be written as

$$\begin{aligned}
\dot{x} &= -\frac{(\beta + My)(-2a(\alpha + kx) + d(\beta + My) + 1) + b(\alpha + kx)^2}{k}, \\
\dot{y} &= x\left(\frac{k(2a\alpha + 2b\beta + 1)}{M} + 2bky\right) + \frac{a\alpha^2 - a\beta^2 + \alpha + 2\alpha b\beta}{M} \\
&\quad +2y(\alpha b - a\beta) + \frac{ak^2x^2}{M} - aMy^2.
\end{aligned} \tag{3.6}$$

In order for this piecewise differential system to be continuous on the circle $x^2 + y^2 = 1$, the differential systems (3.2) and (3.6) must coincide on this circle. This implies that

$$a = 0,\ A = \frac{k}{M},\ b = 0,\ B = 0,\ C = \frac{\alpha}{M},\ d = 0,\ D = \frac{-\beta}{k},\ \omega = 2.$$

Under these conditions system (3.6) becomes

$$\dot{x} = -\frac{\beta}{k} - \frac{My}{k}, \quad \dot{y} = \frac{kx}{M} + \frac{\alpha}{M}. \tag{3.7}$$

Since (3.7) is not a quadratic system we do not have a continuous piecewise differential system with a quadratic center of type (i).

In case (ii) we have $b = -d$, then system (3.3) after the change of variables $x = kX + \alpha$, $y = MY + \beta$, can be written as

$$\begin{aligned} \dot{x} &= -\frac{b(\alpha+kx)^2 + (\beta+My)(c(\alpha+kx)+\beta d + dMy + 1)}{k}, \\ \dot{y} &= \frac{kx(2a\alpha + e(\beta+My)+1) - a(-\alpha+\beta+My)(\alpha+\beta+My)}{M} \\ &\quad + \frac{ak^2x^2 + \alpha + \alpha e(\beta+My)}{M}. \end{aligned} \tag{3.8}$$

In order that this piecewise differential system be continuous on the circle $x^2 + y^2 = 1$, the differential systems (3.2) and (3.8) must coincide on this circle. This implies that

$$a = 0,\ A = \frac{k}{M},\ b = 0,\ B = 0,\ c = 0,\ C = \frac{\alpha}{M},\ D = 0,\ e = 0,\ \omega = \frac{2\sqrt{k}}{\sqrt{M}}.$$

Under these conditions the systems (3.6) and (3.8) are the same and are given in (3.7). We do not have a continuous piecewise differential system with a quadratic center of type (ii).

In case (iii) we have that $c + 2a = e + 3b + 5d = a^2 + bd + 2d^2 = 0$, so system (3.3) after applying the change of variables $x = kX + \alpha$, $y = MY + \beta$, is

$$\begin{aligned} \dot{x} &= \frac{-d\left(\beta + d\left(-2\alpha^2 + \beta^2 - 2k^2x^2 - 4\alpha kx + M^2y^2 + 2\beta My\right) + My\right)}{dk} \\ &\quad + \frac{2ad(\alpha+kx)(\beta+My) + a^2(\alpha+kx)^2}{dk}, \\ \dot{y} &= \frac{d\left(ak^2x^2 + k(2a\alpha x + x) - a(-\alpha+\beta+My)(\alpha+\beta+My) + \alpha\right)}{dM} \\ &\quad + \frac{3a^2(\alpha+kx)(\beta+My) + d^2(\alpha+kx)(\beta+My)}{dM}. \end{aligned} \tag{3.9}$$

However in order that this system be continuous on the circle $x^2 + y^2 = 1$, the solutions obtained, that are not complex, are such that either $d = 0$ or $k = M = 0$. As this contradicts the hypotheses, we have that there is no continuos piecewise differential system in this case. □

Proof of statement (b) of Theorem 31. Now we will work with the quadratic center of type (iv), that is, $c = a = 0$. Under this condition system (3.3) is

$$\dot{x} = -dy^2 - bx^2 - y, \qquad \dot{y} = exy + x. \tag{3.10}$$

Doing a rescaling of the time we can, without loss of generality, work with two subcases $b = 0$ and $b = 1$. In the first case, that is, $b = 0$, there is no continuous piecewise differential system on the circle $x^2 + y^2 = 1$. Indeed, with the change of variables $x = kX + \alpha$, $y = MY + \beta$, we can writte system (3.10) as

$$\begin{aligned} \dot{x} &= -\frac{\beta(\beta d + 1)}{k} - \frac{dM^2y^2}{k} - \frac{My(2\beta d + 1)}{k}, \\ \dot{y} &= x\left(\frac{k(\beta e + 1)}{M} + eky\right) + \frac{\alpha(\beta e + 1)}{M} + \alpha e y. \end{aligned} \tag{3.11}$$

In order for this piecewise differential system to be continuous on the circle $x^2 + y^2 = 1$, the differential system (3.11) and (3.2) must coincide on the circle. This implies that

$$A = \frac{k}{M}, \; B = 0, \; C = \frac{\alpha}{M}, \; d = 0, \; D = \frac{-\beta}{k}, \; e = 0, \; \omega = 2. \tag{3.12}$$

But this solution is such that system (3.11) becomes non quadratic, because by rewriting (3.11) under these conditions, we get the system

$$\dot{x} \;=\; -\frac{\beta}{k} - \frac{My}{k}, \quad \dot{y} = \frac{kx}{M} + \frac{\alpha}{M}. \tag{3.13}$$

It remains to study the case $b = 1$. Then with the change of variables $x = kX + \alpha$, $y = MY + \beta$ and $b = 1$, we can write system (3.10) as

$$\begin{aligned} \dot{x} &= -\frac{\alpha^2 + \beta + \beta^2 d}{k} - \frac{dM^2y^2}{k} - \frac{My(2\beta d + 1)}{k} - kx^2 - 2\alpha x, \\ \dot{y} &= x\left(\frac{k(\beta e + 1)}{M} + eky\right) + \frac{\alpha(\beta e + 1)}{M} + \alpha e y. \end{aligned} \tag{3.14}$$

In order for this piecewise differential system to be continuous on the circle $x^2 + y^2 = 1$, the differential system (3.14) and (3.2) must coincide on this circle. This implies that

$$A = \frac{k}{M}, \; B = C = 0, \; D = \frac{-\beta^2 k^2 - k^2 M^2 - \beta M^2}{kM^2},$$
$$d = \frac{k^2}{M^2}, \; e = \alpha = 0, \; \omega = \frac{2\sqrt{2\beta k^2 + M^2}}{M}. \tag{3.15}$$

Under these conditions system (3.2) becomes

$$\dot{x} = \frac{-\beta^2 k^2 - k^2 M^2 - \beta M^2}{kM^2} - \frac{y\left(2\beta k^2 + M^2\right)}{kM},$$
$$\dot{y} = \frac{kx}{M}. \tag{3.16}$$

with the first integral

$$H_1(x,y) = 4k^2x^2M + 4y^2M\left(2\beta k^2 + M^2\right) - 8y\left(-\beta^2k^2 - k^2M^2 - \beta M^2\right), \tag{3.17}$$

and system (3.14) becomes

$$\dot{x} = -\frac{k\left(\beta^2 + M^2\left(x^2+y^2\right) + 2\beta M y\right)}{M^2} - \frac{\beta + My}{k},$$
$$\dot{y} = \frac{kx}{M}. \tag{3.18}$$

System (3.18) has the first integral $H_2(x,y)$ given by

$$H_2(x,y) = \frac{\exp(2My)\left(2\beta^2k^2 - 2\beta k^2 + 4\beta k^2 My - 2k^2My + k^2 + 2M^3y +\right)}{2k^2M^2}$$
$$+\frac{\exp(2My)M^2(2k^2x^2 + 2k^2y^2 + 2\beta - 1)}{2k^2M^2}. \tag{3.19}$$

Assume that the continuous piecewise differential system has a limit cycle which intersects the circle $x^2 + y^2 = 1$ in the two points (x_1, y_1) and (x_2, y_2). In order to determine how many limit cycles exist for this continuous piecewise differential system formed by systems (3.16) and (3.18) we will analyse how many solutions the system below admits:

$$e_1(x,y) := H_1(x_1,y_1) - H_1(x_2,y_2) = 0,$$
$$e_2(x,y) := H_2(x_1,y_1) - H_2(x_2,y_2) = 0. \tag{3.20}$$

Consider the change of variables given by

$$x_i = \sin t_i \quad y_i = \cos t_i. \tag{3.21}$$

With the change variable (3.21), system (3.20) becomes

$$\begin{aligned} e_1(t_1,t_2) = & \frac{4(\sin t_1 - \sin t_2)\left(2\left(k^2\left(\beta^2+M^2\right)+\beta M^2\right)\right)}{M^3} \\ & + \frac{4(\sin t_1 - \sin t_2)2M\left((2\beta-1)k^2+M^2\right)(\sin t_1 + \sin t_2)}{M^3}, \\ e_2(t_1,t_2) = & \frac{k^2\left(2(\beta-1)\beta+2M^2+1\right)(\exp(2M\sin t_1)-\exp(2M\sin t_2))}{2k^2M^2} \\ & + \frac{(2\beta-1)M^2(\exp(2M\sin t_1)-\exp(2M\sin t_2))}{2k^2M^2} \\ & + \frac{2M(2\beta-1)k^2(\sin t_1\exp(2M\sin t_1)-\sin t_2\exp(2M\sin t_2))}{2k^2M^2} \\ & + \frac{2M^3(\sin t_1\exp(2M\sin t_1)-\sin t_2\exp(2M\sin t_2))}{2k^2M^2}. \end{aligned} \tag{3.22}$$

Notice that $e_1 = 0$ when $\sin t_1 = \sin t_2$, or

$$2\left(k^2\left(\beta^2+M^2\right)+\beta M^2\right)+M\left((2\beta-1)k^2+M^2\right)(\sin t_1+\sin t_2)=0. \tag{3.23}$$

If $\sin t_1 = \sin t_2$ then e_2 is identically zero. Therefore, if there are periodic orbits, we will have a continuum of periodic orbits for system (3.22), that is, no limit cycles.

Now we consider the second case, when (3.23) holds. We obtain two solutions for the variable t_1 given by

$$\begin{aligned} t_1^1 &= \pi - \sin^{-1}\left(\frac{k^2(M\sin t_2 - 2\beta^2 - 2M^2 - 2\beta M\sin t_2) - M^3\sin t_2 - 2\beta M^2}{M\left(2\beta k^2 - k^2 + M^2\right)}\right), \\ t_1^2 &= \sin^{-1}\left(\frac{k^2(M\sin t_2 - 2\beta^2 - 2M^2 - 2\beta M\sin t_2) + M^3(-\sin t_2) - 2\beta M^2}{M\left(2\beta k^2 - k^2 + M^2\right)}\right). \end{aligned} \tag{3.24}$$

Substituting t_1^1 in e_2, we obtain the equation

$$e_2(t_1^1, t_2) = \sum_{i=0}^{3} a_i f_i(t_2) = 0, \tag{3.25}$$

where the coefficients are given by

$$\begin{aligned} a_0 &= \frac{2M\left((2\beta-1)k^2+M^2\right)\left(-2\left(k^2\left(\beta^2+M^2\right)+\beta M^2\right)\right)}{(2\beta-1)k^2M+M^3}, \\ a_1 &= \frac{2M\left((2\beta-1)k^2+M^2\right)\left(-M\left((2\beta-1)k^2+M^2\right)\right)}{(2\beta-1)k^2M+M^3}, \\ a_2 &= k^2\left(2(\beta-1)\beta+2M^2+1\right)+(2\beta-1)M^2, \\ a_3 &= 2M\left((2\beta-1)k^2+M^2\right), \end{aligned} \tag{3.26}$$

and the functions are

$$\begin{aligned} f_0(t_2) &= \exp\left(-\frac{4\left(k^2\left(\beta^2+M^2\right)+\beta M^2\right)}{(2\beta-1)k^2+M^2}-2M\sin t_2\right), \\ f_1(t_2) &= \sin t_2 \exp\left(-\frac{4\left(k^2\left(\beta^2+M^2\right)+\beta M^2\right)}{(2\beta-1)k^2+M^2}-2M\sin t_2\right), \\ f_2(t_2) &= \exp\left(-\frac{4\left(k^2\left(\beta^2+M^2\right)+\beta M^2\right)}{(2\beta-1)k^2+M^2}-2M\sin t_2\right)-\exp(2M\sin t_2), \\ f_3(t_2) &= \sin t_2\left(-\exp(2M\sin t_2)\right). \end{aligned} \tag{3.27}$$

We compute the following Wronskians

$$W(f_0)(t_2) = \exp\left(-\frac{4\left(k^2\left(\beta^2+M^2\right)+\beta M^2\right)}{(2\beta-1)k^2+M^2}-2M\sin t_2\right), \tag{3.28}$$

$$W(f_0, f_1)(t_2) = \cos t_2 \exp\left(-\frac{8\left(k^2\left(\beta^2+M^2\right)+\beta M^2\right)}{(2\beta-1)k^2+M^2}-4M\sin t_2\right), \tag{3.29}$$

$$W(f_0, f_1, f_2)(t_2) = -16M^2\cos^3 t_2 \exp\left(-\frac{8\left(k^2\left(\beta^2+M^2\right)+\beta M^2\right)}{(2\beta-1)k^2+M^2} - 2M\mathrm{sin}t_2\right), \tag{3.30}$$

$$W(f_0, f_1, f_2, f_3)(t_2) = 256M^4\cos^6 t_2 \exp\left(-\frac{8\left(k^2\left(\beta^2+M^2\right)+\beta M^2\right)}{(2\beta-1)k^2+M^2}\right). \tag{3.31}$$

In the interval $(-\pi/2, \pi/2)$ the Wronskians (3.28), (3.29), (3.30) and (3.31) are non zero. Then (f_0, f_1, f_2, f_3) is an Extended Chebyshev system that have at most three solutions. This means that we can has at most three limit cycles.

Since the rank of the 4×3 matrix

$$\begin{bmatrix} \frac{\partial a_0}{\partial k} & \frac{\partial a_0}{\partial M} & \frac{\partial a_0}{\partial \beta} \\ \frac{\partial a_1}{\partial k} & \frac{\partial a_1}{\partial M} & \frac{\partial a_1}{\partial \beta} \\ \frac{\partial a_2}{\partial k} & \frac{\partial a_2}{\partial M} & \frac{\partial a_2}{\partial \beta} \\ \frac{\partial a_3}{\partial k} & \frac{\partial a_3}{\partial M} & \frac{\partial a_3}{\partial \beta} \end{bmatrix} \tag{3.32}$$

cannot be four, the four coefficients are not independent, and consequently we cannot guarantee that the system has three solutions, we only can say that it has at most three solutions. □

3.4 Examples

We will present two examples, both with only one limit cycle. The first one is defined by a quadratic center inside the circle $x^2 + y^2 = 1$ and a linear center outside the circle. In the second example, we obtain the limit cycle regardless of which system is defined inside (or outside) the circle $x^2 + y^2 = 1$.

As we saw in the proof of Theorem 31, the existence of limit cycles is possible for quadratic centers of type ((iv)). So in the examples we will be under the condition $c = a = 0$.

Example 1. Consider the continuous piecewise differential system separated by the unit circle centered at the origin of coordinates

$$Z = \begin{cases} Z_1(x,y) = \left(-\dfrac{2x^2}{3} - \dfrac{2y^2}{3} - \dfrac{77y}{60} - \dfrac{61}{150}, \dfrac{4x}{3}\right), \text{ if } x^2+y^2 \leqslant 1, \\ \\ Z_2(x,y) = \left(-\dfrac{77y}{60} - \dfrac{161}{150}, \dfrac{4x}{3}\right), \text{ if } x^2+y^2 \geqslant 1. \end{cases} \tag{3.33}$$

Note that $Z_1(x,y)$ is the quadratic center (3.18), and $Z_2(x,y)$ is the linear center (3.16), both with $\beta = 1/5$, $k = 2/3$, and $M = 1/2$. The first integrals of Z_1 and Z_2 are respectively

$$\begin{aligned} H_1(x,y) &= \frac{64x^2}{9} + \frac{308y^2}{45} + \frac{2576y}{225}, \\ H_2(x,y) &= \frac{9}{2}\exp(y)\left(\frac{2x^2}{9} + \frac{2y^2}{9} - \frac{y}{60} + \frac{137}{900}\right). \end{aligned} \tag{3.34}$$

This continuous piecewise differential system (3.33) has exactly one limit cycle, because the unique real solution (x_1, y_1, x_2, y_2) of the system

$$\begin{aligned} &H_1(x_1,y_1) - H_1(x_2,y_2) = 0, \\ &H_2(x_1,y_1) - H_2(x_2,y_2) = 0, \\ &x_1^2 + y_1^2 = 1, \\ &x_2^2 + y_2^2 = 1, \end{aligned} \tag{3.35}$$

is

$$(x_1, y_1, x_2, y_2) = (0.886447.., 0.462831.., -0.886447.., 0.462831..). \tag{3.36}$$

See this limit cycle in Figure 12.

Example 2. Consider the linear center

$$\dot{x} = -\frac{49y}{30} - \frac{97}{150}, \qquad \dot{y} = \frac{2x}{3}, \tag{3.37}$$

with first integral

$$H_1(x,y) = \frac{16x^2}{9} + \frac{196y^2}{45} + \frac{776y}{225}. \tag{3.38}$$

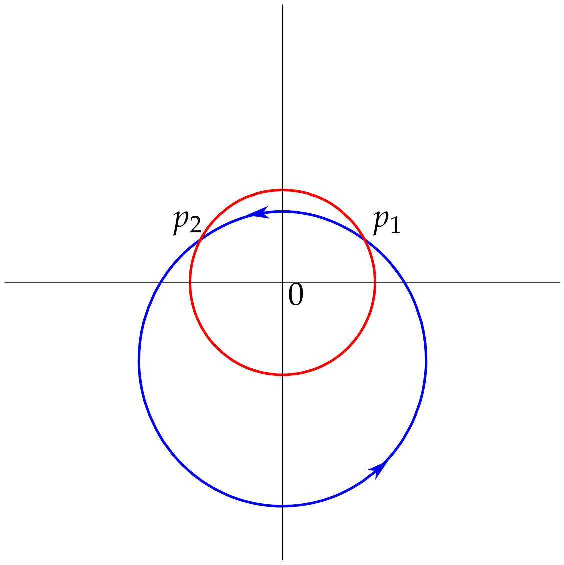


Figure 12 – System (3.33) with its limit cycle, which looks like the big circle in the figure, passing through the points $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ given by the solution (3.36). The small circle of the figure is the circle $x^2 + y^2 = 1$.

Consider the quadratic center

$$\dot{x} = -\frac{x^2}{3} - \frac{y^2}{3} - \frac{49y}{30} - \frac{47}{150}, \qquad \dot{y} = \frac{2x}{3}, \tag{3.39}$$

with first integral

$$H_2(x,y) = \frac{81}{8}\exp(y)\left(\frac{8x^2}{81} + \frac{8y^2}{81} + \frac{116y}{405} - \frac{392}{2025}\right). \tag{3.40}$$

This continuous piecewise differential system (3.33) has exactly one limit cycle, because the unique real solution (x_1, y_1, x_2, y_2) of the system

$$\begin{aligned} &H_1(x_1,y_1) - H_1(x_2,y_2) = 0, \\ &H_2(x_1,y_1) - H_2(x_2,y_2) = 0, \\ &x_1^2 + y_1^2 = 1, \\ &x_2^2 + y_2^2 = 1, \end{aligned} \tag{3.41}$$

is

$$(x_1, y_1, x_2, y_2) = (1, 0, -1, 0). \tag{3.42}$$

Note that we have two possible configurations, one for the linear center inside the circle $x^2 + y^2 = 1$ and the quadratic center outside this circle, and vice versa. In both cases we have one unique limit cycle passing through the solution (3.42). See these two possibilities in Figure 13 and Figure 14.

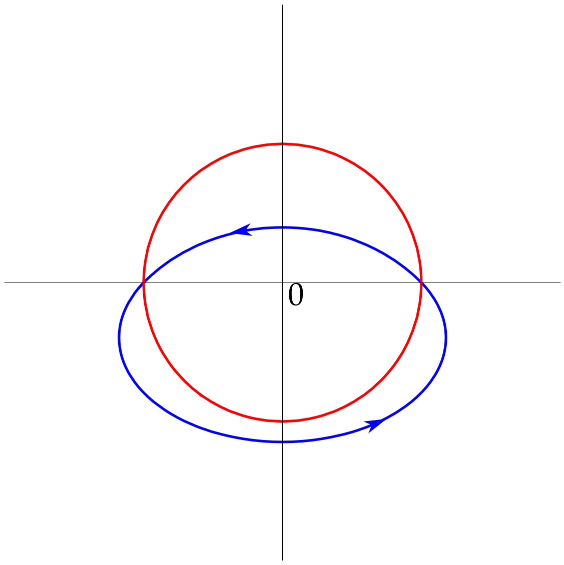


Figure 13 – The limit cycle of the continuous piecewise differential system formed by the linear center (3.37) outside the circle $x^2 + y^2 = 1$, and the quadratic center (3.39) inside the circle.

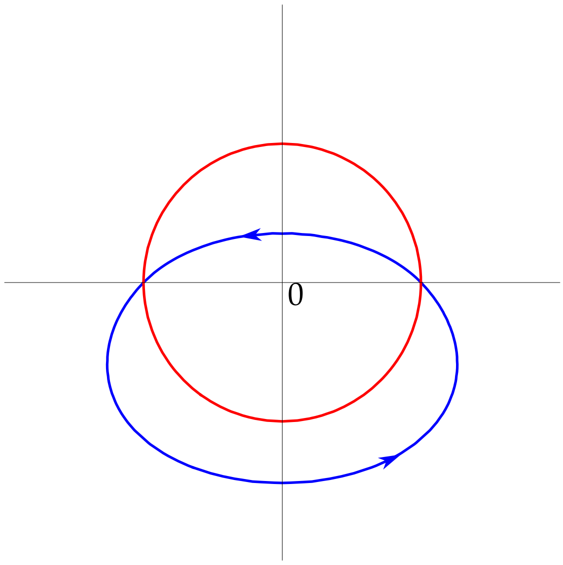


Figure 14 – The limit cycle of the continuous piecewise differential system formed by the linear center (3.37) inside the circle $x^2 + y^2 = 1$, and the quadratic center (3.39) outside the circle.

4 Limit cycles of discontinuous piecewise differential Hamiltonian systems separated by a straight line

The content within this chapter corresponds to paper [6].

4.1 Introduction and statement of the main results

Discontinuous dynamical systems are prevalent in real-world applications, ranging from electrical circuits and mechanical systems to biological processes. These systems often exhibit abrupt changes or discontinuities due to switching phenomena, impacts, or sudden state transitions, see for more details the books of Simpson [52], di Bernardo et al. [9] and, the survey of Makarenkov and Lamb [44]. The Filippov convention provides a powerful framework for modeling and analyzing such systems, enabling a more accurate representation of their dynamics.

The investigation of the existence or absence of limit cycles in the analysis of differential systems holds significant importance in unraveling their dynamic behavior. Consequently, numerous researchers have delved into the exploration of limit cycles in discontinuous piecewise linear differential systems, particularly those characterized by a separation through a straight line. This exploration has been extensively examined by various authors, exemplified in the works of, for instance, [1, 8, 12, 15, 16, 19, 21, 24, 25, 26, 31, 34, 36, 38, 40, 41].

In this chapter we study the limit cycles for the class of discontinuous piecewise differential systems separated by a straight line and formed by two Hamiltonian systems of degree either one, or two, or three. Without loss of generality we can consider that the straight line of discontinuity is $x = 0$, and that the vector field associated to these discontinuous piecewise differential systems is

$$Z(x,y) = \begin{cases} Z_1(x,y), & \text{if } x \leqslant 0, \\ Z_2(x,y), & \text{if } x \geqslant 0, \end{cases} \tag{4.1}$$

where Z_i is the vector field of the Hamiltonian system

$$\dot{x} = \frac{\partial}{\partial y} H_i(x,y), \quad \dot{y} = -\frac{\partial}{\partial x} H_i(x,y),$$

with Hamiltonian $H_i(x,y)$ for $i \in \{1,2\}$.

The behaviour of the piecewise differential system on the line of discontinuity $x = 0$ is defined following Filippov's rules, see [13].

The main result of this chapter is the following one.

Theorem 34. *Consider the discontinuous piecewise differential system (4.1) formed by two arbitrary Hamiltonians $H_1(x,y)$ and $H_2(x,y)$ of degree*

(a) 2, *then system (4.1) has no limit cycles.*

(b) 3, *then system (4.1) has at most one limit cycle.*

(c) 4, *then system (4.1) has at most three limit cycles.*

Moreover, there are differential systems (4.1) formed by two convenient Hamiltonians $H_1(x,y)$ and $H_2(x,y)$ of the corresponding degree realizing the upper bounds on the number of limit cycles of statements (b) and (c).

Theorem 34 is proved in section 4.2.

4.2 Proof of Theorem 34

Proof of statement (a) of Theorem 34. Consider two arbitrary Hamiltonians of degree two as follows

$$H_1(x,y) = a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2,$$
$$H_2(x,y) = b_0 + b_1 x + b_2 y + b_3 x^2 + b_4 xy + b_5 y^2.$$

These Hamiltonians generate the next Hamiltonian systems of degree one

$$\dot{x} = a_2 + a_4 x + 2a_5 y, \quad \dot{y} = -a_1 - 2a_3 x - a_4 y, \tag{4.2}$$

$$\dot{x} = b_2 + b_4 x + 2b_5 y, \quad \dot{y} = -b_1 - 2b_3 x - b_4 y. \tag{4.3}$$

Of course $H_1(x, y)$ and $H_2(x, y)$ are first integrals of systems (4.2) and (4.3), respectively. Now we look for the limit cycles that intersect the straight line $x = 0$ at the points $(0, y_1)$ and $(0, y_2)$ with $y_1 \neq y_2$. To do this we analyse how many solutions the following polynomial system has:

$$\begin{aligned} e_1(y_1, y_2) &:= H_1(0, y_1) - H_1(0, y_2) = 0, \\ e_2(y_1, y_2) &:= H_2(0, y_1) - H_2(0, y_2) = 0. \end{aligned} \tag{4.4}$$

To solve system (4.4) is equivalent to find the solutions of the system

$$\begin{aligned} E_1(y_1, y_2) &:= \frac{e_1(y_1, y_2)}{(y_1 - y_2)} = 0 \Rightarrow a_2 + a_5 y_1 + a_5 y_2 = 0, \\ E_2(y_1, y_2) &:= \frac{e_2(y_1, y_2)}{(y_1 - y_2)} = 0 \Rightarrow b_2 + b_5 y_1 + b_5 y_2 = 0. \end{aligned} \tag{4.5}$$

Since the straight lines $E_1(y_1, y_2) = 0$ and $E_2(y_1, y_2) = 0$ are parallel, it follows that system (4.5) has either no solutions with respect to the variables y_1 and y_2, or infinitely many solutions. In both cases the discontinuous piecewise differential system cannot have limit cycles. □

For the proof of statement (b) of Theorem 34 we shall use the next well-known result. For a proof see for instance [17].

Theorem 35 (Bézout Theorem)**.** *Let f and g be two polynomials in $\mathbb{R}[x, y]$ of degrees n and m respectively. Then if the set $V(f, g) := \{(x, y) \in \mathbb{R}^2 : f(x, y) = g(x, y) = 0\}$ has finitely many solutions, then it has at most nm points.*

Proof of statement (b) of Theorem 34. Consider the following two arbitrary Hamiltonians of degree three

$$\begin{aligned} H_1(x, y) &= a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2 + a_6 x^3 + a_7 x^2 y + a_8 xy^2 + a_9 y^3, \\ H_2(x, y) &= b_0 + b_1 x + b_2 y + b_3 x^2 + b_4 xy + b_5 y^2 + b_6 x^3 + b_7 x^2 y + b_8 xy^2 + b_9 y^3. \end{aligned}$$

These Hamiltonians generate the Hamiltonian systems

$$\begin{aligned} \dot{x} &= a_2 + a_4 x + 2a_5 y + a_7 x^2 + 2a_8 xy + 3a_9 y^2, \\ \dot{y} &= -a_1 - 2a_3 x - a_4 y - 3a_6 x^2 - 2a_7 xy - a_8 y^2, \end{aligned} \tag{4.6}$$

$$\begin{aligned} \dot{x} &= b_2 + b_4 x + 2b_5 y + b_7 x^2 + 2b_8 xy + 3b_9 y^2, \\ \dot{y} &= -b_1 - 2b_3 x - b_4 y - 3b_6 x^2 - 2b_7 xy - b_8 y^2, \end{aligned} \tag{4.7}$$

Again $H_1(x, y)$ and $H_2(x, y)$ are first integrals of systems (4.6) and (4.7), respectively. Now we look for the limit cycles that intersect the straight line $x = 0$ at the points $(0, y_1)$ and $(0, y_2)$, with $y_1 \neq y_2$. So we must analyse how many solutions the system has

$$\begin{aligned} e_1(y_1, y_2) &:= H_1(0, y_1) - H_1(0, y_2) = 0, \\ e_2(y_1, y_2) &:= H_2(0, y_1) - H_2(0, y_2) = 0. \end{aligned} \tag{4.8}$$

Solving system (4.8) is equivalent to finding the solutions of the system

$$\begin{aligned} E_1(y_1, y_2) &:= a_2 + a_5(y_1 + y_2) + a_9(y_1^2 + y_1 y_2 + y_2^2) = 0, \\ E_2(y_1, y_2) &:= b_2 + b_5(y_1 + y_2) + b_9(y_1^2 + y_1 y_2 + y_2^2) = 0, \end{aligned}$$

where $E_i(y_1, y_2) = e_i(y_1, y_2)/(y_1 - y_2)$. Notice that

$$\begin{aligned} E_{12}(y_1, y_2) = & \quad b_9 E_1(y_1, y_2) - a_9 E_2(y_1, y_2) \\ = & \quad b_9 a_2 - a_9 b_2 + (b_9 a_5 - a_9 b_5)(y_1 + y_2). \end{aligned}$$

By Bézout's Theorem the upper bound for the maximum number of solutions of system $E_1(y_1, y_2) = 0$ and $E_{12}(y_1, y_2) = 0$ is 2, whenever this system has finitely many solutions. Note that by the symmetry of these polynomial equations, if (y_1, y_2) is a solution then (y_2, y_1) is also a solution, but these two solutions determine the same periodic orbit. Hence this family of discontinuous piecewise differential systems has at most one limit cycle. This upper bound is reached as can be seen in Example 1 of section 4.3. □

Proof of statement (c) of Theorem 34. Consider two arbitrary Hamiltonians of degree four,

$$\begin{aligned} H_1(x, y) = & \quad a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2 + a_6 x^3 + a_7 x^2 y + a_8 xy^2 \\ & + a_9 y^3 + a_{10} x^4 + a_{11} x^3 y + a_{12} x^2 y^2 + a_{13} xy^3 + a_{14} y^4, \\ H_2(x, y) = & \quad b_0 + b_1 x + b_2 y + b_3 x^2 + b_4 xy + b_5 y^2 + b_6 x^3 + b_7 x^2 y + b_8 xy^2 \\ & + b_9 y^3 a_{10} x^4 + b_{11} x^3 y + b_{12} x^2 y^2 + b_{13} xy^3 + b_{14} y^4. \end{aligned}$$

These Hamiltonians generate the following two Hamiltonian systems

$$\begin{aligned} \dot{x} = &\ a_2 + a_4 x + 2a_5 y + a_7 x^2 + 2a_8 xy + 3a_9 y^2 + a_{11} x^3 + 2a_{12} x^2 y \\ &+3a_{13} xy^2 + 4a_{14} y^3, \\ \dot{y} = &\ 4a_{10} x^3 - a_1 - 2a_3 x - a_4 y - 3a_6 x^2 - 2a_7 xy - a_8 y^2 + 3a_{11} x^2 y \\ &+2a_{12} xy^2 + a_{13} y^3, \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} \dot{x} = &\ b_2 + b_4 x + 2b_5 y + b_7 x^2 + 2b_8 xy + 3b_9 y^2 + b_{11} x^3 + 2b_{12} x^2 y \\ &+3b_{13} xy^2 + 4b_{14} y^3, \\ \dot{y} = &\ 4b_{10} x^3 - b_1 - 2b_3 x - b_4 y - 3b_6 x^2 - 2b_7 xy - b_8 y^2 + 3b_{11} x^2 y \\ &+2b_{12} xy^2 + b_{13} y^3, \end{aligned} \tag{4.10}$$

respectively. The Hamiltonians $H_1(x, y)$ and $H_2(x, y)$ are first integrals of systems (4.9) and (4.10), respectively. Now we look for the limit cycles that intersect the straight line $x = 0$ at the points $(0, y_1)$ and $(0, y_2)$, with $y_1 \neq y_2$. To do that we analyse how many solutions the system

$$\begin{aligned} e_1(y_1, y_2) &:= H_1(0, y_1) - H_1(0, y_2) = 0, \\ e_2(y_1, y_2) &:= H_2(0, y_1) - H_2(0, y_2) = 0, \end{aligned} \tag{4.11}$$

can have. Define

$$E_1(y_1, y_2) := e_1(y_1, y_2)/(y_1 - y_2) \quad \text{and} \quad E_2(y_1, y_2) := e_2(y_1, y_2)/(y_1 - y_2).$$

Since we are interested in the solutions with $y_1 \neq y_2$, system (4.11) is equivalent to system $E_1(y_1, y_2) = E_2(y_1, y_2) = 0$, i.e.

$$\begin{aligned} a_2 + a_5(y_1 + y_2) + a_9(y_1^2 + y_1 y_2 + y_2^2) + a_{14}(y_1^3 + y_1^2 y_2 + y_1 y_2^2 + y_2^3) = 0, \\ b_2 + b_5(y_1 + y_2) + b_9(y_1^2 + y_1 y_2 + y_2^2) + b_{14}(y_1^3 + y_1^2 y_2 + y_1 y_2^2 + y_2^3) = 0. \end{aligned}$$

Notice that

$$\begin{aligned} E_{12}(y_1, y_2) &= b_{14} E_1(y_1, y_2) - a_{14} E_2(y_1, y_2) \\ &= (b_{14} a_5 - a_{14} b_5)(y_1 + y_2) + (b_{14} a_9 - a_{14} b_9)(y_1^2 + y_1 y_2 + y_2^2) \\ &\quad + b_{14} a_2 - a_{14} b_2, \end{aligned}$$

is a polynomial of degree two. By the Bézout Theorem the upper bound for the maximum number of solutions of system $E_1(y_1, y_2) = 0$ and $E_{12}(y_1, y_2) = 0$

is 6, whenever this system has finitely many solutions. Again note that by the symmetry of these polynomial equations, if (y_1, y_2) is a solution then (y_2, y_1) is also a solution, but these two solutions determine the same periodic orbit. This implies that the discontinuous piecewise differential system has at most three limit cycles. This upper bound is reached, see Example 2 of section 4.3. □

4.3 Examples

In this section we provide in example 1 a discontinuous piecewise differential system separated by the straight line $x = 0$ and formed by two Hamiltonians systems of degree 2 having one limit cycle. And in example 2 a discontinuous piecewise differential system separated by the straight line $x = 0$ and formed by two Hamiltonians systems of degree 3 having three limit cycles. Hence these two examples complete the proof of Theorem 34.

Example 1. Consider the following two Hamiltonians of degree three,

$$\begin{aligned} H_1(x,y) &= x^3 - y^3 - y^2 + y, \\ H_2(x,y) &= -x^3 - xy - 8y^3 - y^2 + \frac{7y}{2}. \end{aligned} \tag{4.12}$$

These Hamiltonians generate the Hamiltonian systems

$$\dot{x} = 1 - 2y - 3y^2, \quad \dot{y} = -3x^2, \tag{4.13}$$

$$\dot{x} = -x - 24y^2 - 2y + \frac{7}{2}, \quad \dot{y} = 3x^2 + y, \tag{4.14}$$

respectively. Of course $H_1(x,y)$ and $H_2(x,y)$ are first integrals of systems (4.13) and (4.14), respectively. For this discontinuous piecewise differential system the system (4.11) determines the system

$$\begin{aligned} E_1(y_1,y_2) &= 1 - y_1 - y_1^2 - y_2 - y_1y_2 - y_2^2 = 0, \\ E_2(y_1,y_2) &= \frac{1}{2}\left(7 - 2y_1 - 16y_1^2 - 2y_2 - 16y_1y_2 - 16y_2^2\right) = 0. \end{aligned} \tag{4.15}$$

System (4.15) has the unique real solution

$$(\bar{y}_1, \bar{y}_2) = \left(\frac{1}{28}\left(9 - \sqrt{37}\right), \frac{1}{28}\left(\sqrt{37} + 9\right)\right). \tag{4.16}$$

Then the two points of intersection with $x = 0$ of the limit cycle are $(0, \bar{y}_1)$ and $(0, \bar{y}_2)$. See this limit cycle in Figure 15.

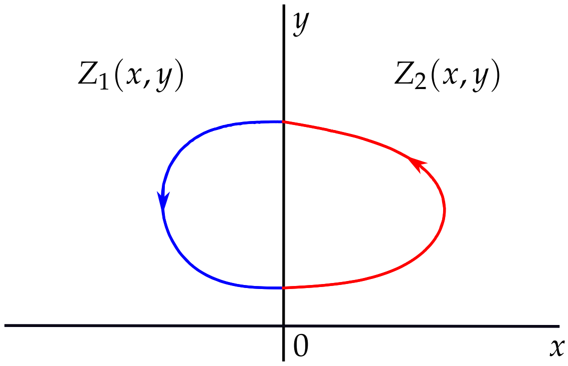


Figure 15 – The limit cycle of the discontinuous piecewise differential system generated by Hamiltonians (4.12) passing through the points $(0, \bar{y}_1)$, and $(0, \bar{y}_2)$, where $\bar{y}_1$, $\bar{y}_2$ are given in (4.16). $H_1(x,y)$ defines system in $x \leqslant 0$, and $H_2(x,y)$ defines system in $x \geqslant 0$.

Example 2. Consider the following two Hamiltonians of degree four

$$\begin{aligned} H_1(x,y) &= 2x^3y + 2x^2 - \frac{4xy}{3} + y^4 - 4y^3 + \frac{51y^2}{10} - \frac{19y}{10}, \\ H_2(x,y) &= 3x^4 + 2x^3 + xy^2 - 2xy + y^4 - \frac{31y^3}{12} + \frac{5y^2}{4} - \frac{y}{6}. \end{aligned} \tag{4.17}$$

These Hamiltonians generate the Hamiltonian systems

$$\dot{x} = 2x^3 - \frac{4x}{3} + 4y^3 - 12y^2 + \frac{51y}{5} - \frac{19}{10}, \quad \dot{y} = -6x^2y + 4x + \frac{4y}{3}, \tag{4.18}$$

$$\dot{x} = 2xy - 2x + 4y^3 - \frac{31y^2}{4} + \frac{5y}{2} - \frac{1}{6}, \quad \dot{y} = -12x^3 - 6x^2 - y^2 + 2y, \tag{4.19}$$

and $H_1(x,y)$ and $H_2(x,y)$ are first integrals for systems (4.18) and (4.19), respectively. For this discontinuous piecewise differential system, system (4.11) has only the following three real solutions

$$\begin{aligned} (\bar{y}_1^1, \bar{y}_2^1) &= (-0.206887, 2.01873), \\ (\bar{y}_1^2, \bar{y}_2^2) &= (0.141455, 0.393626), \\ (\bar{y}_1^3, \bar{y}_2^3) &= (1.41754, 1.67084). \end{aligned} \tag{4.20}$$

Then the two points of intersection with $x = 0$ of each limit cycle are $(0, \bar{y}_1^i)$ and $(0, \bar{y}_2^i)$ for $i \in \{1, 2, 3\}$, see these three limit cycles in Figure 16.

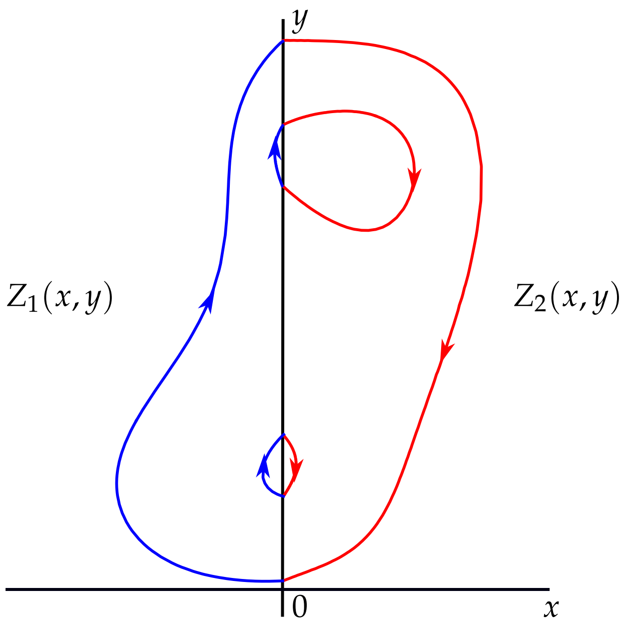


Figure 16 – The three limit cycles of the discontinuous piecewise differential system generated by Hamiltonians (4.17) passing through the points $(0, \bar{y}_i^i)$, $i \in \{1,2,3\}$ where $\bar{y}_i^i$ are given in (4.20).

5 Limit cycles of discontinuous piecewise differential Hamiltonian systems separated by a circle, or a parabola, or a hyperbola

5.1 Introduction and statement of the main results

The importance of Hamiltonian systems in dynamical systems lies in their ability to provide a comprehensive and insightful framework for studying the evolution of physical systems. From the conservation of energy to their applications in physics, engineering, and beyond, Hamiltonian systems continue to be a cornerstone in the exploration of dynamic behaviours in diverse scientific domains.

We are interested in studying the limit cycles of discontinuous piecewise smooth differential systems. A limit cycle of a system is a periodic orbit of that system for which there is no other periodic orbit in some sufficiently small neighbourhood containing it. In this work, we study the limit cycles for a class of discontinuous piecewise differential linear Hamiltonian systems separated by conics.

The discontinuous piecewise differential systems studied here in are particular Filippov systems, so the properties that we know for differential systems cannot be directly used. Therefore, to shed light on the study of these systems we introduce some important notions used in this work.

We make use of the Filippov's convention for discontinuous piecewise smooth vector fields defined on a open set $U \subset \mathbb{R}^2$. We also assume that the discontinuities appear on a differentiable submanifold Σ which can be given as $\Sigma = f^{-1}(0) \cap U$, being zero a regular value of the C^k function $f : U \to \mathbb{R}$ with

$k > 0$. Then the curve Σ splits the open set U in two open sets

$$\Sigma^+ := \{(x,y) \in U : f(x,y) \geqslant 0\} \text{ and } \Sigma^- := \{(x,y) \in U : f(x,y) \leqslant 0\},$$

In this chapter the piecewise smooth differential system is given by the vector fields

$$Z(x,y) = \begin{cases} Z_1(x,y), \text{ if } f(x,y) \leqslant 0, \\ Z_2(x,y), \text{ if } f(x,y) \geqslant 0. \end{cases} \tag{5.1}$$

There exist two types of limit cycles for piecewise smooth differential equations. Those of the first type are called sliding limit cycles and are such that some part of the cycle is contained in the sliding region. On the other hand, those of the second type are called crossing limit cycles and correspond to the ones that touch the discontinuity line only on points of the crossing region.

In this chapter we provide an upper bound for the maximum number of crossing limit cycles, simple limit cycles in what follows, for the discontinuous system (5.1), where Z_i for $i \in \{1,2\}$ are given by,

$$\dot{x} = \frac{\partial}{\partial y} H_i(x,y),$$
$$\dot{y} = -\frac{\partial}{\partial x} H_i(x,y).$$

and H_1, H_2 are Hamiltonians of degree two,

$$\begin{aligned} H_1(x,y) &= a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2, \\ H_2(x,y) &= b_0 + b_1 x + b_2 y + b_3 x^2 + b_4 xy + b_5 y^2. \end{aligned} \tag{5.2}$$

So the Hamiltonians vector fields Z_1 and Z_2 of the discontinuous piecewise differential systems (5.1) that we shall work with in this chapter are

$$\begin{aligned} Z_1(x,y) &= (a_2 + a_4 x + 2a_5 y, -a_1 - 2a_3 x - a_4 y), \\ Z_2(x,y) &= (b_2 + b_4 x + 2b_5 y, -b_1 - 2b_3 x - b_4 y). \end{aligned} \tag{5.3}$$

We consider three distinct discontinuous lines $f(x,y) = 0$, the circle, the parabola and the hyperbola, which uo to an affine change of variables can be written without loss of generality as

$$\begin{aligned} f(x,y) &= x^2 + y^2 - 1, \\ f(x,y) &= x^2 - y, \\ f(x,y) &= 1 - xy, \end{aligned}$$

respectively. See Figure 17.

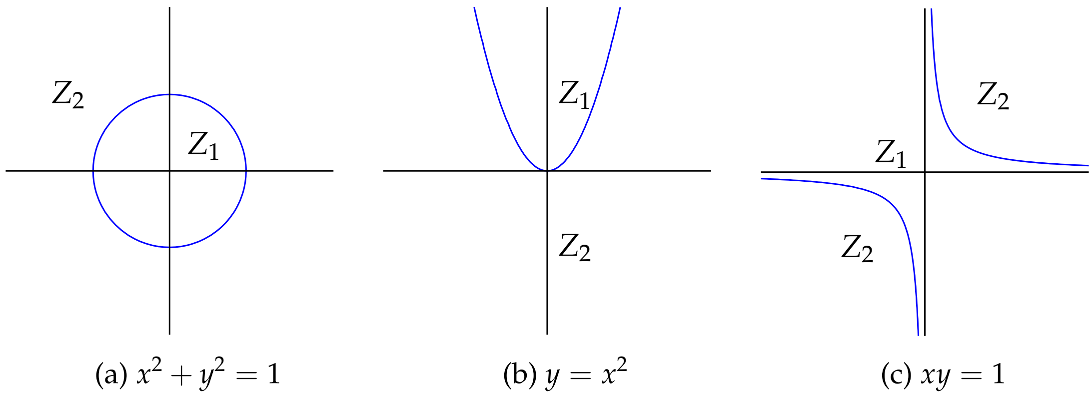


Figure 17 – Lines of discontinuities

Theorem 36. *The maximum number of limit cycles of the piecewise differential systems* (5.1) *with* Z_1 *and* Z_2 *given by* (5.3)*, intersecting the line of discontinuity* $f(x,y) = 0$ *in two points is*

(i) 3*, for* $f(x,y) = x^2 + y^2 - 1$*,*

(ii) 3*, for* $f(x,y) = x^2 - y$*,*

(iii) 2*, for* $f(x,y) = 1 - xy$*.*

These upper bounds are reached.

Theorem 36 is proved in Section 5.2.

5.2 Proof of Theorem 36

Proof of statement (i) of Theorem 36. Notice that $H_1(x,y)$ and $H_2(x,y)$ given in (5.2) are first integrals of the vectors fields $Z_1(x,y)$ and $Z_2(x,y)$ given in (5.3), respectively. If the discontinuous piecewise differential system (5.1) has a limit cycle which intersects the unit circle at the two points

$$p_1 = \left(\frac{2t_1}{t_1^2 + 1}, \frac{1 - t_1^2}{t_1^2 + 1} \right), \quad p_2 = \left(\frac{2t_2}{t_2^2 + 1}, \frac{1 - t_2^2}{t_2^2 + 1} \right),$$

with $t_1 \neq t_2$ then t_1 and t_2 must satisfy the equations

$$\begin{aligned} e_1 &= H_1(p_1) - H_1(p_2) = \frac{2(t_1 - t_2)}{(t_1^2+1)^2(t_2^2+1)^2} E_1 = 0, \\ e_2 &= H_2(p_1) - H_2(p_2) = \frac{2(t_1 - t_2)}{(t_1^2+1)^2(t_2^2+1)^2} E_2 = 0, \end{aligned} \tag{5.4}$$

where

$$\begin{aligned} E_1 = \; & a_1t_1^3t_2^3 + a_1t_1^3t_2 - a_1t_1^2t_2^2 - a_1t_1^2 + a_1t_1t_2^3 + a_1t_1t_2 - a_1t_2^2 - a_1 + a_2t_1^3t_2^2 + a_2t_1^3 \\ & + a_2t_1^2t_2^3 + a_2t_1^2t_2 + a_2t_1t_2^2 + a_2t_1 + a_2t_2^3 + a_2t_2 + 2a_3t_1^3t_2^2 + 2a_3t_1^2t_2^3 - 2a_3t_1 \\ & - 2a_3t_2 - a_4t_1^3t_2^3 + a_4t_1^3t_2 + 3a_4t_1^2t_2^2 + a_4t_1^2 + a_4t_1t_2^3 + 3a_4t_1t_2 + a_4t_2^2 - a_4 \\ & - 2a_5t_1^3t_2^2 - 2a_5t_1^2t_2^3 + 2a_5t_1 + 2a_5t_2 \end{aligned}$$

and E_2 has the same expression of E_1 replacing $(a_1, a_2, a_3, a_4, a_5)$ by $(b_1, b_2, b_3, b_4, b_5)$.

Note that if (s_1, s_2) is also a solution of (5.4), then (s_2, s_1) is a solution, and we call this property the symmetric property. Of course both solutions give rise to the same possible limit cycle.

On the other hand, we know that if (s_1, s_2) is a solution of $E_1 = E_2 = 0$ then s_1 must be a root of the resultant polynomial $R(t_1)$ of E_1 and E_2 with respect to the variable t_2. Similarly s_2 must be a root of the resultant polynomial $R(t_2)$ of E_1 and E_2 with respect to the variable t_1. These resultants $R(t_1)$ and $R(t_2)$ are equal to each other by replacing t_1 by t_2. We have that for

$$R(t_1) = -4\left(t_1^2+1\right)^6 \tilde{R}(t_1),$$

the real solutions of $R(t_1)$ are the real zeros of $\tilde{R}(t_1)$, and the degree of the polynomial $\tilde{R}(t_1)$ is six. By the symmetric property, the discontinuous piecewise differential system (5.1) can have at most three limit limit cycles.

The expression of $\tilde{R}(t_1)$ is huge, so we chose not to write it here, but it can be computed easily using same algebraic manipulators. □

Proof of statement (ii) of Theorem 36. Notice that $H_1(x, y)$ and $H_2(x, y)$ given in (5.2) are first integrals of the vector fields $Z_1(x, y)$ and $Z_2(x, y)$ given by (5.3), respectively. If the discontinuous piecewise differential systems (5.1) has a limit cycle which intersects the parabola at the two points

$$p_1 = \left(t_1, t_1^2\right), \quad p_2 = \left(t_2, t_2^2\right),$$

with $t_1 \neq t_2$, then these two points must satisfy the equations

$$\begin{aligned} e_1 &= H_1(p_1) - H_1(p_2) = (t_1 - t_2)E_1 = 0, \\ e_2 &= H_2(p_1) - H_2(p_2) = (t_1 - t_2)E_2 = 0, \end{aligned} \tag{5.5}$$

where $E_1 = a_1 + t_2(a_2 + a_3 + t_1(a_4 + a_5 t_1)) + t_1(a_2 + a_3 + t_1(a_4 + a_5 t_1)) + t_2^2(a_4 + a_5 t_1) + a_5 t_2^3$ and E_2 has the same expression of E_1 replacing $(a_1, a_2, a_3, a_4, a_5)$ by $(b_1, b_2, b_3, b_4, b_5)$. Note that if (s_1, s_2) is a solution of system (5.5) then (s_2, s_1) is a solution as well, by the symmetric property. Then both give rise to the same possible limit cycle.

On the other hand, we know that if (s_1, s_2) is a solution of $E_1 = E_2 = 0$ then s_1 must be a root of the resultant polynomial $R(t_1)$ of E_1 and E_2 with respect to the variable t_2. Similarly s_2 must be a root of the resultant polynomial $R(t_2)$ of E_1 and E_2 with respect to the variable t_1. These resultants $R(t_1)$ and $R(t_2)$ are equal to each other by replacing t_1 by t_2.

By the symmetric property the discontinuous piecewise differential systems (5.1) can have at most three limit limit cycles.

Once again, expression of $\tilde{R}(t_1)$ is huge, so we do not write it here, but it can be computed easily using algebraic manipulators. □

Proof of statement (iii) of Theorem 36. Suppose that there exists three limit cycles for the discontinuity piecewise differential system generated by the Hamiltonians (5.2). At least two of these limit cycles intersect the same branch of the hyperbola. Assume without loss of generality that these two limit cycles intersect the branch of the hyperbola contained in the first quadrant of the plane $\mathbb{R}^2$. Consider that the points of intersection of these limit cycles with the hyperbola are given by the points

$$p_1 = \left(t_1, \frac{1}{t_1}\right), \quad p_2 = \left(t_2, \frac{1}{t_2}\right), \quad p_3 = \left(t_3, \frac{1}{t_3}\right) \quad \text{and} \quad p_4 = \left(t_4, \frac{1}{t_4}\right).$$

Without loss of generality, assume that $0 < t_1 < t_2 < t_3 < t_4$. So one of the limit cycles passes through p_1, p_4, and the other limit cycle passes through points p_2 and p_3. These points must satisfy the equations

$$
\begin{aligned}
&H_1(p_1) - H_1(p_4) = 0 \Leftrightarrow a_5(t_1 + t_4) - t_1t_4(t_1t_4(a_1 + a_3(t_1 + t_4)) - a_2) = 0,\\
&H_2(p_1) - H_2(p_4) = 0 \Leftrightarrow t_1^2t_4^2b_1 - t_1t_4b_2 + t_1^3t_4^2b_3 + t_1^2t_4^3b_3 - t_1b_5 - t_4b_5 = 0,\\
&H_1(p_2) - H_1(p_3) = 0 \Leftrightarrow a_5(t_2 + t_3) - t_2t_3(t_2t_3(a_1 + a_3(t_2 + t_3)) - a_2) = 0,\\
&H_2(p_2) - H_2(p_3) = 0 \Leftrightarrow t_2^2t_3^2b_1 - t_2t_3b_2 + t_2^3t_3^2b_3 + t_2^2t_3^3b_3 - t_2b_5 - t_3b_5 = 0,
\end{aligned} \tag{5.6}
$$

Solving system (5.6) in terms of the coefficients of the polynomials $H_1(x, y)$ and $H_2(x, y)$ we obtain

$$
\begin{aligned}
a_3 &= \frac{t_1t_2t_3(a_2 - a_1t_2t_3) - (a_1t_2^2t_3^2 - a_2t_2t_3 + a_2t_1(t_2 + t_3))t_4 + a_1t_1^2(t_2 + t_3)t_4^2}{(t_2 + t_3)(t_1 + t_4)(t_2^2t_3^2 - t_1^2t_4^2)},\\
a_5 &= \frac{t_1t_2t_3t_4(a_1t_1t_2t_3(t_2 - t_1 + t_3 - t_4)t_4 + a_2(-t_2^2t_3 - t_2t_3^2 + t_1t_4(t_1 + t_4)))}{(t_2 + t_3)(t_1 + t_4)(t_2^2t_3^2 - t_1^2t_4^2)},\\
b_1 &= \frac{b_5(t_1t_2(t_3 - t_4) - t_1t_3t_4 + t_2t_3t_4) + b_3t_1t_2t_3t_4(t_2^2t_3 + t_2t_3^2 - t_1t_4(t_1 + t_4))}{t_1t_2t_3t_4(-t_2t_3 + t_1t_4)},\\
b_2 &= \frac{b_3t_1^2t_2^2t_3^2(t_2 - t_1 + t_3 - t_4)t_4^2 + b_5(t_1t_2^2t_3^2 + t_2^2t_3^2t_4 - t_1^2(t_2 + t_3)t_4^2)}{t_1t_2t_3t_4(-t_2t_3 + t_1t_4)}.
\end{aligned}
$$

Replacing a_3, a_5, b_1, b_2 in $H_1(x, y)$ we obtain the following expression for $H_1(x, y)$

$$
\begin{aligned}
&\frac{(t_1t_2t_3(a_2 - a_1t_2t_3) - (a_1t_2^2t_3^2 - a_2t_2t_3 + a_2t_1(t_2 + t_3))t_4 + a_1t_1^2(t_2 + t_3)t_4^2)x^2}{(t_2 + t_3)(t_1 + t_4)(t_2t_3 - t_1t_4)(t_2t_3 + t_1t_4)}\\
&+\frac{t_1t_2t_3t_4(a_1t_1t_2t_3(t_2 - t_1 + t_3 - t_4)t_4 - a_2t_2t_3(t_2 + t_3) + a_2t_1t_4(t_1 + t_4))y^2}{(t_2 + t_3)(t_1 + t_4)(t_2t_3 - t_1t_4)(t_2t_3 + t_1t_4)}\\
&+a_1x + a_2y + a_4xy,
\end{aligned} \tag{5.7}
$$

and for $H_2(x, y)$

$$
\begin{aligned}
&\frac{(b_5(t_1t_2(t_3 - t_4) - t_1t_3t_4 + t_2t_3t_4) + b_3t_1t_2t_3t_4(t_2^2t_3 + t_2t_3^2 - t_1t_4(t_1 + t_4)))x}{t_1t_2t_3t_4(-t_2t_3 + t_1t_4)}\\
&+\frac{(b_3t_1^2t_2^2t_3^2(-t_1 + t_2 + t_3 - t_4)t_4^2 + b_5(t_1t_2^2t_3^2 + t_2^2t_3^2t_4 - t_1^2(t_2 + t_3)t_4^2))y}{t_1t_2t_3t_4(-t_2t_3 + t_1t_4)}\\
&+b_4xy + b_5y^2 + b_3x^2.
\end{aligned} \tag{5.8}
$$

The possible limit cycles intersects the hyperbola $xy = 1$ at the points $(x, 1/x)$ and $(X, 1/X)$, their coordinates x, X must satisfy the system

$$e_1(x, X) := H_1(x, 1/x) - H_1(X, 1/X) = 0,$$
$$e_2(x, X) := H_2(x, 1/x) - H_2(X, 1/X) = 0,$$

where H_1 and H_2 are given by (5.7) and (5.8). This system has three solutions of the form (x, X)

$$(t_1, t_4), \quad (t_2, t_3), \quad (\alpha, \beta).$$

Therefore if the three solutions gave rise to a limit cycle, these limit cycles must pass through the points

$$p_1 = \left(t_1, \frac{1}{t_1}\right) \text{ and } p_4 = \left(t_4, \frac{1}{t_4}\right),$$
$$p_2 = \left(t_2, \frac{1}{t_2}\right) \text{ and } p_3 = \left(t_3, \frac{1}{t_3}\right),$$
$$\boldsymbol{\alpha} = \left(\alpha, \frac{1}{\alpha}\right) \text{ and } \boldsymbol{\beta} = \left(\beta, \frac{1}{\beta}\right).$$

Here

$$\alpha = \frac{A - \sqrt{B}}{C}, \qquad \beta = \frac{A + \sqrt{B}}{C},$$

where

$$\begin{aligned} A &= (t_2 + t_3 - t_1 - t_4)(t_2t_3 + t_1t_4)(t_2t_3(t_1 + t_4) - t_1t_4(t_2 + t_3)), \\ B &= (t_1 + t_4 - t_2 - t_3)(t_2t_3(t_1 + t_4) - t_1t_4(t_2 + t_3))\big(4t_1t_2t_3t_4(t_2t_3 - t_1t_4)^2 \\ &\quad + (t_1 + t_4 - t_2 - t_3)(t_2t_3 + t_1t_4)^2(t_2t_3(t_1 + t_4) - t_1t_4(t_2 + t_3))\big), \\ C &= 2(t_2t_3 - t_1t_4)(t_2t_3(t_1 + t_4) - t_1t_4(t_2 + t_3)). \end{aligned}$$

As we are assuming that $0 < t_1 < t_2 < t_3 < t_4$, then the possible distribution for the coordinates $t_1, t_2, t_3, t_4, \alpha$ and β are

$$\begin{aligned} &0 < \alpha < t_1 < t_2 < t_3 < t_4 < \beta, \\ &0 < \beta < t_1 < t_2 < t_3 < t_4 < \alpha, \\ &0 < t_1 < \alpha < t_2 < t_3 < \beta < t_4, \\ &0 < t_1 < \beta < t_2 < t_3 < \alpha < t_4, \\ &0 < t_1 < t_2 < \alpha < \beta < t_3 < t_4, \\ &0 < t_1 < t_2 < \beta < \alpha < t_3 < t_4, \\ &\alpha < \beta < 0 < t_1 < t_2 < t_3 < t_4, \\ &\beta < \alpha < 0 < t_1 < t_2 < t_3 < t_4. \end{aligned} \tag{5.9}$$

In the first six cases of (5.9) we have that

$$\alpha+\beta = \frac{(t_1+t_4-t_2-t_3)(t_2t_3+t_1t_4)}{t_1t_4-t_2t_3} > 0,$$
$$\alpha\beta = \frac{t_1t_2t_3t_4(t_2+t_3-t_1-t_4)}{t_2t_3(t_1+t_4)-t_1t_4(t_2+t_3)} > 0.$$

Therefore $B>0$ and $C<0$. Since $\beta>0$ and $C<0$ it follows that $A+\sqrt{B}<0$, or equivalently $0<\sqrt{B}<-A$. Hence $B<A^2$ and $B-A^2<0$. But from $\alpha\beta>0$ we obtain the contradiction

$$B-A^2 = 4t_1t_2t_3t_4(t_2+t_3-t_1-t_4)^2(t_1+t_4-t_2-t_3)(t_2t_3(t_1+t_4)-t_1t_4(t_2+t_3)) > 0.$$

In summary, the first six cases of (5.9) cannot occur. Assume that the two last cases of (5.9) hold. Then we have that

$$\alpha+\beta = \frac{(t_1+t_4-t_2-t_3)(t_2t_3+t_1t_4)}{t_1t_4-t_2t_3} < 0,$$
$$\alpha\beta = \frac{t_1t_2t_3t_4(t_2+t_3-t_1-t_4)}{t_2t_3(t_1+t_4)-t_1t_4(t_2+t_3)} > 0.$$

Therefore

$$\frac{t_1+t_4-t_2-t_3}{t_1t_4-t_2t_3} < 0 \text{ and } \frac{t_2+t_3-t_1-t_4}{t_2t_3(t_1+t_4)-t_1t_4(t_2+t_3)} > 0. \qquad (5.10)$$

Assume that $t_1+t_4-t_2-t_3<0$, then from the second inequality of (5.10) we get that $t_2t_3(t_1+t_4)-t_1t_4(t_2+t_3)>0$. But from the first inequality of (5.10) we obtain that $t_1t_4>t_2t_3$ and since $t_2+t_3>t_1+t_4$, we get that $t_1t_4(t_2+t_3)>t_2t_3(t_1+t_4)$ in contradiction with $t_2t_3(t_1+t_4)-t_1t_4(t_2+t_3)>0$. Therefore the last two cases of (5.9) cannot occur. In summary, the maximum number of limit cycles for the discontinuous piecewise differential system (5.1) with the line of discontinuity $xy=1$ is two. □

5.3 Examples

To prove that the upper bound is reached we show some examples for Theorem 36.

Example 1. Consider the following two Hamiltonians of degree two, separated by the line of discontinuity $x^2 + y^2 = 1$

$$\begin{aligned} H_1(x,y) &= x^2 - \frac{137xy}{30} - \frac{211x}{100} - \frac{11y^2}{10} + \frac{421y}{100}, \\ H_2(x,y) &= x^2 - \frac{275xy}{303} - \frac{55x}{101} + y^2 + \frac{55y}{101}. \end{aligned} \tag{5.11}$$

These Hamiltonians generate the Hamiltonian vector fields

$$Z_1(x,y) = \left(-\frac{137x}{30} - \frac{11y}{5} + \frac{421}{100}, -2x + \frac{137y}{30} + \frac{211}{100}\right), \tag{5.12}$$

$$Z_2(x,y) = \left(-\frac{275x}{303} + 2y + \frac{55}{101}, -2x + \frac{275y}{303} + \frac{55}{101}\right), \tag{5.13}$$

respectively. Of course $H_1(x,y)$ and $H_2(x,y)$ are first integrals of the vector fields (5.12) and (5.13), respectively. Notice that the limit cycles intersecting the circle at the points (x,y) and (X,Y) must satisfy the system

$$\begin{aligned} &H_1(x,y) = H_1(X,Y), \\ &H_2(x,y) = H_2(X,Y), \\ &x^2 + y^2 = 1, \\ &X^2 + Y^2 = 1. \end{aligned} \tag{5.14}$$

System (5.14) has the three pairs of solutions formed by the points

$$\begin{aligned} &p_1 = \left(-\frac{4}{5}, -\frac{3}{5}\right), \quad P_1 = \left(\frac{3}{5}, -\frac{4}{5}\right), \\ &p_2 = \left(\frac{4}{5}, \frac{3}{5}\right), \quad P_2 = (1,0), \\ &p_3 = (-1,0), \quad P_3 = (0,1). \end{aligned} \tag{5.15}$$

Then the two points of intersection for each limit cycle are p_i and P_i for $i \in \{1,2,3\}$ as we can see in Figure 18.

Example 2. This example provided three limits cycles for the discontinuous piecewise differential system

$$Z(x,y) = \begin{cases} Z_1(x,y), \text{ if } y \leqslant x^2, \\ Z_2(x,y), \text{ if } y \geqslant x^2, \end{cases} \tag{5.16}$$

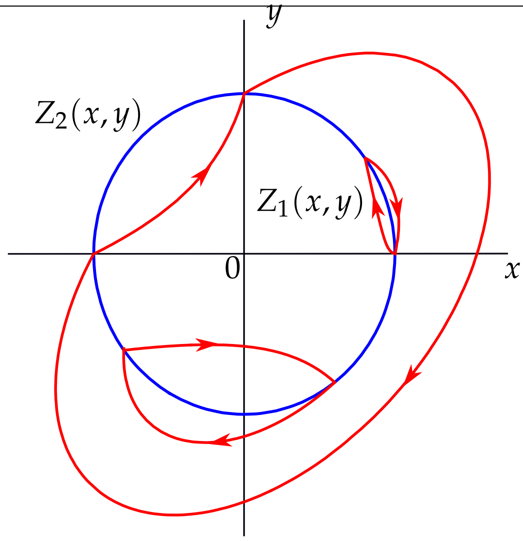


Figure 18 – The three limit cycles of the discontinuous piecewise differential system generated by Hamiltonians (5.11) passing through the points p_i and P_i, $i \in \{1,2,3\}$ given in (5.15).

where

$$Z_1(x,y) = \left(-\frac{577x}{15637} + \frac{103y}{15637} - 1, -\frac{30931x}{15637} + \frac{577y}{15637} - 1\right), \tag{5.17}$$

$$Z_2(x,y) = \left(\frac{103y}{15637} - \frac{577x}{15637}, \frac{343x}{15637} + \frac{577y}{15637} - 1\right). \tag{5.18}$$

The Hamiltonians vector fields Z_1 and Z_2 have, respectively, the Hamiltonians

$$\begin{aligned} H_1(x,y) &= \frac{30931x^2}{31274} - \frac{577xy}{15637} + x + \frac{103y^2}{31274} - y, \\ H_2(x,y) &= -\frac{343x^2}{31274} - \frac{577xy}{15637} + x + \frac{103y^2}{31274}. \end{aligned} \tag{5.19}$$

Of course $H_1(x,y)$ and $H_2(x,y)$ are first integrals of systems (5.17) and (5.18), respectively. Notice that the limit cycles passing through the points (x,y) and (X,Y) of the parabola must satisfy the system

$$\begin{aligned} &H_1(x,y) = H_1(X,Y), \\ &H_2(x,y) = H_2(X,Y), \\ &x^2 = y, \\ &X^2 = Y. \end{aligned} \tag{5.20}$$

System (5.20) has the three pair of solutions formed by the points

$$
\begin{aligned}
&p_1 = (2,4)\,, \quad P_1 = \left(\frac{1}{206}\left(227-\sqrt{1628665}\right), \frac{840097-227\sqrt{1628665}}{21218}\right),\\
&p_2 = (3,9)\,, \quad P_2 = \left(\frac{1}{103}\left(165-\sqrt{493609}\right), \frac{2\left(260417-165\sqrt{493609}\right)}{10609}\right),\\
&p_3 = (-1,1), \quad P_3 = (-4,16).
\end{aligned}
\tag{5.21}
$$

Then the two points of intersection for each limit cycle are p_i and P_i for $i \in \{1,2,3\}$ as we can see in Figure 12.

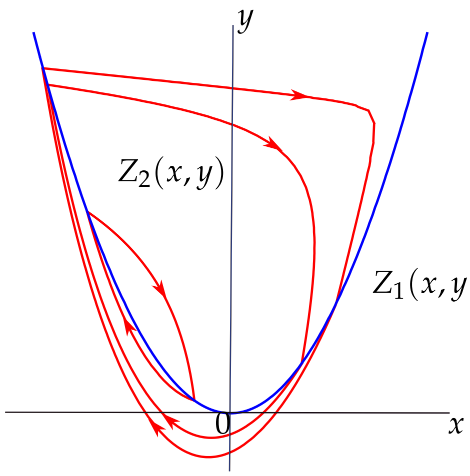


Figure 19 – The three limit cycles of the discontinuous piecewise differential system generated by Hamiltonians (5.19) passing through the points p_i and P_i, $i \in \{1,2,3\}$ given in (5.21).

Example 3. Consider the following two Hamiltonians by degree two,

$$
\begin{aligned}
H_1(x,y) &= \left(x-\frac{5}{4}\right)^2 + \left(y-\frac{4}{5}\right)^2,\\
H_2(x,y) &= -\left(-x^2+1.25507x-0.335138y^2-1.40792y\right).
\end{aligned}
\tag{5.22}
$$

These Hamiltonians generate the Hamiltonian systems

$$Z_1(x,y) = \left(2\left(y-\frac{4}{5}\right), -2\left(x-\frac{5}{4}\right)\right), \tag{5.23}$$

$$Z_2(x,y) = (0.670277y + 1.40792, 1.25507 - 2x)\,, \tag{5.24}$$

respectively. Of course $H_1(x, y)$ and $H_2(x, y)$ are first integrals of the vector fields (5.23) and (5.24), respectively. Notice that the limit cycles passing through the points (x, y) and (X, Y) of the hyperbola must satisfy the system

$$\begin{aligned} &H_1(x, y) = H_1(X, Y), \\ &H_2(x, y) = H_2(X, Y), \\ &xy = 1, \\ &XY = 1. \end{aligned} \tag{5.25}$$

System (5.25) has the two pairs of solutions formed of the points

$$\begin{aligned} p_1 &= (0.9923, 1.007), \quad P_1 = (1.546, 0.6467), \\ p_2 &= (1.053, 0.9517), \quad P_2 = (1.47, 0.6803). \end{aligned} \tag{5.26}$$

Then the two points of intersection for each limit cycle are p_i and P_i for $i \in \{1, 2\}$ as we can see in Figure 20.

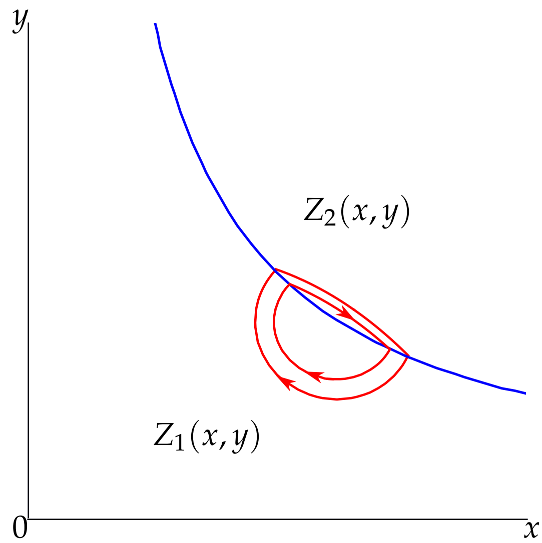


Figure 20 – The two limit cycles of the discontinuous piecewise differential system generated by Hamiltonians (5.22) passing through the points p_i and P_i, $i \in \{1, 2\}$ given in (5.26).

6 Final remarks

This thesis addressed two fundamental themes, each contributing significantly to the understanding and analysis of dynamic systems. Firstly, we dedicated ourselves to the study of singularity indices in vector fields, introducing an innovative index that generalizes the Poincaré index of singularities in continuous vector fields extending its applicability to Filippov vector fields. It is important to note that the properties of the Poincaré index remain valid for this generalization, preserving the values of -1 for saddles, 1 for nodes and foci, and 0 for regular points. This result extends analogously to pseudo-singularities, demonstrating that the characteristics of singularities in continuous fields are shared by pseudo-singularities in Filippov systems specifically, the pseudo-saddle exhibits an index of -1, while the pseudo-node has an index of 1, and regular points in Filippov systems maintain an index of 0.

Besides the contributions presented in this thesis, an interesting avenue for future work would be a deeper investigation into the index of periodic orbits in Filippov vector fields. The aim would be to demonstrate that the index is equals to 1, as in the smooth case. To do so, the idea is to consider the index, similar to Definition 3, as follows:

$$I_\gamma(Z) := \frac{1}{2\pi}\left(J(Z) + \int_{\gamma^+}\omega_W + \int_{\gamma^-}\omega_W\right),$$

where γ is the parametrization of the periodic solution and $\gamma^\pm$ represents the intersection with the $F^\pm$ field. It would follow that the sum of the indices of Filippov singularities inside γ will also be one.

Additionally, we devoted significant attention to investigating the existence of limit cycles in specific vector fields and determining the maximum number of possible limit cycles in these contexts. Our analysis encompassed a variety of vector fields, starting with a case where the vector field is defined by parts but remains continuous. In this configuration, the transition region of the vector field is delineated by a circle, within which we have a distinct vector field compared to the field outside the circle. We explored scenarios where one of the fields has a linear center, and the other has a quadratic center. In this context, we were unable to demonstrate the attainment of the upper bound of three limit cycles, leaving it open to finding an example for this case. Another challenge arose when attempting to consider a general affine transformation in the vector field with a quadratic center. Due to the substantial increase in parameters, specifically six, we encountered limitations in the calculations required for this specific case.

Subsequently, we turned our attention to Filippov vector fields, where the region of discontinuity took various forms, such as a circle, line, parabola, and hyperbola. Considering Hamiltonian vector fields, we determined the maximum number of limit cycles in each scenario, presenting concrete examples that achieve these maximum bounds.

In summary, this thesis represents a substantial contribution to understanding singularity indices in vector fields and exploring the presence of limit cycles in specific configurations. The results obtained, both in the generalization of Poincaré-Hopf indices and the analysis of limit cycles, provide a solid foundation for future research in this domain, offering valuable insights for the understanding and enhancement of fundamental properties of these dynamic systems. I extend my gratitude to all those involved in this academic process, and I hope this work contributes to the continuous advancement of the theory of dynamic systems.

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