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Limit cycles in Generalized Liénard Non-smooth Differential Equations

Ciclos limites em Equações Diferenciais Não-suaves Generalizadas de Liénard

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"Somos seres únicos, a diferença e a diversidade
entre os indivíduos são condições essenciais da natureza humana.
Hoje eu sei que sou uma pessoa melhor, mais
completa, mais corajosa, mais ousada e infinitamente mais livre."
(João W. Nery)

Resumo

O foco dessa dissertação é o estudo de bifurcações de ciclos limites em perturbações descontínuas de centros lineares. A obtenção e demonstração desses resultados baseia-se nos métodos desenvolvidos pela Teoria da Média — ou Teoria *Averaging* — mais especificamente no método de *averaging* periódico de primeira ordem, razão pela qual foi necessário dedicar parte de nossos estudos ao entendimento dessa teoria. No entanto, para que isso pudesse ser atingido, foi necessário antes criar um embasamento de teoria de sistemas dinâmicos suaves e não-suaves, que será apresentado brevemente nos dois primeiros capítulos, como requisito para compreensão do que será desenvolvido nos dos dois últimos capítulos.

Palavras-chave: sistemas dinâmicos. sistemas de Filippov. sistemas de Liénard. ciclos limites. teoria da média. averaging periódico.

Abstract

The focus of this dissertation is the study of bifurcations of limit cycles on discontinuous perturbations of linear centers. To get to these results and prove them, we based ourselves on the methods developed in the Averaging Theory — more specifically on the first order periodic averaging — which is why it was necessary to dedicate a part of our study on understanding this theory. However, for this to be achieved, it was first necessary to build a foundation on the theory of smooth and non-smooth dynamical systems, which will be briefly presented in the first two chapters as a requirement for understanding what will be developed in the last two chapters.

Keywords: dynamical systems. Filippov systems. Liénard systems. limit cycles. averaging theory. periodic averaging.

List of symbols

$\dot{x}$	Derivative of x with respect to a time variable
$\frac{\partial f}{\partial x_j}$	Partial derivative of f with respect to x_j
∇f	Gradient of function f
$V \otimes_{\mathbb{R}} W$	Tensor product of the real vector space V with the vector space W
$V \oplus W$	Direct sum of vector spaces
$span\{\beta\}$	A vector space spanned by the basis β
$\mathcal{L}(V)$	Vector space of linear operators in V

Contents

Introduction

In 1900, the German mathematician David Hilbert proposed a list of mathematical problems for which there was no solution at the time, they became known as the Hilbert's Problems. The 16th problem, which is actually two similar problems from different branches, is one of those that still remain unsolved; what we are interested is on the second part of it: given a certain $n \in \mathbb{N}$, find an upper bound for the number of limit cycles in the system of differential equations

$$\dot{x} = F(x, y),$$
$$\dot{y} = G(x, y),$$

where F and G are polynomials of degree n on x and y. This problem still drive the attention of many mathematicians to the task of determining the number of limit cycles in dynamical systems. Besides that, there are many applications of studying the properties of limit cycles to real life problems, such as in branches of engineering, physics, biology, and many others — see, for instance, references (MAKARENKOV; LAMB, 2012) and (BERNARDO et al., 2008). More recently, the existence and properties of limit cycles in non-smooth dynamical systems, in particular in piecewise smooth cases, have also been extensively studied.

In our study, we mostly restrain Hilbert's 16th problem to a family of differential equations known as the generalized polynomial Liénard differential equations, which are presented in the form

$$\ddot{x} + f(x)\dot{x} + g(x) = 0 \tag{1}$$

where $f(x)$ and $g(x)$ are polynomials of degrees m and n and the dot denotes the derivative with respect to the time t. As we will point out further in this work, specially in Chapter 4, the work of many contemporaneous mathematicians gave rise to various results on the number of limit cycles for different derivations from system (1).

This dissertation is organized as it follows. The first two chapters are dedicated to introducing the foundations of smooth and non-smooth dynamical systems. In them, we focused on outlining definitions and theorems that play an important role on the understanding of the topics that will further be discussed. The general purpose of these chapters is not to provide complete proofs of the theorems, but presenting insights from the proofs, as well as examples that we believe would benefit the understanding of the ideas behind the mathematical objects with which we are dealing and the path we are tracing towards the main goal of our work. However, within the text we point out some references — such as (PERKO, 2012), (HIRSCH; SMALE, 1974), (GUARDIA; SEARA;

TEIXEIRA, 2011) and (FILIPPOV, 1988) — where the rigorous reader can find a more satisfying proof for some theorems.

The third chapter is a more technical one, it is all dedicated to build a complete proof for two theorems: the *First Order Periodic Averaging for Smooth Systems* and the *First Order Periodic Averaging for Non-smooth Systems*. The proofs of these theorems are therein discussed with detail, relying on renowned works such as (SANDERS; VERHULST; MURDOCK, 2007) and (VERHULST, 1990) — for the first theorem — and (LLIBRE; NOVAES; TEIXEIRA, 2015) — for the latter.

The fourth chapter is why we are here for and what gives the title to this dissertation. In this, we presented some of the most recent results obtained by applying the *Averaging Method* to the context of finding lower bounds to the maximum number of limit cycles in various types of systems — in particular, in Liénard-like perturbations. Specifically in the last section, we summarize our study of piecewise smooth Liénard-like systems and generalize the following result of (MARTINS; MEREU, 2014): consider the system

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= -x - \varepsilon(f(x)\cdot y + \operatorname{sgn}(y)(k_1x + k_2)),\end{aligned}$$

where f is a polynomial of degree $n \in \mathbb{N}$ and $k_1, k_2 \in \mathbb{R}$; then, for every $n \geqslant 1$ and $|\varepsilon|$ sufficiently small, the maximum number of limit cycles bifurcating from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$ is $[n/2] + 1$. Moreover, the parameters of the polynomials can be chosen such that this number is actually achieved.

1 Smooth Dynamical Systems

In this chapter, we briefly present some fundamentals of smooth dynamical systems, which will give us a basis to our further study of discontinuous dynamical systems. We start with some thoughts on linear systems, then we head to nonlinear systems, stability and bifurcations. Some results will be stated without further proof, for a more detailed approach we recommend checking out the references (PERKO, 2012) and (HIRSCH; SMALE, 1974).

1.1 Linear Systems

Consider the linear system of ordinary differential equations:

$$\dot{x} = Ax \tag{1.1}$$

where $x \in \mathbb{R}^n$, $\dot{x} = \dfrac{dx}{dt}$ and A is an $n \times n$ matrix.

The Fundamental Theorem for Linear Systems (PERKO, 2012) states that systems such as (1.1), together with an initial condition $x(0) = x_0$, have a unique solution given by

$$x(t) = e^{At}x_0,$$

where e^{At} denotes the matrix exponential, which is defined by applying the Taylor expansion series of e^x to A, i.e.:

$$I_n + At + \frac{A^2t^2}{2!} + \ldots = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} = e^{At}.$$

It is shown that this series converges to an $n \times n$ matrix, so e^{At} is well defined.

The solutions of (1.1) for different initial points describe the trajectories of the points as the time varies; the set of all these solutions form the *phase portrait* of the system, which is an important geometrical tool in our study.

If B is a matrix similar to A – i.e. $B = P^{-1}AP$ for some matrix P – then we can obtain solutions of the system (1.1) by applying the change of variables $x = Py$ to a solution $y(t)$ of the system

$$\dot{y} = By. \tag{1.2}$$

It follows that the phase portrait of (1.1) can be obtained from the phase portrait of (1.2) under a linear transformation, in other words, they're said to be *linearly equivalent*. Furthermore, it's reasonable to infer that the Jordan canonical form of a matrix can give us some insight into the form of the solutions of such systems.

Example 1. Let $A \in \mathcal{M}_2(\mathbb{R})$, we wish to analyze the possible forms of the phase portrait of system (1.1). In order to do that, we consider the nature of the matrix's eigenvalues, denoted by μ and λ, and how this reflects on the Jordan canonical form of A.

- **Case 1**: $\mu, \lambda \in \mathbb{R}$. In this case, the Jordan form of A exists and can either be $J = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ or $J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$; we then split this case into the following:
  - *Case 1.1*: $\lambda < 0 < \mu$. The phase portrait of the system $\dot{x} = Jx$, with $J = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, is a *saddle* at the origin, so the original phase portrait will be a saddle as well.

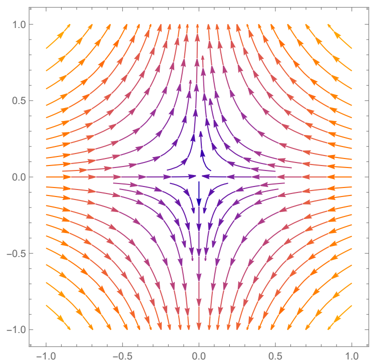


Figure 1 – Saddle at the origin

  - *Case 1.2*: $\lambda < \mu < 0$. The phase portrait of the system (1.2) will be a *stable node* as in Figure 2. If the eigenvalues are rather positive, the arrows will be reversed, giving rise to a *unstable node*.

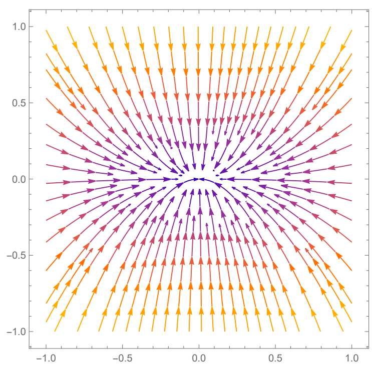


Figure 2 – Stable node at the origin

  - *Case 1.3*: $\lambda = \mu < 0$. This gives us two cases based on the possible dimensions of the eigenspace. If the eigenspace is 2-dimensional - i.e. there are two linearly independents eigenvectors - then the phase portrait of the system will appear as in Figure 3. Conversely, if the eigenspace has dimension one, the phase portrait is a stable degenerated node, as given by Figure 4. In both cases, like the above, changing the signal of the eigenvalue reverse the arrows, resulting on a *unstable (proper or degenerated) node*.

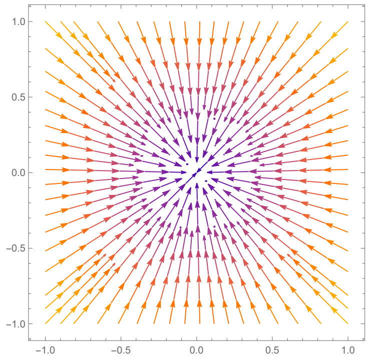


Figure 3 – Proper stable node

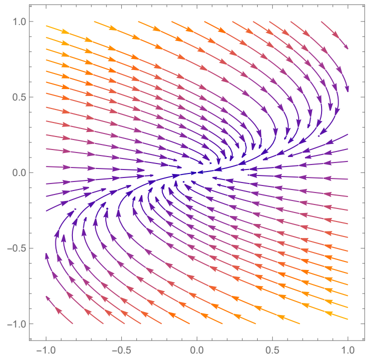


Figure 4 – Degenerated stable node

- **Case 2**: $\mu, \lambda \notin \mathbb{R}$. In this case, μ and λ are complex conjugates. We write $\lambda = a + ib$ and $\mu = a - ib$, $b \neq 0$, then A is similar to a matrix of the form $B = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$; this gives us two possibilities:
  - *Case 2.1*: $a \neq 0$. The origin in the equivalent system is said to be a *focus*, which can be stable or unstable according to the sign of a. If $a < 0$, the orbits will spiral into the origin, which is called a *stable focus* and is represented on Figure 6. Otherwise, if $a > 0$, the trajectories will spiral away from the origin, hence we have an *unstable focus* (Figure 5).

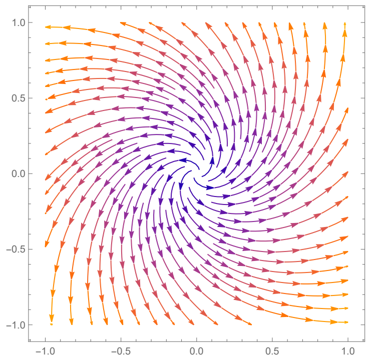


Figure 5 – Unstable Focus

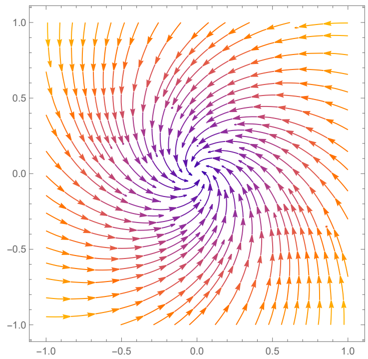


Figure 6 – Stable Focus

  - *Case 2.2*: $a = 0$. Last of all, whenever the eigenvalues of A are purely imaginary, the phase portrait of the system will be linearly equivalent to a *center* at the origin (Figure 7). In this case, the trajectories will be closed orbit around the origin.

Since the real numbers don't form an algebraically closed field, the eigenvalues of a real matrix A need not to be real numbers, as one can observe on the previous

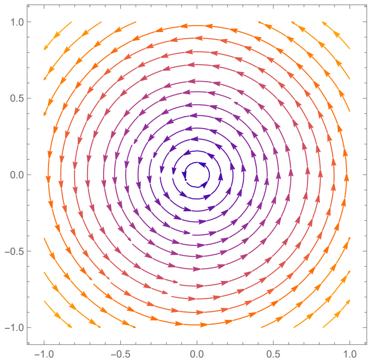


Figure 7 – Center at the origin

example. Let $\lambda_j = a_j + ib_j$ be an eigenvalue of a matrix A, we will denote by $w_j = u_j + iv_j$, $u_j, v_j \in \mathbb{R}^n$, a corresponding generalized eigenvector of A on $\mathbb{R}^n$ over the complex field. Note that if λ_j is real, then $w_j = u_j$.

Theorem 1. *Let $\lambda_j = a_j \in \mathbb{R}$, $1 \leqslant j \leqslant k$, be the real eigenvalues of $A \in \mathcal{M}_n(\mathbb{R})$, counting multiplicity, and $\lambda_j = a_j + ib_j$ for $j = k+1, ..., n$ the complex eigenvalues of A. Then there exists $w_1, ..., w_{k+m}$, with $w_j = u_j + iv_j$, $u_j, v_j \in \mathbb{R}^n$ and $m = \dfrac{n-k}{2}$, such that*

$$B = \{u_1, ..., u_k, u_{k+1}, v_{k+1}, ..., u_{k+m}, v_{k+m}\}$$

is a basis for $\mathbb{R}^n$.

Proof: Set $E_{\mathbb{C}} := E \otimes_{\mathbb{R}} \mathbb{C}$ as the complexification of the real vector space $E = \mathbb{R}^n$; if $T \in \mathcal{L}(E)$, then $T_{\mathbb{C}} \in \mathcal{L}(E_{\mathbb{C}})$ is the complexification of T. Let A be the matrix of both T and $T_{\mathbb{C}}$, and $n = dim(E) = dim(E_{\mathbb{C}})$; since $\mathbb{C}$ is algebraically closed, there exists a basis of $E_{\mathbb{C}}$ formed by generalized eigenvectors in which $T_{\mathbb{C}}$ assumes the Jordan Form. Let $\beta = \{w_1, ..., w_n\}$ be such basis.

If $w_1, ..., w_k$ are generalized eigenvectors associated to real eigenvalues, then we can choose w_j such that it coincides with the real generalized eigenvector, we write $w_j = u_j \in \mathbb{R}^n$. It follows that $E_{\mathbb{C}}$ can be written as $E_1 \oplus E_2$, where $E_1 = span\{u_1, ..., u_k\}$ and $E_2 = span\{w_{k+1}, ..., w_n\}$.

Note that, if λ is an eigenvalue and v is a corresponding eigenvector, then the complex conjugate $\overline{\lambda}$ is an eigenvalue along with the eigenvector $\overline{v}$. In fact, since the matrix A only has real entries:

$$\overline{T_{\mathbb{C}}(v)} = \overline{Av} = \overline{A}\overline{v} = A\overline{v} = T_{\mathbb{C}}(\overline{v})$$

but $T_{\mathbb{C}}(v) = \lambda v \Rightarrow \overline{T_{\mathbb{C}}(v)} = \overline{\lambda v} = \overline{\lambda}\overline{v}$, therefore $\overline{T_{\mathbb{C}}(v)} = \overline{\lambda}\overline{v}$, as we intended to show.

Analogously, suppose v_2 is a generalized eigenvector of A such that $(A - \lambda)v_2 = v$ and note that

$$\overline{(A-\lambda)v_2} = \overline{Av_2 - \lambda v_2} = \overline{Av_2} - \overline{\lambda v_2} = A\overline{v_2} - \overline{\lambda}\overline{v_2} = (A - \overline{\lambda})\overline{v_2}$$

so $\overline{(A-\lambda)v_2} = \overline{v}$, hence $v_2 \in ker(A-\lambda)^2$, i.e. $\overline{v_2}$ is a generalized eigenvector corresponding to $\overline{\lambda}$ and $\overline{v}$.

One can easily see, by induction on $(A-\lambda)^k$, that $\overline{w}$ is a generalized eigenvector whenever w is a generalized eigenvector. Furthermore, we also conclude that, whenever there's a Jordan block J_λ of size p, there's also a $J_{\overline{\lambda}}$ of size p, as well that there are the same number of Jordan blocks corresponding to λ and $\overline{\lambda}$.

With that in mind, we can rearrange the basis of E_2 as the following:

$$w_k, \overline{w_k}, ..., w_{k+m}, \overline{w_{k+m}}$$

where $k + 2m = n$.

Since $w_j = u_j + iv_j$, $\overline{w_j} = u_j - iv_j$; hence u_j and v_j span $\{w_j, \overline{w_j}\}$. Moreover, if $j \neq l$, then $\{u_j, v_j, u_l, v_l\}$ are linearly independent. Indeed, suppose there are $a, b, c, d \in \mathbb{C}$ such that $au_j + bv_j + cu_l + dv_l = 0$ then

$$\begin{aligned} & a\frac{w_j + \overline{w_j}}{2} - ib\frac{w_j - \overline{w_j}}{2} + c\frac{w_l + \overline{w_l}}{2} - id\frac{w_l - \overline{w_l}}{2} = 0 \\ \Rightarrow & \left(\frac{a-ib}{2}\right) w_j + \left(\frac{a+ib}{2}\right) \overline{w_j} + \left(\frac{c-id}{2}\right) w_l + \left(\frac{c+id}{2}\right) \overline{w_l} = 0 \end{aligned}$$

Unless $a = b = c = d = 0$, the above equation yields a contradiction with the fact that β is a basis. Therefore, one can replace $\{w_j, \overline{w_j}, w_l, \overline{w_l}\}$ with $\{u_j, v_j, u_l, v_l\}$. More generally, we have that $\{u_{k+1}, v_{k+1}, ..., u_{k+m}, v_{k+m}\}$ is a basis for E_2, thus $B = \{u_1, ..., u_k, u_{k+1}, v_{k+1}, ..., u_{k+m}, v_{k+m}\}$ is a basis of $E_{\mathbb{C}}$. As $E_{\mathbb{C}}$ has the same dimension of $\mathbb{R}^n$ and B is only formed by real vectors, it follows that B is a basis for $\mathbb{R}^n$.

□

Using the same notation as above, we set

$$\begin{aligned} E^s &= span\{u_j, v_j | a_j < 0\} \\ E^u &= span\{u_j, v_j | a_j > 0\} \\ E^c &= span\{u_j, v_j | a_j = 0\} \end{aligned}$$

E^s, E^u and E^c are, respectively, the *stable, unstable* and *center subspaces* of $\mathbb{R}^n$.

Theorem 2. $\mathbb{R}^n = E^s \oplus E^u \oplus E^c$, *and* E^s, E^u *and* E^c *are invariant with respect to the flow* e^{At}.

Proof: It's easy to see that $\mathbb{R}^n = E^s \oplus E^u \oplus E^c$ is actually a corollary from Theorem 1, together with the definition of E^s, E^u and E^c.

For the second statement, it's enough to prove it for E^s. Let $v \in E^s$, then for some $l \in \mathbb{Z}$, $l \leqslant n$:

$$v = \sum_{j=1}^{l} c_j V_j, \text{ with } V_j = v_j \text{ or } V_j = u_j.$$

By linearity:

$$e^{At} v = e^{At} \left(\sum_{j=1}^{l} c_j V_j \right) = \sum_{j=1}^{l} c_j e^{At} V_j$$

and for each j

$$e^{At} V_j = \lim_{k \to \infty} \left(I_n + At + ... + \frac{A^k t^k}{k!} \right) V_j.$$

Note that if $E(\lambda)$ is the generalized eigenspace corresponding to the eigenvalue λ, then it follows from the definition of E^s, E^u and E^c that $E(\lambda) \subset E^s$ or $E(\lambda) \subset E^u$ or $E(\lambda) \subset E^c$. Since $V_j \in E(\lambda_j)$ for some j and $E(\lambda_j)$ is A-invariant, then $A^k V_j \in E(\lambda_j) \subset E^s$ for every k. Hence

$$\left(V_j + AtV_j + ... + \frac{A^k t^k}{k!} V_j \right) \in E^s$$

so the limit $e^{At} V_j$ is a vector of E^s, therefore $\sum_{j=1}^{l} c_j e^{At} V_j \in E^s$, i.e. E^s is e^{At}-invariant.

□

1.2 Nonlinear Systems

From now on, we shall focus our study on systems of nonlinear differential equations, that is to say systems of the type

$$\dot{x} = f(x), \tag{1.3}$$

where $f : U \to \mathbb{R}^n$ and $U \subset \mathbb{R}^n$ is an open set. In order to do so, we first introduce some basic definitions, then some results of the local theory of nonlinear systems and, finally, we present concepts and results regarding the global theory.

1.2.1 Local Theory

Let $f : U \to \mathbb{R}^n$ be a function from an open set of $\mathbb{R}^n$ to $\mathbb{R}^n$. If f is differentiable, we denote it's derivative by $Df = \left[\frac{\partial f_i}{\partial x_j} \right]$, i.e. the Jacobian matrix. If there exist higher order derivatives of f, we shall write the k-order derivative of f as $D^k f : U \times ... \times U \to \mathbb{R}^n$.

If $D^k f$ exists and it happens to be continuous, then f is said to be a function of class C^k. If f is infinitely differentiable, i.e. $D^k f$ exists for every k, then we say that f is of class C^∞, or a *smooth* function; the system (1.3) is a *smooth system* if $f(x)$ is smooth. Moreover, the *Fundamental Existence-Uniqueness Theorem* guarantees that, if f is at least of class C^1, then for some interval $[-a, a]$, with $a > 0$, the initial value problem

$$\dot{x} = f(x), \quad x(0) = x_0, \tag{1.4}$$

has a unique solution $x(t)$. We may also denote this solution by $\phi_t(x_0) = \phi(t, x_0)$, where $\phi_t(x)$ is called the flow defined by the differential system $\dot{x} = f(x)$.

A point $x_0 \in \mathbb{R}^n$ such that $f(x_0) = 0$ is said to be a *critical point* or a *equilibrium point* of the system (1.3). Taking a look at the Taylor expansion series of f

$$f(x) = Df(x_0)(x - x_0) + \frac{1}{2}(x - x_0)^T D^2 f(x_0)(x - x_0) + ...$$

one may note that the linear function $Df(x_0)x$ is a potential approximation of f near x_0. With that in mind, it will be useful to consider the associated linear system

$$\dot{x} = Ax, \tag{1.5}$$

where $A = Df(x_0)$, on our study of nonlinear systems. Such linear system may be referred to as the *linearization* of the system (1.3).

A critical point x_0 is said to be *hyperbolic* if the real parts of all eigenvalues of $Df(x_0)$ are not zero; otherwise, x_0 is a *nonhyperbolic* equilibrium point. One important result for hyperbolic points in the local theory of nonlinear systems is the *Stable Manifold Theorem*, which shows that the system (1.3) has stable and unstable manifolds, W^s and W^u, that are tangent at the critical point to the stable and unstable subspaces E^s and E^u from the linearized system. Moreover, when a point is nonhyperbolic, the *Center Manifold Theorem* guarantees the existence of a center manifold W^c tangent to the center subspace E^c. These results are discussed in detail throughout the sections 2.7 of (PERKO, 2012).

Another important result in the local theory is the *Hartman-Grobman Theorem*. This theorem shows that, near a hyperbolic critical point, the system (1.3) has the same qualitative structure as the linear system (1.5), as long as f is at least of class C^1. The "same qualitative structure", in this case, refers to the existence of a homeomorphism $H : U \to V$ — where U and V are neighborhoods of the critical point — which maps the trajectories of (1.3) onto the trajectories of (1.5), preserving their orientation by time; we may also refer to this propriety as being *locally topologically equivalent*. More precisely, the theorem can be stated as it follows:

Theorem 3. (Hartman-Grobman) *Let $E \subset \mathbb{R}^n$ be an open set containing the origin, $f \in C^1(E)$ and let ϕ_t be the flow of the system* (1.3). *Suppose that $f(0) = 0$ and that no* *eigenvalue of the matrix A of the linearized system has zero real part. Then there exists a homeomorphism $H : U \to V$ — where U and V are open subsets containing the origin — such that for each $X_0 \in U$ there is an open interval $0 \in I_0 \subset \mathbb{R}$ such that for all $t \in I_0$:*

$$H \circ \phi_t(x_0) = e^{At} H(x_0).$$

Note that the Hartman-Grobman Theorem allows us to determine the local behavior of a system, but only in the neighborhood of hyperbolic equilibrium points. On the other hand, the *Local Center Manifold Theorem* shows that, in case of dealing with a nonhyperbolic critical point, the problem of determining the qualitative structure on its surroundings can be reduced to the study of the system restricted to the center manifold. As this may still be a complicated task, one may wish to simplify the nonlinear part of (1.3), i.e. rewriting the equation as

$$\dot{x} = Jx + F(x), \tag{1.6}$$

where J is the jacobian of f, it may be possible to simplify F. The procedure of reducing the equation (1.6) by annihilating lower nonlinear terms is known as Normal Form Theory, and was first done by Poincaré. In the following example, we illustrate a method to find the normal form of a system.

Example 2. Consider the following system

$$\begin{cases} \dot{x} = x + 3x^2 + y^3, \\ \dot{y} = 3y + y^2. \end{cases}$$

Our goal is to annihilate the quadratic terms to reduce the expression of the vector field in a neighborhood of the origin, which is a equilibrium of the system. In order to achieve this, we introduce a polynomial change of coordinates $(x, y) = (\overline{x}, \overline{y}) + h(\overline{x}, \overline{y})$. To determine the function h of the change of coordinates, note that one can obtain the following relation between the derivatives:

$$(\dot{x}, \dot{y}) = (1 + Dh(\overline{x}, \overline{y}))(\dot{\overline{x}}, \dot{\overline{y}}).$$

Denoting by $f(x, y) = (3x^2 + y^3, y^2)$ and $A = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$, we have that

$$\begin{aligned} (\dot{\overline{x}}, \dot{\overline{y}}) &= (1 + Dh(\overline{x}, \overline{y}))^{-1}(\dot{x}, \dot{y}) \\ &= (1 + Dh(\overline{x}, \overline{y}))^{-1}\left(A \cdot \begin{pmatrix} x \\ y \end{pmatrix} + f(x, y)\right) \\ &= (1 + Dh(\overline{x}, \overline{y}))^{-1}\left(A \cdot \begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} + A \cdot h(\overline{x}, \overline{y}) + f((\overline{x}, \overline{y}) + h(\overline{x}, \overline{y}))\right). \end{aligned}$$

Knowing that the identity $\sum_{k=0}^{\infty}(-Dh(x,y))^k = (1-Dh(x,y))^{-1}$ holds and writing $f = \sum_{k=0}^{\infty} f_k$ and $h = \sum_{k=0}^{\infty} h_k$ as the Taylor expansion of f and h, we obtain the following expression for $(\dot{\overline{x}}, \dot{\overline{y}})$:

$$\begin{aligned}(\dot{\overline{x}}, \dot{\overline{y}}) = A \cdot \begin{pmatrix}\overline{x}\\ \overline{y}\end{pmatrix} + \left(A \cdot h_2(\overline{x},\overline{y}) + f_2(\overline{x},\overline{y}) - Dh_2(\overline{x},\overline{y}) \cdot A \cdot \begin{pmatrix}\overline{x}\\ \overline{y}\end{pmatrix}\right) + \\ + \sum_{k=3}^{\infty}\left(A \cdot h_k(\overline{x},\overline{y}) + g_k(\overline{x},\overline{y}) - Dh_k(\overline{x},\overline{y}) \cdot A \cdot \begin{pmatrix}\overline{x}\\ \overline{y}\end{pmatrix}\right).\end{aligned} \tag{1.7}$$

Where g_k only depends on $f_2, ..., f_{k-1}$ and $h_2, ..., h_{k-1}$. Since we want to eliminate the quadratic term, the following condition must hold:

$$A \cdot h_2(\overline{x},\overline{y}) + f_2(\overline{x},\overline{y}) - Dh_2(\overline{x},\overline{y}) \cdot A \cdot \begin{pmatrix}\overline{x}\\ \overline{y}\end{pmatrix} = 0.$$

in other words, we wish to find some $h(x,y)$ such that

$$f_2(\overline{x},\overline{y}) = A \cdot h_2(\overline{x},\overline{y}) - Dh_2(\overline{x},\overline{y}) \cdot A \cdot \begin{pmatrix}\overline{x}\\ \overline{y}\end{pmatrix}.$$

Define the operator $L_A^{k,2}$ by the following expression

$$L_A^{k,2}(h(x,y)) = Dh_k \cdot A \cdot \begin{pmatrix}x\\ y\end{pmatrix} - A \cdot h_k(x,y) = \begin{pmatrix} x\dfrac{\partial p}{\partial x} + 3y\dfrac{\partial p}{\partial y} - p \\ x\dfrac{\partial q}{\partial x} + 3y\dfrac{\partial q}{\partial y} - 3q \end{pmatrix},$$

where $h(x,y) = (p(x,y), q(x,y))$, and notice that it is linear. Hence we can calculate the matrix of $L_A^{2,2}$ just by applying the transformation to a basis of the linear space of the quadratic polynomials with two entries, which we denote by H_2^2. In order to do that, we consider the following basis of H_2^2

$$\{(0,x^2), (0,xy), (0,y^2), (x^2,0), (xy,0), (y^2,0)\}.$$

Then:

$$\begin{aligned} L_A^{2,2}(0,x^2) &= -(0,x^2), \\ L_A^{2,2}(0,xy) &= (0,xy), \\ L_A^{2,2}(0,y^2) &= 3(0,y^2), \\ L_A^{2,2}(x^2,0) &= (x^2,0), \\ L_A^{2,2}(xy,0) &= 3(xy,0), \\ L_A^{2,2}(y^2,0) &= 5(y^2,0), \end{aligned}$$

which allows us to build the following matrix:

$$L_A^{2,2} = \begin{pmatrix} -1 & & & & & \\ & 1 & & & & \\ & & 3 & & & \\ & & & 1 & & \\ & & & & 3 & \\ & & & & & 5 \end{pmatrix}.$$

Since the matrix is invertible, it's possible to find a coordinate change by solving the homological equation $L_A^{2,2} h_2 = f_2$. Indeed, multiplying the vector $f_2 = (0, 0, 1, 3, 0, 0)$ by $(L_A^{2,2})^{-1}$, we obtain:

$$h_2(x, y) = (3x^2, \frac{y^2}{3})$$

and so the coordinate change will be given by $(x, y) = (\overline{x}, \overline{y}) + h_2(\overline{x}, \overline{y})$; hence:

$$(\dot{\overline{x}}, \dot{\overline{y}}) = (I + Dh_2(\overline{x}, \overline{y}))^{-1} \cdot \left(A \cdot \begin{pmatrix} \overline{x} + 3\overline{x}^2 \\ \overline{y} + \frac{\overline{y}^2}{3} \end{pmatrix} + f\left(\overline{x} + 3\overline{x}^2, \overline{y} + \frac{\overline{y}^2}{3}\right) \right)$$

and then, by doing the calculations, we obtain

$$\begin{aligned} \dot{\overline{x}} &= \overline{x} + 18\overline{x}^3 + \overline{y}^3 - 81\overline{x}^4 - 6\overline{xy}^3 + \overline{y}^4 + \mathcal{O}(|\overline{x}|^5) \\ \dot{\overline{y}} &= 3\overline{y} + 2\frac{\overline{y}^3}{3} - \frac{\overline{y}^4}{3} + \mathcal{O}(|\overline{y}|^5). \end{aligned}$$

Therefore, the normal form of the original system is:

$$\begin{cases} \dot{x} = x + \mathcal{O}(x^3), \\ \dot{y} = 3y + \mathcal{O}(y^3). \end{cases}$$

Notice that if we wished to annihilate higher order terms from the system, i.e. find higher order normal forms, we must then solve the other homological equations that can be obtained by making the expressions from the summation in (1.7) be equal to zero.

1.2.2 Global Theory

In order to approach the main goal of our work, we shall take a closer look into the global theory of nonlinear systems. First of all, we generalize the previous definition of topological equivalence.

Definition 1. (Topological Equivalence) Let $f \in C^1(E_1)$ and $g \in C^1(E_2)$, where $E_1, E_2 \subset \mathbb{R}^n$ are open and consider the two autonomous systems:

$$\begin{aligned} \dot{x} &= f(x), \\ \dot{x} &= g(x). \end{aligned}$$

These systems are said to be topologically equivalent if there's a homeomorphism $H : E_1 \to E_2$ which maps the trajectories from the first system onto the trajectories of the latter and preserves the orientation by time.

Definition 2. Let $\phi(t, x)$ be a trajectory of the system (1.3). The *ω-limit set* of ϕ is the set of all points p for which there is a sequence $t_n \to \infty$ such that

$$\lim_{n\to\infty} \phi(t_n, x) = p.$$

Analogously, we define the *α-limit set* as the set of all points q for which there is a sequence $t_n \to -\infty$ such that

$$\lim_{n\to\infty} \phi(t_n, x) = q.$$

If the α or the ω-limit set are periodic orbits, then we might as well call them *limit cycles*.

By periodic orbits, we mean closed solution curves that are not equilibrium points. Moreover, a periodic orbit Γ is said to be *stable* if for each $\varepsilon > 0$ there is a neighborhood U of the orbit such that, for all $x \in U$ and $t \geqslant 0$, $d(\phi(t, x)), \Gamma) < \varepsilon$; otherwise, Γ is said to be *unstable*.

An important tool to study the stability of periodic orbits is the *Poincaré map*, which will be defined as follows:

Definition 3. (Poincaré Map) Let $\Gamma = \phi(t, x_0)$ denote a periodic orbit of the system

$$\dot{x} = f(x),$$

and let Σ be a hyperplane perpendicular to Γ at x_0. For a neighborhood U of x_0, define

$$\begin{aligned} P : U \cap \Sigma &\to \Sigma, \\ x &\longmapsto P(x), \end{aligned}$$

with $P(x) = \phi(t_1, x)$, where t_1 is the time when the trajectory $\phi(t, x)$, $x \in U \cap \Sigma$, cross Σ again for the first time. The function P is called the *Poincaré Map*.

Moreover, it can be shown, through the implicit function theorem, that the Poincaré map is well-defined, continuous and that t_1 is in fact the period T of the periodic solution.

Theorem 4. (Poincaré-Bendixson) *Let $\dot{x} = f(x)$ be a planar dynamical system, where $f \in C^1(E)$, $E \subset \mathbb{R}^2$ is open. Suppose that Γ is a trajectory of such system, and that $\Gamma^+ \subset K$, where $K \subset E$ is compact and Γ^+ is the trajectory for $t > 0$. If the system admits only a finite number of critical point in K, then $\omega(\Gamma)$ can be: (1) a critical point of the system; (2) a periodic orbit of the system; (3) a finite number of critical points $p_1, ..., p_m$ and a countable number of limit orbits whose α and ω limit sets belong to $p_1, ..., p_m$.*

If on one hand it is possible to study the existence of limit cycles for some planar systems using Theorem 4, on the other hand it does not help us much to determine the exact number of limit cycles for a certain class of systems - this is, in fact, a much more complicated task, and there are many different approaches to this topic. In particular, we are interested in finding upper bounds to the number of limit cycles of certain systems, which is a version of the Hilbert's 16th problem, and we will focus on this task for Liénard-like Systems.

The classical Liénard System is a dynamical system of the form

$$\begin{aligned} \dot{x} &= y - F(x), \\ \dot{y} &= -g(x), \end{aligned} \tag{1.8}$$

which was first studied by Liénard in 1928. Under certain conditions for F and g, Liénard proved the uniqueness of the limit cycle of system (1.8).

An interesting theorem is given by Perko in (PERKO, 2012) providing a necessary and sufficient condition to construct systems like (1.8) that have a desired number m of limit cycles.

Theorem 5. (Perko) *For $\varepsilon \neq 0$ sufficiently small, the system (1.8) with $g(x) = x$ and $F(x) = \varepsilon[a_1 x + a_2 x^2 + \ldots + a_{2m+1}x^{2m+1}]$ has at most m local limit cycles. Furthermore, this system has* ***exactly*** *m limit cycles which are asymptotic to circles of radius r_j, $j = 1, \ldots, m$, centered at the origin as $\varepsilon \to 0$ if, and only if, the mth degree equation*

$$\frac{a_1}{2} + \frac{3a_3}{8}\rho + \frac{5a_5}{16}\rho^2 + \ldots + \binom{2m+2}{m+1}\frac{a_{2m+1}}{2^{2m+2}}\rho^m = 0,$$

has m positive roots $\rho = r_j^2$, $j = 1, \ldots, m$.

In section 4.1, we provide a proof for the first part of this theorem using averaging theory for smooth systems. Despite being an useful tool to construct nice examples, proving the second part of Perko's theorem will not be our goal here; however, the whole proof of this theorem is given in (PERKO, 2012) using Melnikov's Method.

1.3 Bifurcations

The last topic of this chapter aims to introduce the idea of what may happen to a dynamical system when we make perturbations on it. These perturbations are basically variations of the function f of the system (1.3), which we obtain by varying a parameter ε — i.e. we replace $f(x)$ by $f(x, \varepsilon)$; what we want to study is how these changes impact the qualitative behavior of the system. If a system is “stable enough", a small variation of ε shall not cause a big change on the system's phase portrait; however, if the structure of the system is rather unstable, a small variation of the parameter will change the whole structure, yielding a completely different system that is not topologically equivalent to the original one. Let us draw a more formal mathematical approach by presenting some definitions.

Definition 4. (Structural Stability) Let $E \subset \mathbb{R}^n$ be an open set. A vector field $f \in C^1(E)$ is said to be *structurally stable* if there is a $\varepsilon > 0$ for which all vector fields $g \in C^1(E)$ such that

$$||f - g||_1 < \varepsilon,$$

are topologically equivalent to f on E.

In other words, a vector field is structurally stable if it's topologically equivalent to all the vector fields that are close to it. However, if a vector field is not structurally stable, then it's qualitative structure will change when it passes through a *bifurcation* point in the space of the parameters.

Example 3. The linear center

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x, \end{aligned}$$

is **not** structurally stable in any compact set containing the origin. In fact, let K be a compact containing the origin and define the vector fields

$$f(x, y) = \begin{pmatrix} y \\ -x \end{pmatrix} \text{ and } g(x, y) = \begin{pmatrix} y + \mu x \\ -x + \mu y \end{pmatrix},$$

then, taking $||.||_1$ as the C^1 norm:

$$||f - g||_1 = \sup_K |f - g| + \sup_K ||D(f - g)||,$$

where $||D(f-g)|| = \left| \max_{i,j \in \{1,2\}} \left\{ \dfrac{\partial (f-g)_i}{\partial x_j} \right\} \right| = |\mu|$. Let d be the diameter of K, then

$$\sup_K |f - g| = \sup_K \left| \begin{pmatrix} -\mu x \\ -\mu y \end{pmatrix} \right| = |\mu| \cdot d.$$

Hence $||f - g||_1 = |\mu|(d + 1)$. So given $\varepsilon > 0$, consider $|\mu| = \dfrac{\varepsilon}{d+2}$, then $||f - g||_1 < \varepsilon$, i.e. we may take g as close to f as we desire.

However, f and g are **not** topologically equivalent! Indeed, let ϕ_t and ψ_t denote the flows from the vector fields f and g, respectively, and take $\mu < 0$. If f and g were topologically equivalent, then there must exist a homeomorphism $H : \mathbb{R}^2 \to \mathbb{R}^2$ and a time reparametrization $\tau : \mathbb{R} \to \mathbb{R}$ preserving orientation, i.e. a strictly increasing continuous function, such that

$$\phi_t = H^{-1} \circ \psi_{\tau(t)} \circ H.$$

Since f has a center at the origin, $\lim_{t\to\infty} \phi_t(1,0) \neq (0,0)$. However, analyzing the trajectory of the same point through the flow of the vector field g, we have:

$$g(x,y) = \begin{pmatrix} \mu & -1 \\ 1 & \mu \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \lambda = \mu \pm i \text{ are the eigenvalues of the matrix}$$
$$\Rightarrow \psi_t(x_0, y_0) = (e^{(\mu+i)t}x_0, e^{(\mu-i)t}y_0)$$
$$\Rightarrow \psi_t(1,0) = (e^{(\mu+i)t}, 0).$$

But

$$\lim_{t\to\infty} e^{(\mu+i)t} = \lim_{t\to\infty} e^{\mu t}(\cos t - i \sin t) = 0$$
$$\Rightarrow \lim_{t\to\infty} \psi_t(1,0) = (0,0).$$

which yields a contradiction, since a homeomorphism must map the origin onto itself.

Therefore the system is structurally unstable and $\mu = 0$ is a point of bifurcation.

The concept of structural stability can also be extended and applied on a compact differentiable manifold. The next theorem gives a complete characterization of a structurally stable C^1 vector field on a compact, two dimensional, differentiable manifold:

Theorem 6. (Peixoto, (PERKO, 2012)) *Let f be a C^1 vector field on a compact, two dimensional, differentiable manifold M. Then f is structurally stable on M if, and only if:*

1. *the number of critical points and cycles is finite and each one is hyperbolic;*
2. *there are no trajectories connecting saddle points, and*
3. *the nonwandering set Ω consists only of critical points and limit cycles.*

By *nonwandering set* Ω we mean that, for every neighborhood U of any point $x \in \Omega$ and any $T > 0$, there is a $t > T$ such that

$$\phi_t(U) \cap U \neq \varnothing$$

where ϕ_t is the flow defined by (1.3). From this theorem, one can conclude that *saddles, nodes and foci* are structurally stable, while linear centers are not, as we showed in the previous example. We shall focus on the cases where the system is structurally unstable, i.e. when bifurcations do occur; furthermore, a certain type of bifurcation is particularly of our interest in this study.

Definition 5. A *Hopf bifurcation* is a critical point in the parameter space where a structural unstable system changes its behavior and a limit cycle arises.

Theorem 7. (Generic Hopf) *Consider the system given by*

$$\begin{aligned}\dot{x} &= f(x, y, \mu),\\ \dot{y} &= g(x, y, \mu),\end{aligned}$$

where μ is a parameter. Let $(x, y) = (x_0, y_0)$ denote a equilibrium point of the system and $\lambda_{1,2}(\mu) = \alpha(\mu) \pm i\beta(\mu)$ be the eigenvalues of the Jacobian at (x_0, y_0).$For \mu = \mu_0$, suppose that the following conditions hold:

1. *$\alpha(\mu_0) = 0$ and $\beta(\mu_0) \neq 0$, i.e. the equilibrium is nonhyperbolic;*
2. *$\dot{\alpha}(\mu_0) = d \neq 0$, i.e. the eigenvalues cross the imaginary axis transversely;*
3. *$a = \dfrac{1}{16}(f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}) + \dfrac{1}{16\beta(\mu_0)}(f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy}) \neq 0$*

Then a Hopf bifurcation occurs at the fixed point of the system (x_0, y_0) at the value $\mu = \mu_0$.

Moreover, if $ad < 0$, then a unique stable limit cycle bifurcates from the equilibrium as $\mu > \mu_0$ and, if $ad > 0$, then a unique unstable limit cycle bifurcates from the equilibrium as $\mu < \mu_0$.

Condition 3 is derived by calculating the normal form of the original system; for more details of the proof, check out (GUCKENHEIMER, 2002).

Example 4. Consider the following differential equation

$$\ddot{x} - (\alpha - x^2)\dot{x} + x = 0. \tag{1.9}$$

Let $\dot{x} = y$, then $\dot{y} = \ddot{x} = -x + (\alpha - x^2)y$; we can write the equation (1.9) as the planar system

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= -x + (\alpha - x^2)y,\end{aligned} \tag{1.10}$$

which is a specific case of the Liénard system that we presented in the previous section. Notice that the origin is the only equilibrium point of (1.10), so we compute the Jacobian of the system there:

$$J(\alpha) = \begin{pmatrix} 0 & 1 \\ -1 & \alpha \end{pmatrix}$$

It follows that the eigenvalues of $J(\alpha)$ are given by

$$\begin{vmatrix} -\lambda & 1 \\ -1 & \alpha - \lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - \alpha\lambda + 1 = 0$$
$$\therefore \lambda_{1,2} = \frac{\alpha \pm \sqrt{\alpha^2 - 4}}{2}.$$

By Theorem 6, if $Re(\lambda) = 0$, i.e. the origin is nonhyperbolic, then the system will be structurally unstable. Since $Re(\lambda) = \alpha$, we have that $\alpha = 0$ is a point of bifurcation. Moreover, notice that $\dfrac{d(Re(\lambda))}{d\alpha} = 1$, $Im(\lambda)(0) = 1$, $f(x, y, 0) = y$ and $g(x, y, 0) = -x - x^2 y$, then

$$\begin{aligned} a &= \frac{1}{16}(f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}) \\ &+ \frac{1}{16Im(\lambda)(0)}(f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy}) \\ &= \frac{1}{16}(-2) + \frac{1}{16}(-2x(-2y))|_{(x,y)=(0,0)} = -\frac{1}{8} \neq 0. \end{aligned}$$

Therefore the conditions of the theorem 7 hold and a Hopf bifurcation occurs at $\alpha = 0$. Since $d = 1$ and $a = \dfrac{-1}{8}$, $ad < 0$, then the periodic solutions must occur for $\alpha > 0$. On the figures 8, 9 and 10, we illustrate the phase portrait and the trajectory of the point $(x_0, y_0) = (3, 3)$ through the flow in each case of α.

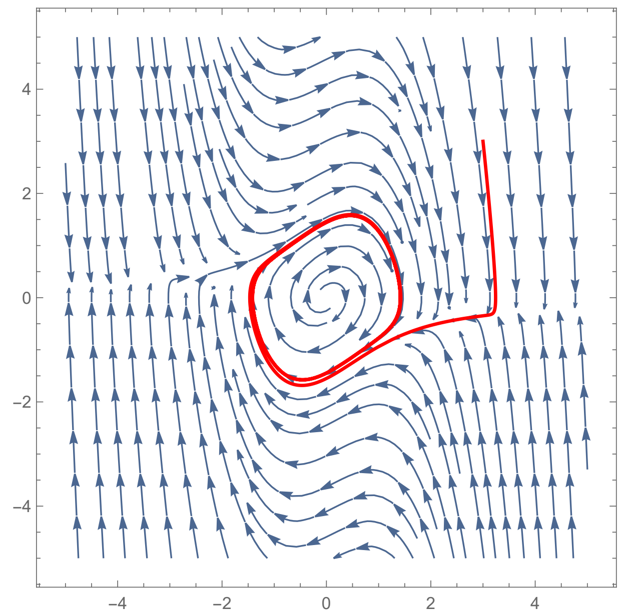


Figure 8 – Limit cycle when $\alpha = 0.5$

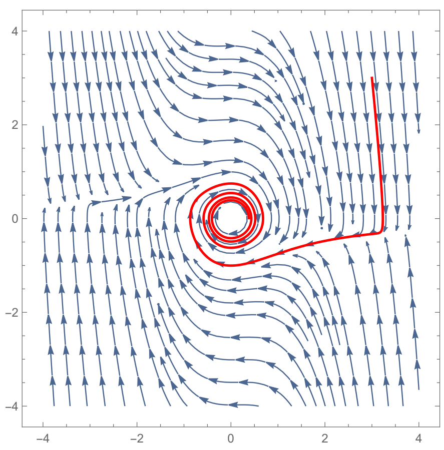


Figure 9 – System (1.9) with $\alpha = 0$

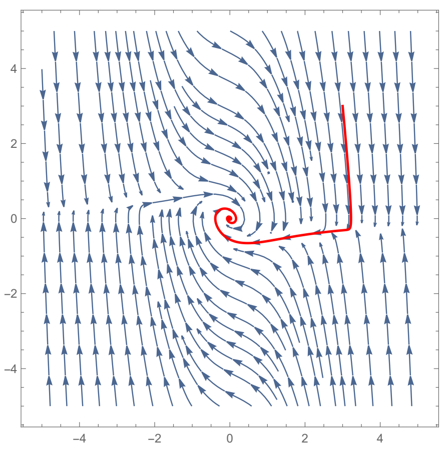


Figure 10 – System (1.9) with $\alpha = -0.5$

2 Non-smooth Dynamical Systems

2.1 Foundations

The kind of non-smooth system we will focus are Filippov Systems, which are systems discontinuous along a hypersurface in the phase space - in particular, we will be restricting these to planar systems with a Liénard-like perturbation. Some concepts developed for smooth systems need to be adapted to suit this new class of dynamical systems; the formalization of those generalizations is the main goal of this section.

Let U be an open neighbourhood of 0 where the vector field we shall study is defined, and let Σ be the hypersurface along which the discontinuities occur. Since any embedded hypersurface is locally the inverse image of a regular value, let $\Sigma = f^{-1}(0) \cap U$, where f is the *germ* of a C^r function with $r > 1$ and which has 0 as a regular value. Here the *germ of a function* refers to the equivalence class of all functions which are locally equal to one another.

Note that the hypersurface Σ splits U into the following open sets:

$$\Sigma^+ = \{(x,y) \in U : f(x,y) > 0\} \text{ and } \Sigma^- = \{(x,y) \in U : f(x,y) < 0\}.$$

We may then define the germs of a discontinuous vector field as

$$Z(x,y) = \begin{cases} X(x,y), & \text{if } (x,y) \in \Sigma^+ \\ Y(x,y), & \text{if } (x,y) \in \Sigma^- \end{cases}, \tag{2.1}$$

and we denote the vector field by $Z = (X, Y)$.

On the regions Σ^+ and Σ^-, the trajectories can be defined by the vector fields X and Y in the same way as for smooth systems. Troubles may arise from what happens along Σ, hence we must take a careful look into these possibilities.

Let $Xf(p) = X(p) \cdot \nabla f(p)$ denote the Lie derivative of f with respect to X at the point p, and define:

$$\begin{aligned} \Sigma^c &= \{p \in \Sigma : Xf(p) \cdot Yf(p) > 0\} \\ \Sigma^s &= \{p \in \Sigma : Xf(p) < 0, Yf(p) > 0\} \\ \Sigma^e &= \{p \in \Sigma : Xf(p) > 0, Yf(p) < 0\} \end{aligned}$$

which we name *crossing region, sliding region* and *escaping region*, respectively. Figures 11, 12 and 13 illustrate each of these possibilities.

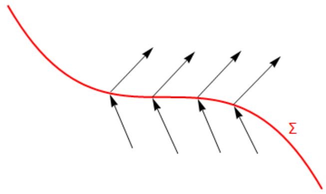


Figure 11 – Crossing region

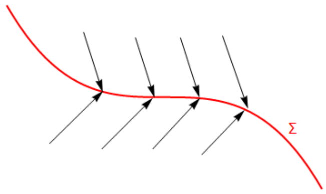


Figure 12 – Sliding region

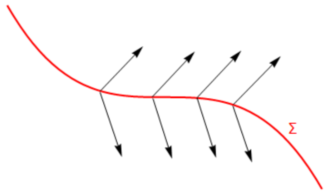


Figure 13 – Escaping region

If $Xf(p) = 0$ or $Yf(p) = 0$, then p is a *tangency point*; we assume that these tangency points are isolated in Σ.

Note that if $p \in \Sigma^c$, i.e. p is in a crossing region, then we can define the trajectory through p by simply matching the trajectories defined by X and Y. However, if $p \in \Sigma^s$ or Σ^e, the trajectory can't be defined so directly; in this case, we shall use the Filippov convention We define the sliding vector field Z^s as the linear convex combination of X and Y tangent to Σ:

$$Z^s(p) = \frac{1}{Yf(p) - Xf(p)}(Yf(p)X(p) - Xf(p)Y(p)). \tag{2.2}$$

Definition 6. Let $\varphi_X(t, p)$ denote the flow of a smooth autonomous vector field X such that $\varphi_X(0, p) = p$. The trajectory of the vector field (2.1) through a point p is defined as follows:

1. For $p \in \Sigma^+$ or $p \in \Sigma^-$, the trajectory is given by $\varphi_Z(t, p) = \varphi_X(t, p)$ and $\varphi_Z(t, p) = \varphi_Y(t, p)$, respectively, for $t \in I \subset \mathbb{R}$

2. For $p \in \Sigma^c$, if $Xf(p), Yf(p) > 0$ the trajectory is defined as $\varphi_Z(t, p) = \varphi_X(t, p)$ for $t \geqslant 0$ and $\varphi_Z(t, p) = \varphi_Y(t, p)$ for $t \leqslant 0$; if $Xf(p), Yf(p) < 0$, the definition is the same but reversing time.

3. For $p \in \Sigma^e \cup \Sigma^s$, if $Z^s(p) \neq 0$, then $\varphi_Z(t, p) = \varphi_{Z^s}(t, p)$, where Z^s is the sliding vector field defined in (2.2).

4. For $p \in \partial\Sigma^c \cup \partial\Sigma^e \cup \partial\Sigma^s$, if the defined trajectories for point in Σ in both sides of p can be extended to p and coincide, then this is the trajectory through p. These are said to be the *regular tangency points*.

5. For any other point, $\varphi_Z(t, p) = p$ for all $t \in I$. This is the case of the *singular tangency points*, i.e. $p \in \partial\Sigma^c \cup \partial\Sigma^e \cup \partial\Sigma^s$ which is not regular, as well as the critical points of X, Y and Z^s in Σ^+, Σ^- and $\Sigma^s \cup \Sigma^e$, respectively.

Notice that the item 2 of the definition is the formalization of saying that the trajectory through p in Σ^c is defined by joining the regular trajectories through p

given by each vector field X and Y. The orbit of a point p is then defined as usually, i.e. $\gamma(p) = \{\varphi_Z(t,p) : t \in I\}$. Another thing we must define is what are the singularities of a Filippov vector field.

Definition 7. A point p is said to be a singularity of the system (2.1) if it fits in one of the following conditions

1. $p \in \Sigma^{\pm}$ and p is an equilibrium point of X or Y;
2. $p \in \Sigma^s \cup \Sigma^e$ such that $Z^s(p) = 0$, in this case we say that p is a pseudoequilibrium;
3. $p \in \partial\Sigma^c \cup \partial\Sigma^e \cup \partial\Sigma^s$.

Definition 8. We define a *regular orbit* of Z as a piecewise smooth curve φ satisfying the following criteria:

1. $\varphi \cap \Sigma^+$ and $\varphi \cap \Sigma^-$ are (a union of) orbits of the smooth vector fields X and Y, respectively.
2. $\varphi \cap \Sigma$ consists only of crossing points (Σ^c) and regular tangency points ($\partial\Sigma^c$).
3. φ is maximal with respect to the above criteria.

We define a *sliding orbit* of Z as smooth curve φ contained in $\overline{\Sigma^s} \cup \overline{\Sigma^e}$ such that it is maximal in Z^s.

Definition 9. A point $p \in \Sigma$ is said to be a *generic point of discontinuity* if there exists a neighborhood $V_p \in I \times D$ containing p such that $\Sigma_p = \Sigma \cap V_p$ is a C^k embedded hypersurface in $I \times D$.

The next Theorem — that was adapted from chapter 2, section 10, of reference (FILIPPOV, 1988) — guarantees the existence and uniqueness of solutions passing through a point in Σ^c. Let $p \in \Sigma^c$ be a generic point of discontinuity and V_p a neighborhood of p, we write $V_p^+ = V_p \cap \Sigma^+$ and $V_p^- = V_p \cap \Sigma^-$.

Theorem 8. *For every point $p \in \Sigma^c$ there's a unique solution passing either from V_p^- into V_p^+ or from V_p^+ into V_p^-.*

Proof: From the Theorem of Existence and Uniqueness for smooth systems, the solutions with initial points in V_p^+ or V_p^- exist and are unique; then, extending continuously these solutions for $\overline{V_p^+}$ and $\overline{V_p^-}$, and by item 2. of definition 6, the solution passing through p is uniquely defined. □

To end this section we present two examples of planar Filippov Systems with a line of discontinuity, showing how to handle the set Σ and compute the sliding vector field, when it exists.

Example 5. Consider the following planar system

$$\dot{x} = \operatorname{sgn}(y) \cdot y,$$
$$\dot{y} = x.$$

It's easy to see that the smooth components of this system are

$$X(x, y) = \begin{pmatrix} y \\ x \end{pmatrix} \text{ and } Y(x, y) = \begin{pmatrix} -y \\ x \end{pmatrix},$$

and that $\Sigma = \{(x, 0) : x \in \mathbb{R}\}$. Notice that in this case Σ can be described as $f^{-1}(0)$, where $f(x, y) = y$; lets see what happens near the points of discontinuity.

If $p \in \Sigma$ we write $p = (x_0, 0)$; computing the Lie derivatives:

$$Xf(p) = X(x_0, 0) \cdot \nabla f(x_0, 0) = (0, x_0) \cdot (0, 1) = x_0,$$
$$Yf(p) = Y(x_0, 0) \cdot \nabla f(x_0, 0) = (0, x_0) \cdot (0, 1) = x_0.$$

hence $Xf(p)Yf(p) = x_0^2 \geqslant 0$. This indicates that all points in Σ are crossing points, except for the origin, which is a (singular) tangency point. Figure 14 shows the phase portrait of this system.

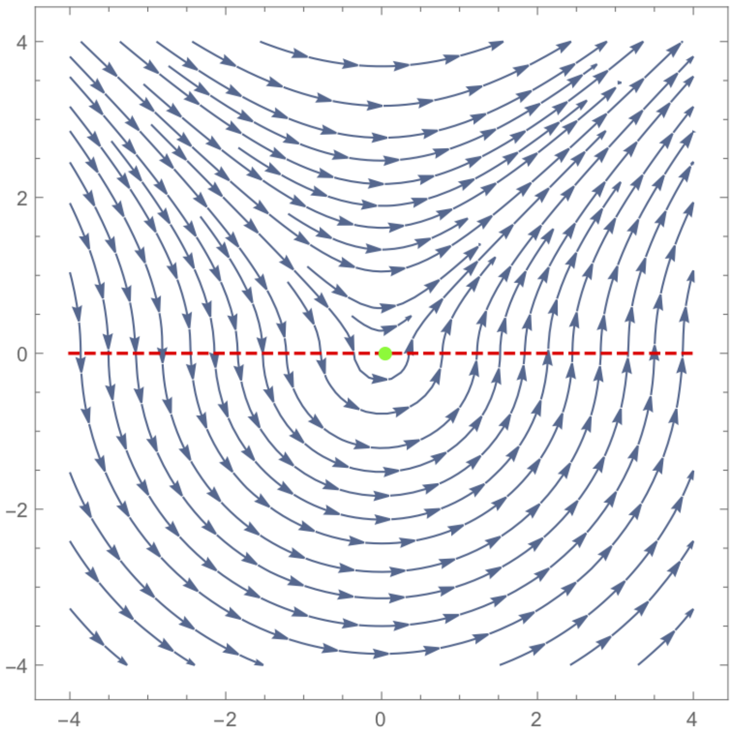


Figure 14 – System from Example 5; the red traced line is Σ^c and the green dot is the origin, which is a point in $\partial\Sigma^c$

.

Example 6. Let $X(x,y) = \begin{pmatrix} 1 \\ 2x \end{pmatrix}$ and $Y(x,y) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and define

$$Z(X,Y) = \begin{cases} X(x,y), & \text{if } y > 0, \\ Y(x,y), & \text{if } y < 0, \end{cases}$$

then $\Sigma = f^{-1}(0)$, where f is as in the previous example. For $p = (x_0, 0) \in \Sigma$, we compute the Lie derivatives:

$$Xf(p) = X(x_0,0) \cdot \nabla f(x_0,0) = (1, 2x_0) \cdot (0,1) = 2x_0,$$
$$Yf(p) = Y(x_0,0) \cdot \nabla f(x_0,0) = (1,1) \cdot (0,1) = 1.$$

This implies that $Xf(p)Yf(p) = 2x_0$, which gives us the following cases:

- **$x_0 > 0$:** In this case $Xf(p)Yf(p) > 0$, hence $p \in \Sigma^c$;
- **$x_0 = 0$:** When p is the origin, $Xf(p)Yf(p) = 0$, thus it is tangency point;
- **$x_0 < 0$:** In this case $Xf(p)Yf(p) < 0$ yields that p is in $\Sigma^s \cup \Sigma^e$; more specifically, since $Yf(p) > 0$ and $Xf(p) < 0$, $p \in \Sigma^s$.

  Furthermore, we can compute the sliding vector field in p using the Filippov convention, explicitly given in (2.2):

$$\begin{aligned} Z^s(p) &= \frac{1}{1-2x_0} \cdot \left[1 \cdot \begin{pmatrix} 1 \\ 2x_0 \end{pmatrix} - 2x_0 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right] \\ &= \frac{1}{1-2x_0} \begin{pmatrix} 1-2x_0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{aligned}$$

  Thus $Z^s(p)$ doesn't depend on p (except for the fact that we must have $x_0 < 0$), and points towards the origin, as illustrated in Figure 15.

Remark 1. *Note that if the system is non-autonomous we can always go up one dimension and treat it like an autonomous system, hence the definitions regarding the discontinuity region Σ make sense in these cases. This should be clear in section 3.3, where we consider the time variable to compute the Lie derivative.*

2.2 Regularization Method

A method to study discontinuous systems through approximations of regular systems was presented by Sotomayor and Teixeira in 1996 (see reference (SOTOMAYOR; TEIXEIRA, 1996)). This approach consists in defining a transition function, which will

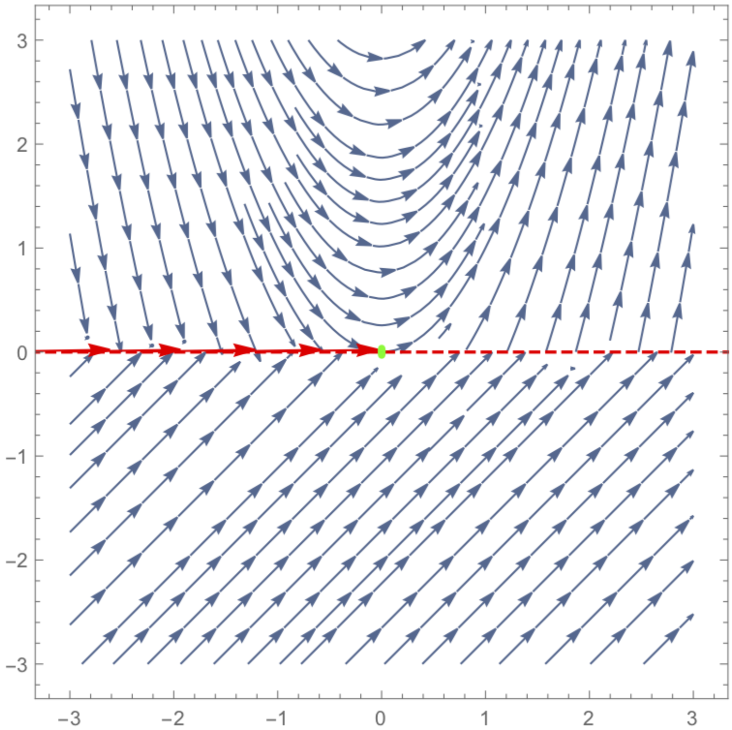


Figure 15 – Phase portrait of the system in Example 6

be used as a weigh in the combination of the vector fields X and Y to obtain a regular approximation of $Z = (X, Y)$. The regularization method was used in many papers in order to apply the averaging theory to discontinuous systems, yielding nice results despite the drawback of demanding extensive calculations.

Since a version of the averaging theorem for discontinuous systems was proven in (LLIBRE; MEREU; NOVAES, 2015), it is now more convenient to apply it without the need of regularizing the system. However, the regularization method is still an important tool in the study of discontinuous systems, and was used in the early calculations of this work; for this reason, we present briefly in this section the basis of this method.

Definition 10. A *transition function* $\varphi : \mathbb{R} \to \mathbb{R}$ is a smooth function such that $\varphi(t) = 0$ if $t \leqslant 0$, $\varphi(t) = 1$ if $t \geqslant 1$ and $\varphi'(t) > 0$ for $t \in (0, 1)$.

Definition 11. A φ_δ-regularization of $Z = (X, Y)$ is the family of vector fields Z_δ given by

$$Z_\delta(q) = (1 - \varphi_\delta(f(q)))Y(q) + \varphi_\delta(f(q))X(q),$$

where $\varphi_\delta(t) = \varphi(\frac{t}{\delta})$.

Let's see how this works with an example:

Example 7. Consider the following Filippov system:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x + \operatorname{sgn}(y) \cdot (x^4 + 3x^3 + 2x^2 - 2x + 1). \end{aligned} \tag{2.3}$$

Define

$$g(t) = \begin{cases} e^{-\frac{1}{t}} & \text{if} \quad t > 0, \\ 0 & \text{if} \quad t \leqslant 0, \end{cases}$$

and let $\varphi(t) = \dfrac{g(t)}{g(t) + g(1-t)}$. Notice that $\varphi(t) = 0$ for $t \leqslant 0$, $\varphi(t) = 1$ for $t \geqslant 1$ and $\varphi'(t) > 0$ for $t \in (0,1)$; moreover φ is smooth since g is smooth and the denominator doesn't vanish at any point, hence φ is a transition function.

Then the regularization of system (2.3), considering that X is the vector field when $y > 0$ and Y is when $y < 0$, will be

$$\begin{aligned} \dot{x} &= y + \varepsilon \cdot (2\varphi_\delta(y) - 1)p(x), \\ \dot{y} &= -x, \end{aligned} \tag{2.4}$$

where $\varphi_\delta(t) = \varphi(\dfrac{t}{\delta})$, and $p(x) = x^4 + 3x^3 + 2x^2 - 2x + 1$.

Note that $2\varphi_\delta(y) - 1$ approximates the sign function when $\delta \to 0$, since $2\varphi_\delta(y) - 1 = -1, \forall\delta$ if $y < 0$ and $\lim\limits_{\delta \to 0} 2e^{-\frac{\delta}{y}} - 1 = 1$ for $y > 0$. This is also illustrated on the figures 16 and 17.

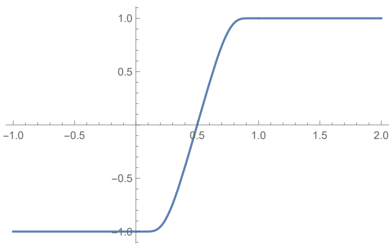


Figure 16 – $2\varphi_\delta(y) - 1$ for $\delta = 1$

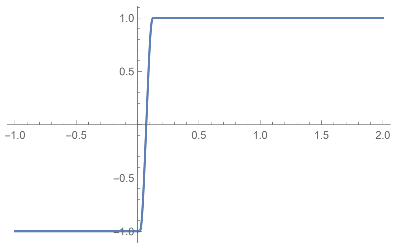
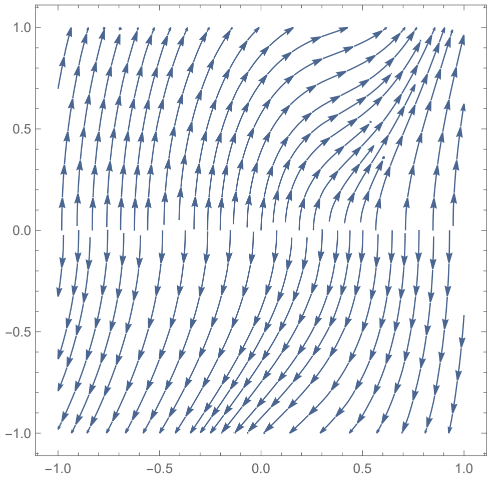


Figure 17 – $2\varphi_\delta(y) - 1$ for a small δ

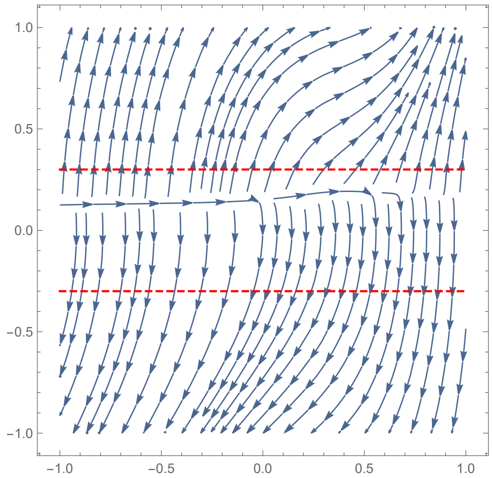
As expected, this approximation yields a regular phase portrait that approximates itself to the phase portrait of the discontinuous system; see figures 18a and 18b.

2.3 Limit cycles on piecewise smooth systems

The definition of limit cycles is one of the most important concepts of smooth dynamical systems that we wish to generalize to the piecewise smooth case. The study of bifurcations in general, and the existence of limit cycles in particular, on non-smooth systems is very important in real life applications, such as models in engineering and physics involving friction — check reference (MAKARENKOV; LAMB, 2012) for some examples. Furthermore, determining the maximum number of the limit cycles of a planar polynomial system is known to be part of Hilbert's 16th problem, which — due to it's generality — has been a recurrent topic for several studies, focusing in more specific families of systems. In this section we will comment some recent study in this field; but, first, let's define what *cycle* means in this new context.



(a)



(b)

Figure 18 – Discontinuous system (a) and regularized system (b), the traced line shows the interval $(-\delta, \delta)$, $\delta = 0.3$

.

Definition 12. We define three types of what we may call a *Cycle*:

1. A *regular periodic orbit* is a regular orbit $\alpha = \varphi_Z(t, p) : t \in \mathbb{R}$ that satisfies $\varphi_Z(t + T, p) = \varphi_Z(t, p)$ for some $T > 0$;

2. A *sliding periodic orbit* is a sliding orbit $\alpha = \varphi_Z(t, p) : t \in \mathbb{R}$ that satisfies $\varphi_Z(t + T, p) = \varphi_Z(t, p)$ for some $T > 0$, note that it only occurs when the whole Σ is a periodic orbit;

3. A *periodic cycle* is the closure of a finite set of pieces of orbits $\alpha_1, ...\alpha_n$ combining pieces of sliding orbits α_{2k} and maximal regular orbits α_{2k+1} such that the departing and arriving points of α_i belong to the closures of α_{i-1} and α_{i+1}, respectively.

On the cases presented in this section, as well as on chapters 3 and 4 of this dissertation, when dealing with limit cycles, we will be referring specifically to regular periodic orbits; this will be guaranteed by hypotheses on the local behavior of the system that exclude the cases involving sliding dynamics.

There are many approaches to study the maximum number of limit cycles in piecewise smooth differential equations; when regarding the family of studied systems, it is expected that the simplest case is formed by piecewise linear systems. In (HUAN; YANG, 2012), the authors studied the case of planar piecewise linear differential systems with two regions sharing the same equilibrium. Using the Poincaré map induced by the discontinuity, the authors make a very complete analysis of the cases where there are at most 2 limit cycles by splitting the original problem in cases according to the parameters yielded from the construction of what they call the full Poincaré map — which is obtained by composing the right and left Poincaré map.

Further in this paper, the authors introduce a specific system

$$\dot{X} = \begin{cases} A^+X, & \text{if } x \geqslant 1, \\ A^-X, & \text{if } x < 1, \end{cases}$$

with

$$A^+ = \begin{pmatrix} \frac{19}{500} & -\frac{1}{10} \\ \frac{1}{10} & \frac{19}{500} \end{pmatrix}, \; A^- = \begin{pmatrix} 1 & -5 \\ \frac{377}{1000} & -\frac{13}{10} \end{pmatrix} \text{ and } X = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Using numerical methods, the authors show through this example that it is possible to obtain a piecewise linear system with a straight line of discontinuity with 3 limit cycles — proving to be untrue a previous conjecture that established the maximum number of limit cycles in such type of systems being 2. A formal proof of the existence and respective stability of these 3 limit cycles was later presented in the paper (LLIBRE; PONCE, 2012). As far as we know, the question of 3 being the maximum number of limit cycles in piecewise linear systems with a straight line of discontinuity still remains open; although the reference (LI; LLIBRE, 2019) provides us a very detailed description of different configurations of such systems and their respective lower bounds for the maximum number of limit cycles.

However, if we drop the assumption that the discontinuity is a straight line, then three is not a maximum number of limit cycles for piecewise linear systems with two regions; in (BRAGA; MELLO, 2014) the authors prove the existence of such systems having four, five, six and seven limit cycles. From this, they conjectured that, for any $n \in \mathbb{N}$, there should exist a system with theses specifications having exactly n limit cycles. This conjecture was proved to be true in the paper (NOVAES; PONCE, 2015) by introducing perturbations to the discontinuity of the system:

$$\dot{X} = \begin{cases} A^+X, & \text{if } H(X) < 0, \\ A^-X, & \text{if } H(X) \geqslant 0, \end{cases} \tag{2.5}$$

where $A^{\pm}$ is given by the following normal form for some $\gamma > 0$:

$$A^{\pm} = \begin{pmatrix} \pm 2\gamma & -1 \\ \gamma^2 + 1 & 0 \end{pmatrix},$$

and H is defined as

$$H(X) = \begin{cases} x, & \text{if } y \leqslant 0, \\ x - h(y), & \text{if } y > 0, \end{cases}$$

for a C^1-function $h(y)$ with $h(0) = 0$ fulfilling the following hypotheses for $y > 0$

1. $|h(y)| < y/\gamma$;

2. $h(y)(2\gamma - (1+\gamma^2)h'(y)) < y$;

3. $h(y)(2\gamma(1+\gamma^2)h'(y)) > -y$.

Assuming all the above definitions and hypotheses, the authors proved that for a positive real number $y*$ there exists a periodic solution of system (2.5) passing through $(h(y*), y*)$ if and only if $h(y*) = 0$, and, in this case, it would cut the y-axis at the points $(0, y*)$ and $(0, -e^{-\gamma\pi}y*)$. The proof of the Braga-Mello conjecture comes, then, as a corollary taking $0 < \gamma < \sqrt{3/13}$ and

$$h(y) = \frac{2\gamma}{(\gamma^2+1)\pi}\begin{cases} 1 - \cos \pi y & \text{if } 0 \leqslant y \leqslant 2n+1, \\ 2 & \text{if } y > 2n+1. \end{cases}$$

3 Averaging Theory

3.1 History and Motivations

The theory that we will explore in this chapter is focused on finding periodic solutions of the problem

$$\dot{x} = \varepsilon F(t, x) + \varepsilon^2 R(t, x, \varepsilon). \tag{3.1}$$

Before we introduce formal definitions, theorems and calculations on this, we give a brief presentation of the history and motivations behind the method. For more details on this topic, take a look at the Appendix A of reference (SANDERS; VERHULST; MURDOCK, 2007).

The first known works that built the foundations for what we nowadays call *Averaging Theory* goes back to the 18th century, when perturbation methods for differential equations were studied on an attempt to fill the gap between Newton's theory of gravitation and the new recent astronomic observations at the time. For it became clear that the dynamics of the solar system were not accurately explained by successive two-body motions, the effects of other objects - such as satellites, large planets and other effects - started to be taken into account in the dynamics between the sun and a given planet. These considerations lead to the reformulation of a perturbed two-body motion problem, which didn't have an available solution by then.

In the first half of the 18th century, the first attempts of solving such problem took place with numerical methods involving the calculation of increments of position and velocity within small intervals of time. In the second half of the century, the works of Clairaut, Lagrange and Laplace brought to sight some new ideas on the topic. Despite not being possible to establish a priority order of who did what first, it is known that Clairaut focused on the solution of a particular problem, while Lagrange did some progress on extending and generalizing it. Using our notation, what Lagrange did was to expand the function $F(t, x)$ by what is now known as the Fourier series, and then keep only the first time-independent term of the expansion to yield a new equation.

Throughout the 19th and 20th century, some improvement of these approaches to perturbation theory and averaging can be seen in the works of Jacobi, Poincaré and Van der Pol. In particular, Van der Pol was concerned with studying nonlinear oscillations, which lead to, more specifically, finding an approximate solution of the system

$$\ddot{x} + x = \varepsilon \cdot (1 - x^2)\dot{x}.$$

The approach given by him to this was based on Lagrange's variation of constants; introducing the transformation $x \mapsto r \cdot \sin(t+\phi)$, the equation can be written as

$$\frac{dr^2}{dt} = \varepsilon \cdot r^2(1 - \frac{r^2}{4}) + ...$$

which he then solved by omitting the higher order terms.

Until the early 20th century, these techniques were applied to many works regardless of the lack of a formal proof of their asymptotic validity. The first proof in that direction was then given by Fatou; his proof assumes periodicity with respect to the time variable and continuous differentiability of the vector field, which resembles the conditions we will assume on Theorem 9 in the next section. Fatou used the Picard-Lindelöf iteration in order to obtain estimations for $\mathcal{O}(\varepsilon)$ on *time scale* $\frac{1}{\varepsilon}$.

3.2 Averaging on Smooth Systems

We start this section introducing the Gronwall Inequality and providing a more formal definition of some concepts regarding approximations; then we will state the First-Order Averaging Theorem for the periodic case, which will be proved based on the classical proof, which can be found in Chapter 2 from (SANDERS; VERHULST; MURDOCK, 2007).

Lemma 1. (The Gronwall Inequality) *Let φ and β be continuous functions with $\beta(t) > 0$. Suppose that for $t_0 \leqslant t \leqslant t_0 + T$ the following inequality holds*

$$\varphi(t) \leqslant \alpha + \int_{t_0}^{t} \beta(s)\varphi(s)ds,$$

then

$$\varphi(t) \leqslant \alpha exp\left[\int_{t_0}^{t} \beta(s)ds\right].$$

Definition 13. A function $\delta(\varepsilon)$ is said to be an *order function* if $\delta(\varepsilon)$ is continuous and positive in $(0, \varepsilon_0]$ and if $\lim_{\varepsilon\to 0^+} \delta(\varepsilon)$ exists.

Notice that $\{\varepsilon^n\}_{n=1}^{\infty}$ is a set of order functions; in particular, these are the ones in which we are interested.

Definition 14. Let $\varphi(t,\varepsilon)$ be a function defined for $\varepsilon > 0$ and for $t \in I_\varepsilon$. We say that $\varphi(t,\varepsilon) = \mathcal{O}(\delta(\varepsilon))$ for $\varepsilon \to 0^+$ if there exist positive constants ε_0 and k such that $||\varphi(t,\varepsilon)|| \leqslant k|\delta(\varepsilon)|$ for all $t \in I_\varepsilon$ and $0 < \varepsilon < \varepsilon_0$.

For instance, we have that $\varepsilon^n = \mathcal{O}(\varepsilon^m)$ for $\varepsilon \to 0^+$ if $n > m$; for this reason, whenever we calculate the Taylor expansion series of order n of a function $f(x)$ around the origin, we write $+\mathcal{O}(x^n)$ to denote the sum of the higher order terms. With these definitions, we can formulate our first theorem of Averaging.

Theorem 9. (First-order Averaging Theorem, (LLIBRE; TEIXEIRA, 2014)) *Consider the following system*

$$\dot{x} = \varepsilon F(t,x) + \varepsilon^2 R(t,x,\varepsilon), \tag{3.2}$$

where $x \in D \subset \mathbb{R}^n$, D is bounded, connected and open, $t \in [0,\infty)$, $\varepsilon \in (0,\varepsilon_0)$ and F and R are T-periodic on the variable t (i.e. $F(t+T,x) = F(t,x)$ and $R(t+T,x,\varepsilon) = R(t,x,\varepsilon)$, for some $T > 0$). Define:

$$\begin{aligned} F_0(z) &= \frac{1}{T}\int_0^T F(t,z)dt, \\ \dot{z} &= \varepsilon F_0(z). \end{aligned}$$

Suppose that F is C^2, R is C^1 and $z(t) \in D$ for $t \in [0, 1/\varepsilon]$. Then:

1. *For $t \in [0, 1/\varepsilon]$, we have $||x(t) - z(t)|| = \mathcal{O}(\varepsilon)$ as $\varepsilon \to 0^+$.*
2. *If $a \neq 0$ is such that $F_0(a) = 0$ and $det(D_z F(a)) \neq 0$, then there exists a periodic solution $\phi(t,\varepsilon)$ of period T of the system (3.2) which is close to a, i.e. $||\phi(0,\varepsilon) - a|| = \mathcal{O}(\varepsilon)$ as $\varepsilon \to 0$.*

The proof of this theorem will be made through the proof of several lemmas but, first, we introduce two more definitions of which we will make use in this demonstration.

Definition 15. We say that $\varphi_\varepsilon(t) = \mathcal{O}(\delta(\varepsilon))$ as $\varepsilon \to 0^+$ on the *time scale* $\delta(\varepsilon)^{-1}$ if the estimate holds for $0 \leqslant \delta(\varepsilon)t \leqslant L$, where L is a constant independent of ε.

Definition 16. A *near-identity transformation* is a family of transformations of the form

$$x = U(t,y,\varepsilon) = y + \varepsilon \cdot u(t,y,\varepsilon), \tag{3.3}$$

where u is periodic in t with period T and y is the new vector variable that replaces x.

The idea of the proof is to choose u such that this change of variables transforms the original equation 3.2 into the full averaged equation

$$\dot{y} = \varepsilon F_0(y) + \varepsilon^2 R_*(t,y,\varepsilon), \tag{3.4}$$

where R_* is induced by the transformation, from which we obtain the (truncated) averaged equation

$$\dot{z} = \varepsilon F_0(z), \tag{3.5}$$

by deleting this last term. We start proving the following lemma:

Lemma 2. *Let f be a C^1 function and K be a compact subset of its domain, then f is Lipschitz on K.*

Proof: Since K is compact and f is C^1, then $\dot{f}$ exists and is continuous on K, hence f and $\dot{f}$ are both bounded on K, i.e. there exist $M, M' \in \mathbb{R}$ such that $||f(x) - f(y)|| \leqslant M$ and $||\dot{f}(x) - \dot{f}(y)|| \leqslant M'$, for all $x, y \in K$.

Let $\{B_\alpha\}_{\alpha \in A}$ be a cover of K by open balls; since K is compact, there exists a finite subcover of $\{B_\alpha\}_{\alpha \in A}$ of K, let $B_1, ..., B_n$ be such subcover.

Because each B_i is convex, we can apply the Mean Value Inequality. Let x_i and y_i be arbitrary points in B_i, then write $[x_i, y_i] = \{x_i \cdot (1-t) + y_i \cdot t : t \in [0,1]\}$; the Mean Value Inequality guarantees that

$$||f(x_i) - f(y_i)|| \leqslant \sup_{c \in [x_i, y_i]} ||\dot{f}(c)|| \cdot ||x_i - y_i||,$$

but since $\dot{f}$ is bounded

$$||f(x_i) - f(y_i)|| \leqslant M' \cdot ||x_i - y_i||, \forall x_i, y_i \in B_i.$$

Now, let $y \in K \backslash B_i$, then because f is bounded

$$||f(x_i) - f(y)|| \leqslant M = \frac{M}{r_i} \cdot r_i \leqslant \frac{M}{r_i} \cdot ||x_i - y||,$$

where r_i is the radius of the ball B_i. Since there is a finite number of such balls covering K, we can take $\lambda_f = \max\{M', \frac{M}{r_1}, ..., \frac{M}{r_n}\}$; then for any $x, y \in K$ it's true that

$$||f(x) - f(y)|| \leqslant \lambda_f \cdot ||x - y||.$$

Therefore f is Lipschitz on K. $\square$

With this, notice that since F is C^2, u is a smooth function and D is a bounded connected set, then F and u are Lipschitz (on the vector variable) on $\overline{D}$ with constants λ_F and λ_u, respectively. In particular, since they are periodic functions on the time variable, then the Lipschitz condition shall hold for all time. We use the Lipschitz condition of u to prove the existence of its inverse on the following lemma.

Lemma 3. *Let $y \mapsto U(t, y, \varepsilon)$ as in (3.3) be a smooth map depending on t and ε. Suppose $D \subset \mathbb{R}^n$ is a bounded connected open set; then there exists ε_0 such that, $\forall t \in \mathbb{R}$ and for all $0 < \varepsilon < \varepsilon_0$, the map $U|_D : D \to U(D, t, \varepsilon)$ is bijective. Moreover, the inverse mapping has the form*

$$y = V(t, x, \varepsilon) = x + \varepsilon v(t, x, \varepsilon)$$

and is smooth in (t, x, ε).

Proof: To prove that U is one-to-one for a small ε, suppose that $U(y_1, t, \varepsilon) = U(y_2, t, \varepsilon)$ for $0 < \varepsilon < 1/\lambda_u$, then

$$\begin{aligned} y_1 + \varepsilon \cdot u(y_1, t, \varepsilon) &= y_2 + \varepsilon \cdot u(y_2, t, \varepsilon) \\ \Rightarrow ||y_1 - y_2|| &= \varepsilon \cdot ||u(y_1, t, \varepsilon) - u(y_2, t, \varepsilon)|| \leqslant \varepsilon\lambda_u ||y_1 - y_2||, \end{aligned}$$

if $y_1 \neq y_2$, this implies that $\varepsilon\lambda_u \geqslant 1$; however, since $\varepsilon < 1/\lambda_u$, we have that $\varepsilon\lambda_u < 1$, therefore we must have $y_1 = y_2$, hence U is one-to-one in D. This completes the proof of the bijection, since we're restraining the codomain of $U|_D$ to the image of D by U.

To prove the form and the smoothness of the inverse of U, we use the inverse function theorem: since $D_y U(t, y_0, 0)$ is the identity matrix for every y_0, it follows that $U(t, y, \varepsilon)$ is locally invertible for a small ε_{y_0} and in a neighborhood B_{y_0} of y_0, moreover this local inverse is smooth and has the desired form, i.e., $y = x + \varepsilon v(t, x, \varepsilon)$. Let $\{B_y\}_{y \in \mathbb{R}^n}$ be a cover of $\overline{D}$; since $\overline{D}$ is compact, it can be covered by a finite number k of such neighborhoods, which we denote by $B_1, ..., B_k$ and with the respective bounds of ε: $\varepsilon_1, ..., \varepsilon_k$. Then, taking $\varepsilon_0 = \min\{1/\lambda_u, \varepsilon_1, ..., \varepsilon_k\}$, for $\varepsilon < \varepsilon_0$ the local inverses exist, are smooth and have the desired form. By the uniqueness of the inverse, the global inverse - which we've proven to exist, since U is bijective - must coincide with the local inverses on each of these neighborhoods, therefore $V(t, x, \varepsilon)$ is smooth and has the desired form. $\square$

Our next step is to verify if such transformation is indeed what we were looking for.

Lemma 4. *There exist mappings U carrying the original equation (3.2) to the full averaged equation (3.4).*

Proof: Let $\dot{x} = f(t, x, \varepsilon)$ and $\dot{y} = g(t, y, \varepsilon)$ be differential equations on the variables $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$, which are related by the transformation $x = U(t, y, \varepsilon)$. Differentiating both sides from the latter equality gives us, by the chain rule:

$$\dot{x} = \frac{\partial U}{\partial t} + D_y U \cdot \frac{dy}{dt}, \tag{3.6}$$

where $D_y U$ denotes the partial derivative of U with respect to the spatial variable y.

Now, we wish to find U such that f and g yield the equations (3.2) and (3.4), respectively. In order to do that, we replace f, g and U by the desired expressions just to derive an equation for $u(t, y, \varepsilon)$; then we do the inverse, i.e. by showing that such equation can be solved and that it yields the desired result, the Lemma is proved.

So if we were to have $f(t, x, \varepsilon) = F(t, x) + \varepsilon \cdot R(t, x, \varepsilon)$, $g(t, y, \varepsilon)$ and $U(t, y, \varepsilon) = y + \varepsilon u(t, y, \varepsilon)$, the equation (3.6) would turn into

$$\varepsilon F(t, x) + \varepsilon^2 R(x, t, \varepsilon) = \varepsilon \frac{\partial u}{\partial t} + (Id + \varepsilon D_y u) \left(\varepsilon F_0(y) + \varepsilon^2 F_*(t, y, \varepsilon) \right), \tag{3.7}$$

where Id is the identity matrix. Notice that the left-hand side of the equation is on the spatial variable x, whilst the right-hand side is on the variable y; to solve this, notice that since $x = y + \varepsilon u(t, y, \varepsilon)$, if we expand the series of F and R for ε around 0, we have

$$\begin{aligned} F(t, y + \varepsilon u(t, y)) &= F(t, y) + \mathcal{O}(\varepsilon), \\ R(t, y + \varepsilon u(t, y), \varepsilon) &= R(t, y, 0) + \mathcal{O}(\varepsilon), \end{aligned}$$

then, rewriting the equation (3.7):

$$\begin{aligned} \varepsilon F(t, y) + \mathcal{O}(\varepsilon^2) + \varepsilon^2(R(y, t, 0) + \mathcal{O}(\varepsilon)) = \varepsilon \left(\frac{\partial u}{\partial t} + F_0(y) \right) &+ \varepsilon^2(F_*(t, y, \varepsilon) + \\ &D_y u(F_0(y) + \varepsilon F_*(t, y, \varepsilon))), \end{aligned}$$

or simply

$$\varepsilon F(t, y) + \mathcal{O}(\varepsilon^2) = \varepsilon \left(\frac{\partial u}{\partial t} + F_0(y) \right) + \mathcal{O}(\varepsilon^2).$$

Hence, to find $u(t, y, \varepsilon)$, we must solve the following equation:

$$\frac{\partial u}{\partial t} = F(t, y) - F_0(y).$$

In fact, this yields

$$u(y, t) = \int_0^t F(y, s) - F_0(y) ds + k(y). \tag{3.8}$$

We easily check that — for any $k(y)$ — the function (3.8) fulfills the requirements for U to be our near-identity transformation. First, notice that $u(t, y, \varepsilon)$ is T-periodic for any $k(y)$:

$$\begin{aligned} u(T, y, \varepsilon) &= \int_0^T F(y, s) - F_0(y) ds + k(y) \\ &= \int_0^T F(y, s) ds - \int_0^T F_0(y) ds + k(y) \\ &= T \cdot F_0(y) - T \cdot F_0(y) + k(y) \\ &= k(y) = u(0, y, \varepsilon). \end{aligned}$$

Moreover, replacing $U(t, y, \varepsilon) = y + \varepsilon u(t, y, \varepsilon)$ in the equation (3.6):

$$\begin{aligned} &\varepsilon F(t, y) + \varepsilon^2 R(t, y, 0) + \mathcal{O}(\varepsilon^2) = \varepsilon(F(t, y) - F_0(y)) + (Id + \varepsilon D_y u) \cdot \frac{dy}{dt} \\ &\Rightarrow (Id + \varepsilon D_y u) \cdot \frac{dy}{dt} = \varepsilon F_0(y) + \varepsilon^2 R(t, y, 0) + \mathcal{O}(\varepsilon^2). \end{aligned}$$

Since $u(t,y,\varepsilon)$ is a smooth function, then $D_y u$ is also smooth; hence, by Lemma 3, $(Id+\varepsilon D_y u)$ is locally invertible and $(Id+\varepsilon D_y u)^{-1}=(Id+\varepsilon T)$ for some T. Then:

$$\begin{aligned}\dot{y} &= (Id+\varepsilon T)\varepsilon F_0(y)+\varepsilon^2 R(t,y,0)+\mathcal{O}(\varepsilon^2)\\ &= \varepsilon F_0(y)+\varepsilon^2(R(t,y,0)+T\cdot F_0(y)+\varepsilon TR(t,y,0)))+\mathcal{O}(\varepsilon^2)\\ &= \varepsilon F_0(y)+\varepsilon^2 F_*(t,y,\varepsilon),\end{aligned}$$

which ends the proof of this lemma. □

Now, let $x(t,\varepsilon)$ denote the solution of equation (3.2) with initial condition $x(0,\varepsilon)=a$, $y(t,\varepsilon)$ the solution of the full averaged equation (3.4) with initial condition $y(0,\varepsilon)=V(a,0,\varepsilon)=a+\varepsilon b(\varepsilon)$ and $z(t,\varepsilon)$ the solution of the truncated averaged equation (3.5) with $z(0,\varepsilon)=a$. The next two lemmas will prove the order of the differences yield by these approximations.

Lemma 5. *The solutions $y(t,\varepsilon)$ and $z(t,\varepsilon)$ satisfy the estimate*

$$||y(t,\varepsilon)-z(t,\varepsilon)||=\mathcal{O}(\varepsilon)$$

for time scale $\mathcal{O}(1/\varepsilon)$.

Proof: Expanding the expressions for $y(t,\varepsilon)$ and $z(t,\varepsilon)$:

$$\begin{aligned}y(t,\varepsilon) &= a+\varepsilon b(\varepsilon)+\int_0^t[\varepsilon F_0(y(s,\varepsilon))+\varepsilon^2 F_*(y(s,\varepsilon),s,\varepsilon)]ds\\ z(t,\varepsilon) &= a+\int_0^t \varepsilon F_0(z(s,\varepsilon))ds,\end{aligned}$$

then

$$\begin{aligned}&||y(t,\varepsilon)-z(t,\varepsilon)|| = \left|\left|\varepsilon b(\varepsilon)+\int_0^t[\varepsilon F_0(y(s,\varepsilon))-\varepsilon F_0(z(s,\varepsilon))+\varepsilon^2 F_*(y(s,\varepsilon),s,\varepsilon)]ds\right|\right|\\ &\leqslant ||\varepsilon b(\varepsilon)||+\varepsilon\int_0^t ||F_0(y(s,\varepsilon))-F_0(z(s,\varepsilon))||ds+\int_0^t ||\varepsilon^2 F_*(y(s,\varepsilon),s,\varepsilon)||ds\\ &\leqslant \varepsilon||b(\varepsilon)||+\varepsilon\lambda_F\int_0^t ||y(s,\varepsilon)-z(s,\varepsilon)||ds+\varepsilon^2 M_* t,\end{aligned}$$

where M_* is a bound for F_* and the Lipschitz constant λ_F is yield from

$$\begin{aligned}||F_0(y(s,\varepsilon))-F_0(z(s,\varepsilon))|| &= \left|\left|\frac{1}{T}\int_0^T(F(y(s,\varepsilon),r)-F(z(s,\varepsilon),r))dr\right|\right|\\ &\leqslant \frac{1}{T}\int_0^T ||F(y(s,\varepsilon),r)-F(z(s,\varepsilon),r)||dr\\ &\leqslant \frac{1}{T}\int_0^T \lambda_F||y(s,\varepsilon)-z(s,\varepsilon)||dr\\ &= \lambda_F||y(s,\varepsilon)-z(s,\varepsilon)||.\end{aligned}$$

The end of the proof follows from the Gronwall Inequality: taking $\alpha = \dfrac{\varepsilon^2 M_*}{\varepsilon\lambda_F} + \varepsilon||b(\varepsilon)||$, $\beta = \varepsilon\lambda_F$ and $\varphi(t) = ||y(t,\varepsilon) - z(t,\varepsilon)|| + \dfrac{\varepsilon^2 M_*}{\varepsilon\lambda_F}$, we have

$$||y(t,\varepsilon) - z(t,\varepsilon)|| + \frac{\varepsilon^2 M_*}{\varepsilon\lambda_F} \leqslant \varepsilon||b(\varepsilon)|| + \varepsilon\lambda_F \int_0^t ||y(s,\varepsilon) - z(s,\varepsilon)|| + \frac{\varepsilon^2 M_*}{\varepsilon\lambda_F} ds + \frac{\varepsilon^2 M_*}{\varepsilon\lambda_F},$$

then

$$\begin{aligned} ||y(t,\varepsilon) - z(t,\varepsilon)|| &\leqslant \left(\frac{\varepsilon^2 M_*}{\varepsilon\lambda_F} + \varepsilon||b(\varepsilon)||\right) \cdot e^{\varepsilon\lambda_F t} - \frac{\varepsilon^2 M_*}{\varepsilon\lambda_F} \\ &= \varepsilon\left(\frac{M_*}{\lambda_F} + ||b(\varepsilon)||\right) \cdot e^{\varepsilon\lambda_F t} - \frac{\varepsilon M_*}{\lambda_F} \\ &= \mathcal{O}(\varepsilon). \end{aligned}$$

□

Lemma 6. *The solutions $x(t,\varepsilon)$ and $z(t,\varepsilon)$ satisfy the estimate*

$$||x(t,\varepsilon) - z(t,\varepsilon)|| = \mathcal{O}(\varepsilon),$$

for time scale $\mathcal{O}(1/\varepsilon)$.

Proof: Applying the triangle inequality:

$$||x(t,\varepsilon) - z(t,\varepsilon)|| \leqslant ||x(t,\varepsilon) - y(t,\varepsilon)|| + ||y(t,\varepsilon) - z(t,\varepsilon)||.$$

By lemmas 3 and 4, $||x(t,\varepsilon) - y(t,\varepsilon)||$ is $\mathcal{O}(\varepsilon)$ for all time and, by lemma 5, $||y(t,\varepsilon) - z(t,\varepsilon)||$ is $\mathcal{O}(\varepsilon)$ for time $\mathcal{O}(1/\varepsilon)$; therefore $||x(t,\varepsilon) - z(t,\varepsilon)|| = \mathcal{O}(\varepsilon)$ for time $\mathcal{O}(1/\varepsilon)$. □

This completes the proof of part 1 of Theorem 9. To complete the proof of part 2 from the Theorem 9, we need to show the following lemma:

Lemma 7. *If $a \neq 0$ is such that $F_0(z)$ and $det(D_z F_0(a)) \neq 0$, then there exists a solution of the system (3.2) near a that is $T-$periodic.*

Proof: Let $a \in \mathbb{R}^n$ be such that $F_0(a) = 0$ and $det(D_z F_0(a)) \neq 0$, and let $x(t, a_\varepsilon, \varepsilon)$ and $z(t, a_\varepsilon, \varepsilon)$ be the solutions of (3.2) and (3.5), respectively, such that $x(0, a_\varepsilon, \varepsilon) = a_\varepsilon$ and $z(0, a_\varepsilon, \varepsilon) = a_\varepsilon$, where $a_\varepsilon \in B_{\varepsilon_0}(a)$ for some $\varepsilon_0 > 0$. Then:

$$\begin{aligned} x(t, a_\varepsilon, \varepsilon) &= x(0, a_\varepsilon, \varepsilon) + \int_0^t \varepsilon F(s, x(s, a_\varepsilon, \varepsilon) + \varepsilon^2 R(s, x(s, a_\varepsilon, \varepsilon), \varepsilon) ds \\ &= x(0, a_\varepsilon, \varepsilon) + \varepsilon \int_0^t F(s, x(s, a_\varepsilon, \varepsilon) ds + \mathcal{O}(\varepsilon^2), \end{aligned}$$

since $F(t,x)$ is Lipschitz, then by Lemma 6

$$||F(s,x(s,a_\varepsilon,\varepsilon)) - F(s,z(s,a_\varepsilon,\varepsilon))|| \leqslant \lambda_F ||x(s,a_\varepsilon,\varepsilon)) - z(s,a_\varepsilon,\varepsilon)|| = \mathcal{O}(\varepsilon),$$

hence

$$x(t,a_\varepsilon,\varepsilon) = x(0,a_\varepsilon,\varepsilon) + \varepsilon\int_0^t F(s,z(s,a_\varepsilon,\varepsilon)ds + \mathcal{O}(\varepsilon^2),$$

now if $F_0(a_\varepsilon) = 0$, then $z(s,a_\varepsilon,\varepsilon) = a_\varepsilon$ for all s, which implies that

$$x(T,a_\varepsilon,\varepsilon) - x(0,a_\varepsilon,\varepsilon) = \varepsilon F_0(a_\varepsilon) + \mathcal{O}(\varepsilon^2) = \mathcal{O}(\varepsilon^2),$$

then, for $|\varepsilon|$ sufficiently small, $||x(T,a_\varepsilon,\varepsilon) - x(0,a_\varepsilon,\varepsilon)|| = 0$, which implies that $x(t,a_\varepsilon,\varepsilon)$ is $T-$periodic. However, $\det D_z F_0(a) \neq 0$ implies that there's a neighborhood where a is the only zero of $F_0(z)$, therefore $a_\varepsilon \to a$ when $\varepsilon \to 0$. □

Example 8. (Van der Pol's oscillator) Consider the Van der Pol's equation

$$\ddot{x} + x = \varepsilon \cdot (1 - x^2)\dot{x}, \tag{3.9}$$

introducing the change of variables $y = \dot{x}$, we get the equivalent system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x + \varepsilon(1 - x^2)y. \end{aligned}$$

As we wish to have periodic time functions to average, it makes sense to rewrite this in polar coordinates and then take the angle as the time variable; in fact, applying the coordinate change

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \cdot \begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix}$$

and isolating $\dot{r}$ and $\dot{\theta}$, the equation (3.9) turns into the system

$$\begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \varepsilon(r\sin^2\theta - r^3\cos^2\theta\sin^2\theta) \\ -1 + \varepsilon(\sin\theta\cos\theta - r^2\cos^3\theta\sin\theta) \end{pmatrix}.$$

Now, taking θ to be the new time variable and expanding the expression for ε around $\varepsilon_0 = 0$ using the Taylor series, we get:

$$\frac{dr}{d\theta} = \varepsilon(r^3\cos^2\theta\sin^2\theta - r\sin^2\theta) + \mathcal{O}(\varepsilon^2). \tag{3.10}$$

Notice that, taking $F(\theta,r) = r^3\cos^2\theta\sin^2\theta - r\sin^2\theta$ and $\varepsilon^2 R(\theta,r,\varepsilon) = \mathcal{O}(\varepsilon^2)$, the functions satisfy the hypotheses from Theorem 9, so this is a system we can average. Hence, calculating $F_0(r)$:

$$\begin{aligned} F_0(r) &= \frac{1}{2\pi}\int_0^{2\pi} r^3\cos^2\theta\sin^2\theta - r\sin^2\theta d\theta \\ &= \frac{1}{8}r(r^2 - 4). \end{aligned}$$

Therefore, the averaged system will be:

$$\dot{\overline{r}} = \varepsilon\frac{1}{8}\overline{r}(\overline{r}^2 - 4) \tag{3.11}$$

and, according to Theorem 9, this indicates that the system (3.10) has a periodic solution around $r_0 = 2$ for a sufficiently small ε. Indeed, the general solution of system (3.11) is

$$\overline{r}(t) = \frac{2}{\sqrt{1 + e^{t\varepsilon}(\frac{4}{r_0^2} - 1)}},$$

where r_0 is the initial value $\overline{r}(0)$; then the solution of the equation (3.10) will be

$$r(t) = \frac{2}{\sqrt{1 + e^{t\varepsilon}(\frac{4}{r_0^2} - 1)}} + \mathcal{O}(\varepsilon)$$

and returning to the original coordinates, we get the following solution for equation (3.9):

$$x(t) = \frac{2}{\sqrt{1 + e^{t\varepsilon}(\frac{4}{r_0^2} - 1)}} \cdot \cos t + \mathcal{O}(\varepsilon).$$

Since $x(t) = r(t)\cos t$, $x(0) = r_0 = 2$ and $\dot{x}(0) = 0$, hence the solution of the initial value problem will be

$$x(t) = 2 \cdot \cos t + \mathcal{O}(\varepsilon)$$

and it's easy to see that this will be a periodic orbit for a sufficiently small ε. In fact, taking $\varepsilon = \dfrac{1}{10}$ and solving the original system numerically using *Wolfram Mathematica*, we easily detect the limit cycle near $(2, 0)$. On Figure 19, one can see that the trajectory starting on point $(2.2, 0)$ is being attracted to the origin, while on Figure 20 the trajectory of the point $(1.8, 0)$ is being repelled from the origin; therefore, it follows from Theorem 4 that there must be a limit cycle between those two points.

3.3 Averaging on Non-smooth Systems

The next step in developing a theory for studying limit cycles on non-smooth systems is to adapt the averaging methods to these type of systems. The first attempts to do this were through regularization methods, which is the approach on many works from our references, like (LLIBRE; TEIXEIRA, 2014) and (MARTINS; MEREU, 2014); the validity of this type of averaging is systematically proved in reference (LLIBRE; NOVAES; TEIXEIRA, 2015). However, another theory for averaging on piecewise smooth systems has also been developed without the need of passing through the process of regularization; we have chosen this as our main approach for it yields simpler calculations, allowing us to focus on other aspects of our work.

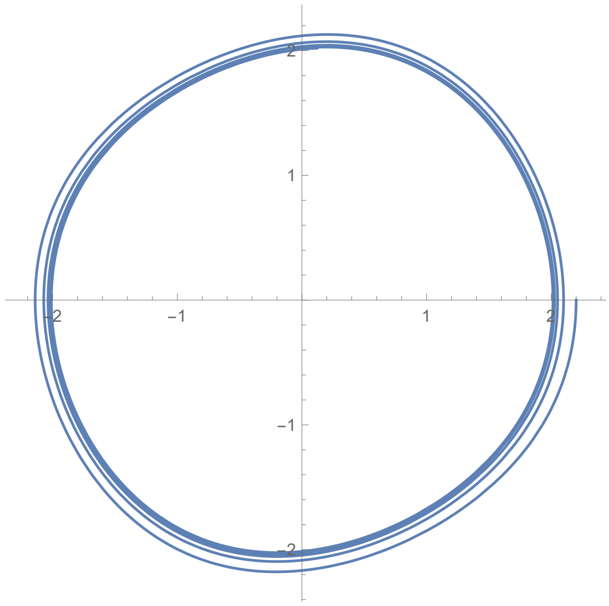


Figure 19 – Solution of system (3.9) with initial value $x(0) = 2.2$, $\dot{x} = 0$ and $\varepsilon = \dfrac{1}{10}$.

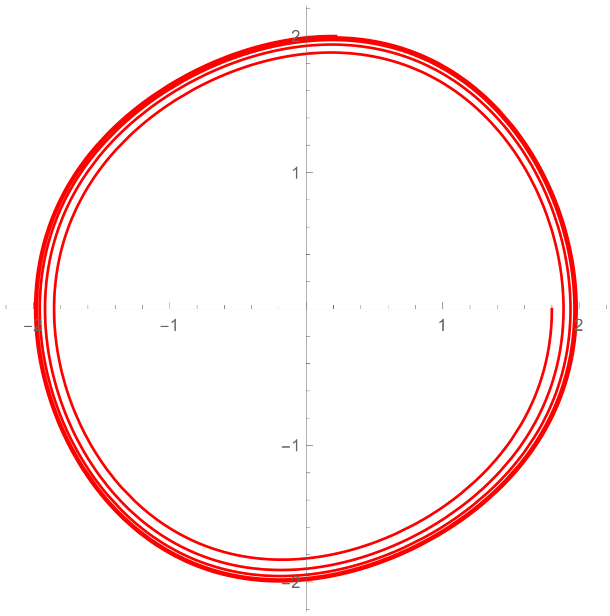


Figure 20 – Solution of system (3.9) with initial value $x(0) = 1.8$, $\dot{x} = 0$ and $\varepsilon = \dfrac{1}{10}$.

The main reference on which we will rely is paper (LLIBRE; MEREU; NOVAES, 2015). On their work, the periodic averaging methods of orders 1 and 2 are developed for systems with arbitrary number of discontinuities. Here, however, we shall focus on the periodic averaging of first order for systems with one hypersurface of discontinuity (in particular, we will further handle only planar systems with a line of discontinuity); despite that, we remark that the calculations of this section can quite easily be extended to the case of M-piecewise smooth systems. Before bringing up the main theorem of this section, we present briefly the definition the Brouwer degree function, which will be needed further when proving the averaging theorem.

Theorem 10. *Let $X = Y = \mathbb{R}^n$. For bounded open sets $V \subset X$, consider continuous maps $f : \overline{V} \to Y$ and points $y_0 \in Y$ such that $y_0 \notin f(\partial V)$, where ∂V denotes the boundary of V. Then to each triple (f, V, y_0) , there corresponds an integer $d_B(f, V, y_0)$ having the following properties:*

1. *If $d_B(f, V, y_0) \neq 0$, then $y_0 \in f(V)$. If $f_0 : X \to Y$ is the identity map, then for every bounded open set and $\forall y_0 \in V$, we have:*

$$d_B(f_0|_V, V, y_0) = \pm 1.$$

2. *(Additivity) If V_1 and V_2 are a pair of disjoint open subsets of V such that*

$$y_0 \notin f(\overline{V} \backslash (V_1 \cup V_2)),$$

then

$$d_B(f_0, V, y_0) = d_B(f_0, V_1, y_0) + d_B(f_0, V_2, y_0).$$

3. *(Invariance under homotopy) Consider a continuous homotopy $\{f_t : 0 \leqslant t \leqslant 1\}$ of maps from $\overline{V}$ to Y. Let $\{y_t : 0 \leqslant t \leqslant 1\}$ be a continuous curve in Y such that $y_t \notin f_t(\partial V)$ for any $t \in [0, 1]$, then $d_B(f_t, V, y_t)$ is constant for every $t \in [0, 1]$.*

Moreover, the degree function $d_B(f, V, y_0)$ is uniquely determined by these conditions.

The Theorem 10 guarantees the existence and uniqueness of the degree function $d_B(f, V, y_0))$; this is presented as two theorems on Browder's paper, (BROWDER, 1983), where the reader can also find the corresponding proofs.

Now, recall from chapter 2 that the discontinuity hypersurface Σ splits the space into two regions, namely Σ^+ and Σ^-, over which we will define the smooth pieces of the equation. The differential system that we're interested on averaging will have the following form

$$\dot{x}(t) = \varepsilon F(t, x) + \varepsilon^2 R(t, x, \varepsilon), \tag{3.12}$$

with

$$F(t, x) = \begin{cases} F^1(t, x), & \text{if } x \in S_1, \\ F^2(t, x), & \text{if } x \in S_2, \end{cases}$$

$$R(t, x, \varepsilon) = \begin{cases} R^1(t, x, \varepsilon), & \text{if } x \in S_1, \\ R^2(t, x, \varepsilon), & \text{if } x \in S_2, \end{cases}$$

where $F^{1,2} : I \times S_{1,2} \to \mathbb{R}^n$, $R^{1,2} : I \times S_{1,2} \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ and $S_{1,2} = D \cap \Sigma^{\pm}$. Or, alternatively, using the function $\mathcal{X}(A) = \begin{cases} 1, \text{ if } x \in A \\ 0, \text{ if } x \notin A \end{cases}$, we can rewrite those as:

$$F(t,x) = \mathcal{X}_{S_1} F^1(t,x) + \mathcal{X}_{S_2} F^2(t,x),$$
$$R(t,x,\varepsilon) = \mathcal{X}_{S_1} R^1(t,x,\varepsilon) + \mathcal{X}_{S_2} R^2(t,x,\varepsilon).$$

Then, if $F^{1,2}$ and $R^{1,2}$ are T-periodic functions, we can define the averaged function $F_0(z)$ as

$$F_0(z) = \frac{1}{T}\int_0^T F(t,z)dt, \tag{3.13}$$

which is exactly how we've defined when treating the smooth case; the only difference here is that, within the calculations, there will be the need of splitting this integral whenever we reach a discontinuity.

As we have seen before in chapter 2, when generalizing concepts from smooth differential equations to the piecewise smooth case, we need to pay attention to the discontinuity Σ; more specifically, we must take special care with what happens along the sliding region Σ^s, for the dynamics on this region is more complex that just connecting the flows defined on Σ^+ and Σ^- — like we do on Σ^c. With that in mind, we add an extra hypothesis to handle the case of averaging on piecewise smooth systems, which will basically discard these more problematic cases.

Theorem 11. (First order averaging for discontinuous systems, (LLIBRE; MEREU; NOVAES, 2015)) *Assuming the following hypothesis:*

H1 There exists an open bounded set $C \subset D$ such that, for each $z \in \overline{C}$, the curve $\{(t,z) : t \in I = \mathbb{S}^1 = \mathbb{R}/T\}$ reaches transversely the set Σ and only at generic points of discontinuity;

H2 For $j = 1, 2$, the continuous functions F^j and R^j are locally Lipschitz with respect to x, and T-periodic with respect to the time variable t;

H3 For $a \in C$ with $F_0(a) = 0$, there exists a neighborhood $U \subset C$ of a such that $F_0(z) \neq 0$ for all $z \in \overline{U}\backslash\{a\}$ and $d_B(F_0, U, 0) \neq 0$.

Then for $|\varepsilon| \neq 0$ sufficiently small, there exists a T-periodic solution $x(t,\varepsilon)$ of system (3.12) such that $x(0,\varepsilon) \to a$ as $\varepsilon \to 0$.

The proof of this theorem will be made through the proof of 4 lemmas, based on how it is done in (LLIBRE; MEREU; NOVAES, 2015); we start with an immediate consequence of hypothesis *H1*:

Lemma 8. *The hypothesis* H1 *implies that, for $\varepsilon \neq 0$ sufficiently small, every solution of* (3.12) *starting in $\overline{C}$ reaches the set Σ only at Σ^c.*

Proof: For $z \in \overline{C}$, let $p = (t_0, z) \in \Sigma$ be a generic point of discontinuity; then there is a neighborhood V_p of p such that $\Sigma_p = V_p \cap \Sigma$ is an embedded C^k-hypersurface of $\mathbb{S}^1 \times \mathbb{R}^n$, $k \geqslant 1$. This implies that Σ_p can be locally described by the inverse image of a regular value C^k function $h_p : V_p \to \mathbb{R}$; i.e. we can write

$$\Sigma_p \cap \tilde{V}_p = h_p^{-1}(0) \cap \Sigma,$$

for an open set $p \in \tilde{V}_p \subset V_p$. Then, we describe the system locally as:

$$\dot{x} = \begin{cases} F_\varepsilon^+(t,x) = \varepsilon F^1(t,x) + \varepsilon^2 R^1(t,x,\varepsilon), \text{ if } h_p(t,x) > 0, \\ F_\varepsilon^-(t,x) = \varepsilon F^2(t,x) + \varepsilon^2 R^2(t,x,\varepsilon), \text{ if } h_p(t,x) < 0. \end{cases}$$

Recall from chapter 2 that

$$\begin{aligned} &p \in \Sigma^c \text{ if } F_\varepsilon^+ h_p(p) \cdot F_\varepsilon^- h_p(p) > 0, \\ &p \in \Sigma^s \cup \Sigma^e \text{ if } F_\varepsilon^+ h_p(p) \cdot F_\varepsilon^- h_p(p) < 0, \end{aligned}$$

where $F_\varepsilon^\pm h_p(p) = F_\varepsilon^\pm(p) \cdot \nabla h_p(p)$ denote the Lie derivative of h_p with respect to $F_\varepsilon^\pm$ at the point p. Calculating $F_\varepsilon^+ h_p(p) \cdot F_\varepsilon^- h_p(p)$:

$$\begin{aligned} F_\varepsilon^+ h_p(p) &= \left(1, \varepsilon F_1^1 + \varepsilon^2 R_1^1, \ldots, \varepsilon F_n^1 + \varepsilon^2 R_n^1\right) \cdot \left(\frac{\partial h_p}{\partial t}, \frac{\partial h_p}{\partial x_1}, \ldots, \frac{\partial h_p}{\partial x_n}\right) \\ &= \frac{\partial h_p}{\partial t} + \sum_{i=1}^n (\varepsilon F_i^1 + \varepsilon^2 R_i^1) \left(\frac{\partial h_p}{\partial x_i}\right), \end{aligned}$$

$$\begin{aligned} F_\varepsilon^- h_p(p) &= \left(1, \varepsilon F_1^2 + \varepsilon^2 R_1^2, \ldots, \varepsilon F_n^2 + \varepsilon^2 R_n^2\right) \cdot \left(\frac{\partial h_p}{\partial t}, \frac{\partial h_p}{\partial x_1}, \ldots, \frac{\partial h_p}{\partial x_n}\right) \\ &= \frac{\partial h_p}{\partial t} + \sum_{i=1}^n (\varepsilon F_i^2 + \varepsilon^2 R_i^2) \left(\frac{\partial h_p}{\partial x_i}\right), \end{aligned}$$

$$\begin{aligned} \therefore F_\varepsilon^+ h_p(p) \cdot F_\varepsilon^- h_p(p) = &\left(\frac{\partial h_p}{\partial t}\right)^2 + \frac{\partial h_p}{\partial t}\left(\sum_{i=1}^n (\varepsilon F_i^2 + \varepsilon^2 R_i^2)\left(\frac{\partial h_p}{\partial x_i}\right) + (\varepsilon F_i^1 + \varepsilon^2 R_i^1)\left(\frac{\partial h_p}{\partial x_i}\right)\right) \\ &+ \left(\sum_{i=1}^n (\varepsilon F_i^1 + \varepsilon^2 R_i^1)\left(\frac{\partial h_p}{\partial x_i}\right)\right) \cdot \left(\sum_{j=1}^n (\varepsilon F_j^2 + \varepsilon^2 R_j^2)\left(\frac{\partial h_p}{\partial x_j}\right)\right) \end{aligned}$$

thus

$$F_\varepsilon^+ h_p(p) \cdot F_\varepsilon^- h_p(p) = \left(\frac{\partial h_p}{\partial t}\right)^2 + \mathcal{O}(\varepsilon).$$

Notice that the hypothesis of the curve $\{(t, z) : t \in \mathbb{S}\}$ reaching Σ transversely implies that $\dfrac{\partial h_p}{\partial t}(t_0, z) \neq 0$ (if $\dfrac{\partial h_p}{\partial t}(t_0, z) = 0$, the curve will be tangent to Σ at p). Therefore $\left(\dfrac{\partial h_p}{\partial t}\right)^2 > 0$, which means that, for a sufficiently small $\varepsilon \neq 0$, $F_\varepsilon^+ h_p(p) \cdot F_\varepsilon^- h_p(p) > 0$, i.e. $p \in \Sigma^c$. $\square$

Remark 2. *By proving this lemma, we reassure that we'll be dealing only with regular periodic orbits.*

Lemma 9. *The averaged function* (3.13) *is continuous.*

Proof: Let $z_0 \in C$ and let V be a bounded open neighborhood of z_0 such that $\overline{V} \subset C$. Define $\Delta(z, z_0) = F_0(z) - F_0(z_0)$, we wish to show that

$$|\Delta(z, z_0)| \to 0, \text{ whenever } z \to z_0.$$

Define $I_z^{1,2} = \{t \in [0, T] : (t, z) \in S_{1,2}\}$ and $I_z^0 = \{t \in [0, T] : (t, z) \in \Sigma\}$; then, the expression of $F_0(z)$ can be split into:

$$F_0(z) = \int_0^T F(t, z)dt = \int_{I_z^1} F(t, z)dt + \int_{I_z^2} F(t, z)dt + \int_{I_z^0} F(t, z)dt.$$

By hypothesis H_1, we have that the set I_z^0 has measure zero, so the integral over it will be zero, thus there's no need to worry with how $F(t, z)$ is defined at the set of discontinuity Σ. Furthermore, it's clear that $F|_{I_z^1} = F^1$ and $F|_{I_z^2} = F^2$ by definition; hence, this yields the following expression for $F_0(z)$:

$$F_0(z) = \int_{I_z^1} F^1(t, z)dt + \int_{I_z^2} F^2(t, z)dt$$

hence

$$\begin{aligned}
|F_0(z) - F_0(z_0)| &= \left|\int_{I_z^1} F^1(t, z)dt + \int_{I_z^2} F^2(t, z)dt - \int_{I_{z_0}^1} F^1(t, z_0)dt - \int_{I_{z_0}^2} F^2(t, z_0)dt\right| \\
&\leqslant \int_{I_z^1 \cap I_{z_0}^1} |F^1(t, z) - F^1(t, z_0)|dt + \int_{I_z^2 \cap I_{z_0}^2} |F^2(t, z) - F^2(t, z_0)|dt \\
&\quad \left|\int_{I_z^1 \setminus I_{z_0}^1} F^1(t, z) + \int_{I_{z_0}^1 \setminus I_z^1} F^1(t, z_0) + \int_{I_z^2 \setminus I_{z_0}^2} F^2(t, z) + \int_{I_{z_0}^2 \setminus I_z^2} F^2(t, z_0)\right|,
\end{aligned}$$

first, notice that

$$\begin{aligned}
&\int_{I_z^1 \cap I_{z_0}^1} |F^1(t, z) - F^1(t, z_0)|dt + \int_{I_z^2 \cap I_{z_0}^2} |F^2(t, z) - F^2(t, z_0)|dt \\
&\qquad \leqslant \int_{I_z^1 \cap I_{z_0}^1} \lambda_{F^1} ||z - z_0||dt + \int_{I_z^2 \cap I_{z_0}^2} \lambda_{F^2} ||z - z_0||dt \\
&\qquad \leqslant (\lambda_{F^1} + \lambda_{F^2}) \cdot T \cdot ||z - z_0||,
\end{aligned}$$

where λ_{F^1} and λ_{F^2} are yielded from hypothesis *H2*, i.e. they are the Lipschitz constants of F^1 and F^2, respectively. Hence, when $z \to z_0$, it's guaranteed that

$$\left(\int_{I_z^1 \cap I_{z_0}^1} |F^1(t,z) - F^1(t,z_0)| dt + \int_{I_z^2 \cap I_{z_0}^2} |F^2(t,z) - F^2(t,z_0)| dt \right) \to 0.$$

Now, for the second part of the expression, let

$$L = \max\{F^j(t,z) : (t,z) \in [0,T] \times \overline{V}, j \in 1,2\}$$

and notice that

$$\begin{aligned} &\left| \int_{I_z^1 \backslash I_{z_0}^1} F^1(t,z) + \int_{I_{z_0}^1 \backslash I_z^1} F^1(t,z_0) + \int_{I_z^2 \backslash I_{z_0}^2} F^2(t,z) + \int_{I_{z_0}^2 \backslash I_z^2} F^2(t,z_0) \right| \\ &\leqslant L \cdot (\mu(I_z^1 \backslash I_{z_0}^1) + \mu(I_z^2 \backslash I_{z_0}^2) + \mu(I_{z_0}^1 \backslash I_z^1) + \mu(I_{z_0}^2 \backslash I_z^2)), \end{aligned}$$

where $\mu(.)$ denotes the Lebesgue measure of the set. Then $z \to z_0$ implies that $\mu(I_z^j \backslash I_{z_0}^j) \to 0$ and $\mu(I_{z_0}^j \backslash I_z^j) \to 0$ for $j \in \{1,2\}$, what zeros this part of the expression as well. Therefore $|\Delta(z,z_0)| \to 0,$ if $z \to z_0$, i.e. $F_0(z)$ is continuous on z. □

Lemma 10. *Let $x(t,z,\varepsilon) : [0,t_z) \to \mathbb{R}^n$ be the solution of system (3.12) with $x(0,z,\varepsilon) = z$, then, under the assumptions of Theorem 11 and for $t_z \leqslant T$, we have that the following equality holds:*

$$x(t,z,\varepsilon) = z + \varepsilon \cdot \int_0^t F(s,z) ds + \mathcal{O}(\varepsilon^2).$$

Proof: Let $z \in C$; the function $t \in [0,t_z) \mapsto x(t,z,\varepsilon)$ is piecewise differential. From hypothesis *H1*, for some ε sufficiently small, we assume the solution crosses Σ and write

$$x(t,z,\varepsilon) = \begin{cases} x_1(t,z,\varepsilon), & \text{if } 0 \leqslant t \leqslant t_\varepsilon, \\ x_2(t,z,\varepsilon), & \text{if } t_\varepsilon \leqslant t \leqslant t_z, \end{cases}$$

where t_ε is the time when the solution reaches Σ and depends on ε. Moreover we have that

$$\begin{aligned} x_1(0,z,\varepsilon) &= z, \\ x_2(t_\varepsilon,z,\varepsilon) &= x_1(t_\varepsilon,z,\varepsilon). \end{aligned} \tag{3.14}$$

Since $x(t,z,\varepsilon)$ is a solution of the system (3.12), then:

$$\begin{aligned} \frac{\partial x_1}{\partial t}(t,z,\varepsilon) &= \varepsilon F^1(t, x_1(t,z,\varepsilon)) + \varepsilon^2 R^1(t, x_1(t,z,\varepsilon), \varepsilon), \\ \frac{\partial x_2}{\partial t}(t,z,\varepsilon) &= \varepsilon F^2(t, x_2(t,z,\varepsilon)) + \varepsilon^2 R^2(t, x_2(t,z,\varepsilon), \varepsilon), \end{aligned} \tag{3.15}$$

thus, each line of the expression (3.15) define with the corresponding line of (3.14) a initial value problem, for which we can apply the Theorem 1.2.4 from (SANDERS;

VERHULST; MURDOCK, 2007) that guarantees the existence and uniqueness of a solution for the interval $[0, \inf(T, d/M_i)$, where $M_i = \sup_G ||\varepsilon F^i(t,x) + \varepsilon^2 R^i(t,x,\varepsilon)||$, $G = [0,T] \times D \times [-\varepsilon_0, \varepsilon_0]$ and d is such that $||x - z|| < d \; \forall x \in D$. This means that we are able to choose ε_0^i sufficiently small such that d/M_i is large enough to make $\inf(T, d/M_i) = T$ for $i = 1, 2$, which guarantees the existence of these solutions on the interval $[0, T]$ for a sufficiently small ε.

Moving forward, by solving the initial value problems we get:

$$\begin{aligned} x_1(t,z,\varepsilon) &= x_1(0,z,\varepsilon) + \int_0^t (\varepsilon F^1(s, x_1(s,z,\varepsilon)) + \varepsilon^2 R^1(s, x_1(s,z,\varepsilon),\varepsilon))ds \\ &= z + \varepsilon \int_0^t F^1(s, x_1(s,z,\varepsilon))ds + \varepsilon^2 \int_0^t R^1(s, x_1(s,z,\varepsilon),\varepsilon)ds \\ &= z + \varepsilon \int_0^t F(s, x(s,z,\varepsilon))ds + \varepsilon^2 \int_0^t R(s, x(s,z,\varepsilon),\varepsilon)ds, \end{aligned}$$

for $t \in [0, t_\varepsilon]$. Analogously:

$$x_2(t,z,\varepsilon) = x_2(t_\varepsilon,z,\varepsilon) + \varepsilon \int_{t_\varepsilon}^t F^2(s, x_2(s,z,\varepsilon)) + \varepsilon^2 \int_{t_\varepsilon}^t R^2(s, x_2(s,z,\varepsilon),\varepsilon)ds,$$

for $t \in [t_\varepsilon, t_z)$; but

$$x_2(t_\varepsilon,z,\varepsilon) = x_1(t_\varepsilon,z,\varepsilon) = z + \varepsilon \int_0^{t_\varepsilon} F^1(s, x_1(s,z,\varepsilon))ds + \varepsilon^2 \int_0^{t_\varepsilon} R^1(s, x_1(s,z,\varepsilon),\varepsilon)ds.$$

hence:

$$\begin{aligned} x_2(t,z,\varepsilon) = z &+ \varepsilon \int_0^{t_\varepsilon} F^1(s, x_1(s,z,\varepsilon))ds + \varepsilon^2 \int_0^{t_\varepsilon} R^1(s, x_1(s,z,\varepsilon),\varepsilon)ds \\ &+ \varepsilon \int_{t_\varepsilon}^t F^2(s, x_2(s,z,\varepsilon)) + \varepsilon^2 \int_{t_\varepsilon}^t R^2(s, x_2(s,z,\varepsilon),\varepsilon)ds, \end{aligned}$$

$$\therefore x_2(t,z,\varepsilon) = z + \varepsilon \int_0^t F(s, x(s,z,\varepsilon))ds + \varepsilon^2 \int_0^t R(s, x(s,z,\varepsilon),\varepsilon)ds,$$

for $t \in [t_\varepsilon, t_z)$. Since the expressions of $x(t,z,\varepsilon)$ coincide for both intervals, we can simply write:

$$x(t,z,\varepsilon) = z + \varepsilon \int_0^t F(s, x(s,z,\varepsilon))ds + \varepsilon^2 \int_0^t R(s, x(s,z,\varepsilon),\varepsilon)ds, \tag{3.16}$$

for all $t \in [0, t_z)$.

Now, let $\varepsilon_0 = \min\{\varepsilon_0^1, \varepsilon_0^2\}$, such that the solution $x(t,z,\varepsilon)$ exists for every $t \in [0,T]$. From the continuity of $x(t,z,\varepsilon)$ and compactness of $[0,T] \times \overline{C} \times [-\varepsilon_0, \varepsilon_0]$, there exists a compact $K \subset D$ such that the solution $x(t,z,\varepsilon)$ is contained in K for every

$t \in [0,T]$, $z \in \overline{C}$ and $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ and, since R is piecewise continuous, it follows that $R(t,x,\varepsilon)$ is bounded by a constant N on $[0,T] \times K \times [-\varepsilon_0, \varepsilon_0]$. Then, for every $t \leqslant T$:

$$\left| \int_0^t R(s, x(s,z,\varepsilon), \varepsilon) ds \right| \leqslant \int_0^T |R(s, x(s,z,\varepsilon), \varepsilon)| ds \leqslant T \cdot N,$$

thus $\int_0^t R(s, x(s,z,\varepsilon), \varepsilon) ds = \mathcal{O}(1)$. Then, expression (3.16) becomes

$$x(t,z,\varepsilon) = z + \varepsilon \int_0^t F(s, x(s,z,\varepsilon)) ds + \mathcal{O}(\varepsilon^2),$$

which still isn't what we wanted to achieve — notice the spatial argument of F. But, since $F^i(t,x)$ is Lipschitz for $i = 1,2$, we have that:

$$|F^i(t, x_i(t,z,\varepsilon)) - F^i(t,z)| \leqslant L_i |x_i(t,z,\varepsilon) - z| = \mathcal{O}(\varepsilon),$$

hence $F^i(t, x_i(t,z,\varepsilon)) = F^i(t,z) + \mathcal{O}(\varepsilon)$; then

$$\begin{aligned} \int_0^t F(s, x(s,z,\varepsilon)) ds &= \int_0^{t_\varepsilon} F^1(s, x_1(s,z,\varepsilon)) ds + \int_{t_\varepsilon}^t F^2(s, x_2(s,z,\varepsilon)) ds \\ &= \int_0^{t_\varepsilon} F^1(s,z) + \mathcal{O}(\varepsilon) ds + \int_{t_\varepsilon}^t F^2(s,z) + \mathcal{O}(\varepsilon) ds \\ &= \int_0^{t_\varepsilon} F^1(s,z) ds + \int_{t_\varepsilon}^t F^2(s,z) ds + \mathcal{O}(\varepsilon). \end{aligned}$$

Making $\varepsilon \to 0$, $t_\varepsilon \to 0$, we wish to bound the error of the integral; notice that:

$$\begin{aligned} \int_0^{t_\varepsilon} F^1(s,z) ds + \int_{t_\varepsilon}^t F^2(s,z) ds = &\int_0^{t_0} F^1(s,z) ds + \int_{t_0}^{t_\varepsilon} F^1(s,z) ds \\ &+ \int_{t_0}^t F^2(s,z) ds - \int_{t_0}^{t_\varepsilon} F^2(s,z) ds. \end{aligned}$$

Notice that $E(\varepsilon) = \int_{t_0}^{t_\varepsilon} F^1(s,z) - F^2(s,z) ds \leqslant \tilde{E} |t_\varepsilon - t_0|$ for some constant $\tilde{E}$. We wish to prove that $E(\varepsilon) = \mathcal{O}(\varepsilon)$; in order to do so, let h_p be as in the proof of Lemma 8 and write $H_i = h_p(t, x_i(t,z,\varepsilon))$. Then $H_i(t_0, 0) = 0$ and

$$\frac{\partial H_i}{\partial t}(t_0, 0) = \frac{\partial h_p}{\partial t}(t_0, x_i(t_0, z, 0)) + \frac{\partial h_p}{\partial z}(t_0, x_i(t_0, z, 0)) \cdot \frac{\partial x_i}{\partial t}(t_0, z, 0),$$

but $\frac{\partial x_i}{\partial t}(t_0, z, 0) = 0$ and, by hypothesis *H1*, $\frac{\partial h_p}{\partial t}(t_0, x_i(t_0, z, 0)) \neq 0$, hence

$$\frac{\partial H_i}{\partial t}(t_0, 0) \neq 0,$$

then it follows from the Implicit Function Theorem that $\varepsilon \mapsto t_\varepsilon$ is a C^k function, hence

$$t_\varepsilon = t_0 + \mathcal{O}(\varepsilon)$$

and, since $\int_0^{t_0} F^1(s,z)ds + \int_{t_0}^t F^2(s,z)ds = \int_0^t F(s,z)ds$, this implies

$$\int_0^t F(s,x(s,z,\varepsilon))ds = \int_0^t F(s,z)ds + \mathcal{O}(\varepsilon).$$

Therefore

$$x(t,z,\varepsilon) = z + \varepsilon \int_0^t F(s,z)ds + \mathcal{O}(\varepsilon^2).$$

□

Lemma 11. *Let $U \subset \mathbb{R}^n$ be a bounded open set and let $f : \overline{U} \times [-\varepsilon_0, \varepsilon_0] \to \mathbb{R}^n$ be a continuous function. If $f(x,0) \neq 0$ for all $x \in \partial U$, then — for $\varepsilon = \tilde{\varepsilon} \neq 0$ sufficiently small and $V = U \times [-\tilde{\varepsilon}, \tilde{\varepsilon}]$— $d_B(f(x,\varepsilon),V,0)$ is well defined and $d_B(f(x,\varepsilon),V,0) = d_B(f(x,0),V,0)$.*

Proof: $d_B(f(x,\varepsilon),V,0)$ will be well defined if it satisfies the hypotheses of Theorem 10. By our assumptions, $f(x,\varepsilon) : \overline{U} \times [-\varepsilon_0, \varepsilon_0]$ is continuous and $U \times [-\varepsilon_0, \varepsilon_0]$ is bounded, so what is left is to prove that $0 \notin f(\partial U, \varepsilon)$, for $\varepsilon \in [-\tilde{\varepsilon}, \tilde{\varepsilon}]$.

Let $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ and consider the continuous homotopy

$$f_t(x,\varepsilon) = f(x,0) + t(f(x,\varepsilon) - f(x,0)). \tag{3.17}$$

Suppose that there are sequences $(\varepsilon_i) \subset [-\varepsilon_0, \varepsilon_0]$, $(x_i) \subset \partial U$ and $(t_i) \subset [0,1]$ with $\lim_{i\to\infty} \varepsilon_i = 0$ such that $f_{t_i}(x_i, \varepsilon_i) = 0$. Since ∂U is the boundary of a bounded set, it is compact, and so is $[0,1]$; then, there exist subsequences of (x_i) and (t_i) converging to points of the sets ∂U and $[0,1]$, respectively, i.e. $t_{i_j} \to \tilde{t} \in [0,1]$ and $x_{i_j} \to \tilde{x} \in \partial U$. Hence

$$\begin{aligned}
& f(x_{i_j},0) + t_{i_j}(f(x_{i_j},\varepsilon_{i_j}) - f(x_{i_j},0)) = 0 \\
& \Rightarrow \lim_{j\to\infty}(f(x_{i_j},0) + t_{i_j}(f(x_{i_j},\varepsilon_{i_j}) - f(x_{i_j},0))) = 0 \\
& \Rightarrow f(\tilde{x},0) + \tilde{t}(f(\tilde{x},0) - f(\tilde{x},0)) = 0 \\
& \therefore f(\tilde{x},0) = 0,
\end{aligned}$$

which yields a contradiction to the hypothesis of $f(x,0) \neq 0$ for all $x \in \partial U$.

To avoid such contradiction, there must be some $0 \neq \tilde{\varepsilon} \in [-\varepsilon_0, \varepsilon_0]$ such that $0 \notin f_t(\partial U, \varepsilon)$ for all $\varepsilon \in [-\tilde{\varepsilon}, \tilde{\varepsilon}]$. In particular, for $t = 1$, $0 \notin f(\partial U, \varepsilon)$, hence $d_B(f(x,\varepsilon),V,0)$ is well defined.

Now, by the property *3.* from Theorem 10, we conclude that $d_B(f(x,\varepsilon),V,0)$ must be invariant by the homotopy defined in (3.17), therefore

$$d_B(f(x,\varepsilon),V,0) = d_B(f(x,0),V,0) \text{ for all } \varepsilon \in [-\tilde{\varepsilon}, \tilde{\varepsilon}].$$

□

Proof of Theorem 11:

We recall that we desire to find a periodic solution from system (3.12), i.e. $x(t,z,\varepsilon)$ such that $x(T,z,\varepsilon) = z$. So, to complete this proof, let $g(z,\varepsilon)$ be a function such that $\varepsilon g(z,\varepsilon) = x(T,z,\varepsilon) - z$. First, notice that g is well-defined, since the solution $x(t,z,\varepsilon)$ is defined $\forall t \in [0,T]$, as guaranteed by Lemma 10. Moreover, also by Lemma 10:

$$x(T,z,\varepsilon) - z = \varepsilon g(z,\varepsilon) = \varepsilon \cdot \int_0^T F(s,z)ds + \mathcal{O}(\varepsilon^2)$$
$$\Rightarrow g(z,\varepsilon) = \int_0^T F(s,z)ds + \mathcal{O}(\varepsilon),$$

for $\varepsilon \neq 0$; hence $g(z,\varepsilon)$ is continuous by Lemma 9. By the hypothesis *H3*, if $a \in C$ is such that $F_0(a) = \dfrac{1}{T}\displaystyle\int_0^T F(s,a)ds = 0$, then there is a neighborhood $U \subset C$ such that $F_0(z) \neq 0$ for all $z \in \overline{U}\backslash\{a\}$ and $d_B(F_0,U,0) \neq 0$; then, by Lemma 11, it follows that for $\varepsilon \neq 0$ sufficiently small

$$d_B(F_0(z),U,0) = d_B(g(z,\varepsilon),U,0) \neq 0.$$

Thus, by property *1.* from Theorem 10, $0 \in g(U,\varepsilon)$ for ε sufficiently small, i.e. there exists $a_\varepsilon \in U$ such that $g(a_\varepsilon,\varepsilon) = 0$, therefore $x(t,a_\varepsilon,\varepsilon)$ is a periodic solution of (3.12). Furthermore, since $F_0(z) \neq 0$ for all $z \in \overline{U}\backslash\{a\}$, then $a_\varepsilon \to a$ when $\varepsilon \to 0$ and the proof is complete. □

4 Limit Cycles on Liénard-like Perturbations

We start this chapter by introducing an important theorem that will be extensively used to find a lower bound for the maximum number of limit cycles of a Liénard-like perturbed system. First of all, we remark the limitations of our calculations: even though we aim to find a *maximum number* of limit cycles of a system, we can only guarantee a *lower bound of the maximum number* of limit cycles, in the sense of it being the maximum number of cycles *that can be found through first-order averaging.* That being said, we shall focus on determining the maximum number of roots of the averaged function defined in the previous chapter and, in order to do so, we introduce the following theorem, as stated in (LLIBRE; TEIXEIRA, 2014):

Theorem 12. (Descartes' Theorem) *Consider the real polynomial* $p(x) = a_{i_1}x^{i_1} + ... + a_{i_r}x^{i_r}$*, with* $r > 1$*,* $0 \leqslant i_1 < ... < i_r$ *and the numbers* a_{i_j} *are not simultaneously zeros for* $j \in \{1, 2, ..., r\}$*. If* $a_{i_j} \cdot a_{i_{j+1}} < 0$*, we say that they have a variation of sign. If the number of variations of signs is* m*, then* $p(x)$ *has at most* m *positive real roots. Moreover, it's always possible to choose the coefficients of* $p(x)$ *in such a way that* $p(x)$ *has exactly* $r - 1$ *positive real roots.*

Proof: We shall split this proof in two parts. First we prove the statement

If the number of variations of signs is m*, then* $p(x)$ *has at most* m *positive real roots.*

Suppose, without loss of generality, that $a_{i_1} > 0$ (the case of $a_{i_1} < 0$ is analogous). This gives us two possibilities:

1. $a_{i_1} \cdot a_{i_r} > 0$: in this case $a_{i_r} > 0$. Notice that $p(0) \geqslant 0$ and $p(x) \to +\infty$ when $x \to +\infty$, hence the graph of p must cross the positive part of the x-axis an even number of times, i.e. there are a even number of roots with odd multiplicity. Furthermore, notice that if the graph touches the x-axis at a point $x = x_0$, without crossing it, this root must have even multiplicity. Therefore p has an even number of positive roots counting multiplicity.

2. $a_{i_1} \cdot a_{i_r} < 0$: in this case $a_{i_r} < 0$, then $p(0) \geqslant 0$ and $p(x) \to -\infty$ when $x \to +\infty$, hence the graph of p must cross the positive part of the x-axis an odd number of times; i.e. p has an odd number of positive roots counting multiplicity.

Moreover, notice that, if $a_{i_1} \cdot a_{i_r} > 0$, the number of sign changes of the coefficients is even; otherwise, if $a_{i_1} \cdot a_{i_r} < 0$, then it is odd. If s denote the number of variation of sign of the coefficients of p and z the number of zeros of p counting multiplicity, then we know that $z \equiv s(\text{mod } 2)$. We proceed now by induction on r and, without loss of generality, suppose $i_1 = 0$.

If $r = 2$, then we write $p(x) = a \cdot x^n + b$ and note that p will have at most one root. Indeed, if $a > 0$, then $\dot{p}(x) > 0 \; \forall x > 0$, hence p has one root if $b < 0$ $(z = s = 1)$ and zero roots if $b > 0$ $(z = s = 0)$; for $a < 0$, the same argument is valid changing the signs.

Assume that for $r \leqslant k - 1$ the statement is true, i.e. $z(p) \leqslant s(p)$. Then for $r = k$ consider the following cases:

1. If $a_{i_1} \cdot a_{i_2} > 0$, then $s(\dot{p}) = s(p)$. Since $deg(\dot{p}) = deg(p) - 1 = k - 1$, by the induction hypothesis $z(\dot{p}) \leqslant s(\dot{p})$. Then, by the Mean Value Theorem, we get that $z(\dot{p}) \geqslant z(p) - 1$, hence:

$$s(p) = s(\dot{p}) \geqslant z(\dot{p}) \geqslant z(p) - 1 \Rightarrow s(p) \geqslant z(p) - 1.$$

   However, because $z(p) \equiv s(p)(\text{mod } 2)$ we can't have $s(p) = z(p) - 1$, hence $s(p) > z(p) - 1$, so $s(p) \geqslant z(p)$.

2. If $a_{i_1} \cdot a_{i_2} < 0$, then $s(\dot{p}) + 1 = s(p)$, hence

$$s(p) > s(\dot{p}) \geqslant z(\dot{p}) \geqslant z(p) - 1 \Rightarrow s(p) > z(p) - 1,$$

   which yields again $s(p) \geqslant z(p)$.

Therefore, the number of roots of p is, at most, equal to number of variation of sign of p.

Now, using as reference the Appendix III from (LLIBRE; TEIXEIRA, 2014), we'll show that the following is also true:

Its always possible to choose the coefficients of $p(x)$ in such a way that $p(x)$ has exactly $r - 1$ positive real roots.

We first prove the following lemma:

Lemma 12. *Let $f_1, ..., f_r : A \to \mathbb{R}$ be linear independent functions. There exist $b_1, ..., b_r$ such that the r vectors:*

$$\begin{pmatrix} f_1(b_1) \\ f_1(b_2) \\ \vdots \\ f_1(b_r) \end{pmatrix}, \begin{pmatrix} f_2(b_1) \\ f_2(b_2) \\ \vdots \\ f_2(b_r) \end{pmatrix} \cdots \begin{pmatrix} f_r(b_1) \\ f_r(b_2) \\ \vdots \\ f_r(b_r) \end{pmatrix}$$

are linearly independents.

Proof of Lemma 12: We will prove this by induction.

For $r = 2$, notice that it is trivially true since f_1 and f_2 are linearly independent.

Assume that this is true for $r = k - 1$ and, by contradiction, suppose that this is not true for $r = k$. Then there exist $\alpha_1, ..., \alpha_k$ not all equal to zero such that

$$\alpha_1(b)\begin{pmatrix} f_1(b_1) \\ f_1(b_2) \\ \vdots \\ f_1(b_{k-1}) \\ f_1(b) \end{pmatrix} + \alpha_2(b)\begin{pmatrix} f_2(b_1) \\ f_2(b_2) \\ \vdots \\ f_2(b_{k-1}) \\ f_2(b) \end{pmatrix} + ... + \alpha_k(b)\begin{pmatrix} f_r(b_1) \\ f_r(b_2) \\ \vdots \\ f_k(b_{k-1}) \\ f_k(b) \end{pmatrix} = 0.$$

By the induction hypothesis, we have necessarily $\alpha_k(b) \neq 0$; then split in two cases:

1. There exists $i \in \{1, ..., k-1\}$ such that $f_k(b_i) \neq 0$. In this case, $\dfrac{\alpha_j(b)}{\alpha_k(b)}$ doesn't depend on b; then for every b, $f_k(b) = \displaystyle\sum_{j=1}^{k-1} \frac{\alpha_j(b)}{\alpha_k(b)} f_j(b)$, contradicting the linear independence of $f_1, ..., f_k$.
2. For every $i \in \{1, ..., k-1\}$, $f_k(b_i) = 0$. In this case $\alpha_j(b) = 0$ for every $j \in \{1, ..., k-1\}$, but then $\alpha_k(b) \equiv 0$, which contradicts the hypothesis.

This completes the proof of the Lemma. □

Now, notice that taking

$$f_1(x) = x^{i_1}, f_2(x) = x^{i_2}, ..., f_r(x) = x^{i_r},$$

such that $0 \leqslant i_1 < ... < i_r$, the functions are linearly independent. Taking $b_1, ..., b_r$ like in Lemma 12, the matrix

$$A = \begin{pmatrix} f_1(b_1) & f_2(b_1) & ... & f_r(b_1) \\ f_1(b_2) & f_2(b_2) & ... & f_r(b_2) \\ \vdots & \vdots & & \vdots \\ f_1(b_r) & f_2(b_r) & ... & f_r(b_r) \end{pmatrix},$$

will be invertible; hence the equation $A \cdot \vec{\alpha} = (0, ..., 0, 1)^T$ has a unique solution $\vec{\alpha}$. In particular, there exists $\alpha_1, \alpha_2, ..., \alpha_r$ such that

$$\begin{pmatrix} f_1(b_i) & f_2(b_i) & ... & f_r(b_i) \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_r \end{pmatrix} = 0,$$

for $i = 1, 2, ..., r-1$. In other words, we can choose coefficients of the polynomial such that we obtain exactly $r-1$ roots in a certain set, in particular the set of the positive real numbers. This finally completes the proof of Theorem 12. $\square$

4.1 Smooth Perturbations

In this section, we will apply the averaging theory developed in section 3.2 to obtain a result on the number of limit cycles on a generalized Liénard smooth system; we present a proof of the first part of Theorem 5.

Proof of the first part of Theorem 5: Consider the system

$$\begin{aligned} \dot{x} &= y - \varepsilon \cdot p(x), \\ \dot{y} &= -x, \end{aligned} \tag{4.1}$$

where $p(x) = a_1 x + a_2 x^2 + ... + a_{2m+1} x^{2m+1}$. First we shall rewrite it in polar coordinates, i.e. $x = r\cos\theta$ and $y = r\sin\theta$. Notice that

$$\begin{aligned} \dot{x} &= \dot{r}\cos\theta - r\sin\theta \cdot \dot{\theta} \\ \dot{y} &= \dot{r}\sin\theta + r\cos\theta \cdot \dot{\theta} \end{aligned} \Rightarrow \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \cdot \begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix}.$$

Then, by inverting the coordinate change matrix, we get

$$\begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix} = \frac{1}{r} \begin{pmatrix} r\cos\theta & r\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} r\sin\theta - \varepsilon \cdot p(r\cos\theta) \\ -r\cos\theta \end{pmatrix},$$

which yields the following system:

$$\begin{aligned} \dot{r} &= -\varepsilon \cdot \cos\theta \cdot p(r\cos\theta), \\ \dot{\theta} &= -1 + \frac{\varepsilon}{r}\sin\theta \cdot p(r\cos\theta). \end{aligned}$$

Admitting θ as the new independent variable, we can rewrite the above system as the following equation:

$$\frac{dr}{d\theta} = \frac{\varepsilon \cdot \cos\theta \cdot p(r\cos\theta)}{1 - \frac{\varepsilon}{r}\sin\theta \cdot p(r\cos\theta)}.$$

In order to apply the averaging theorem, we shall first compute the Taylor expansion of the function $g(\varepsilon) = \dfrac{\varepsilon \cdot \cos\theta \cdot p(r\cos\theta)}{1 - \frac{\varepsilon}{r}\sin\theta \cdot p(r\cos\theta)}$. Just to make it simpler, let $a = \cos\theta \cdot p(r\cos\theta)$ and $b = \dfrac{1}{r}\sin\theta \cdot p(r\cos\theta)$; then:

$$g(\varepsilon) = \frac{\varepsilon \cdot a}{1 - \varepsilon \cdot b}.$$

The expansion series of $g(\varepsilon)$ near 0 will then be:

$$g(\varepsilon) = g(0) + g'(0) \cdot \varepsilon + \mathcal{O}(\varepsilon^2),$$

where $g(0) = 0$ and

$$g'(0) = \left.\frac{a(1 - \xi \cdot b) + \xi \cdot a(\xi)}{(1 - \xi \cdot b)^2}\right|_{\xi=0} = a.$$

Finally, the original system (4.1) can be written as

$$\frac{dr}{d\theta} = \varepsilon \cdot \cos\theta \cdot p(r\cos\theta) + \mathcal{O}(\varepsilon^2). \tag{4.2}$$

Since $F(r,\theta) = \cos\theta \cdot p(r\cos\theta)$ is just a sum and product of smooth functions, it is itself a smooth function; moreover, it is a periodic function on θ, since θ appears only as the argument of cosines. Writing $\mathcal{O}(\varepsilon^2) = \varepsilon^2 \cdot R(r,\theta)$, notice that, analogously to the case of $F(r,\theta)$, $R(r,\theta)$ is also smooth and periodic on the second variable, so the conditions from theorem 9 hold. Write

$$F_0(r) = \frac{1}{2\pi}\int_0^{2\pi} \cos\theta \cdot p(r\cos\theta)d\theta,$$

since $p(r\cos\theta) = \displaystyle\sum_{i=1}^{2m+1} a_i r^i \cos^i\theta$, then

$$F_0(r) = \frac{1}{2\pi}\int_0^{2\pi}\sum_{i=1}^{2m+1} a_i r^i \cos^{i+1}\theta d\theta = \frac{1}{2\pi}\sum_{i=1}^{2m+1} a_i r^i \int_0^{2\pi}\cos^{i+1}\theta d\theta.$$

Note that

$$\int_0^{2\pi}\cos^{2k+1}\theta d\theta = 0,$$
$$\int_0^{2\pi}\cos^{2k}\theta d\theta = b_k \neq 0,$$

so all the even terms of the summation (which yield odd terms in the integral) will vanish, in such way that there will be left only the odd terms of the summation, i.e.:

$$\begin{aligned}\sum_{i=1}^{2m+1} a_i r^i \int_0^{2\pi}\cos^{i+1}\theta d\theta &= \sum_{k=0}^{m} a_{2k+1} r^{2k+1}\int_0^{2\pi}\cos^{2k+2}\theta d\theta \\ &= \sum_{k=0}^{m} a_{2k+1} b_k r^{2k+1}.\end{aligned}$$

By item 2 of the Theorem 9, if $\alpha \neq 0$ is such that $F_0(\alpha) = 0$ and $det(D_r F(\alpha)) \neq 0$, then there exists a periodic solution $\phi(\theta,\varepsilon)$ of period T of the system (4.2) which is close to α. On the other hand, since $F_0(r) = \dfrac{1}{2\pi}\displaystyle\sum_{k=0}^{m} a_{2k+1} b_k r^{2k+1}$ is a polynomial with $m+1$ terms, then by Theorem 12 $F_0(r)$ has at most m roots.

Therefore, m is a lower bound for the maximum number of limit cycles of system (4.2). Moreover, since Descartes' Theorem guarantees that we can always choose proper coefficients of $F_0(r)$ such that this maximum number of roots is achieved, this also guarantees that we can choose $p(x)$ such that the system (4.1) has m limit cycles. □

4.2 Non-smooth Perturbations

In this section, we apply the method of averaging for non-smooth systems, in particular non-smooth Liénard-like perturbations. We'll start by outlining the results from some more recent works on the topic and, by the end of the chapter, we will have stated and proved the main result of our work, which is a generalization of Martins and Mereu's results from (MARTINS; MEREU, 2014).

One of the works that started the study of limit cycles on discontinuous Liénard polynomial differential systems was done by Llibre and Teixeira and is summarized by the paper (LLIBRE; TEIXEIRA, 2014). In their work, they proposed the study of the system

$$\begin{aligned} \dot{x} &= y + \varepsilon \cdot \operatorname{sgn}(g_m(x,y)) \cdot f(x), \\ \dot{y} &= -x, \end{aligned} \tag{4.3}$$

where $f(x)$ is a polynomial of degree n and the zero set of the function $\operatorname{sgn}(g_m(x,y))$, $m \in \{0, 2, 4, 6...\}$, is the union of $m/2$ distinct straight lines passing through the origin, dividing the plane in sectors of angles $2\pi/m$. They managed to prove — using the periodic averaging method for regularized discontinuous systems — that for $m = 0, 2$ and 4 the lower bounds for the maximum number of limit cycles of system (4.3) are, respectively, $\left[\frac{n-1}{2}\right]$, $\left[\frac{n}{2}\right]$ and $\left[\frac{n-1}{2}\right]$. They also left unproved the conjecture that, for $m \geqslant 6$, a lower bound for the maximum number of limit cycles of this system should be $\left[\frac{1}{2}\left(n - \frac{m-2}{2}\right)\right]$. This was then proven to be true in (DONG; LIU, 2017).

In (MARTINS; MEREU, 2014), the studied system was

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x - \varepsilon(f(x) \cdot y + \operatorname{sgn}(y)(k_1 x + k_2)), \end{aligned} \tag{4.4}$$

where f is a polynomial of degree $n \in \mathbb{N}$ and $k_1, k_2 \in \mathbb{R}$. The approach chosen by the authors was also based on the ideas of the regularization method — specifically by introducing a piecewise linear function of the type

$$\varphi_w(y) = \begin{cases} -1 & \text{if } y < -w \\ \dfrac{y}{w} & \text{if } -w < y < w \\ 1 & \text{if } y > w \end{cases}.$$

The original system (4.4) is then replaced by

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x - \varepsilon(f(x)\cdot y + \varphi_w(y)(k_1 x + k_2)), \end{aligned} \tag{4.5}$$

and it's easy to see that, taking $w \to 0$, $\varphi_w(y) \to \mathrm{sgn}(y)$. The main result of their work is presented as the following theorem:

Theorem 13. (Martins-Mereu) *For every $n \geqslant 1$ and $|\varepsilon|$ sufficiently small, the maximum number of limit cycles of the system (4.5) bifurcating from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$ is $[n/2]+1$. Moreover, there are systems (4.5) having exactly $[n/2]+1$ limit cycles.*

By proving this case, the case for the differential system (4.4) comes as a corollary, taking the limit when $w \to 0$. We generalize this theorem replacing $k_1 x + k_2$ by an arbitrary real polynomial $g(x)$ of degree $m \geqslant 1$. As mentioned before in section 3.3, our proof will not pass through the regularization process — in one hand, this should ease the process of calculating the averaged function but, in the other hand, we need to take a more careful look at the conditions of the functions we are averaging.

Theorem 14. (Main Result) *Let $f(x)$ and $g(x)$ be real polynomials of degrees $n \geqslant 1$ and $m \geqslant 1$, respectively, and consider the system*

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x - \varepsilon\cdot(f(x)\cdot y + sgn(y)\cdot g(x)). \end{aligned} \tag{4.6}$$

Then, for $|\varepsilon|$ sufficiently small, the number $\left[\frac{n}{2}\right]+\left[\frac{m}{2}\right]+1$ is a lower bound to the maximum number of limit cycles of the system (4.6) bifurcating from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$. Moreover, we can choose f and g such that this number of cycles is indeed achieved.

Proof of the Main Theorem:

Let $f(x) = \displaystyle\sum_{i=0}^{n} a_i x^i$ and $g(x) = \displaystyle\sum_{j=0}^{m} b_j x^j$. We start rewriting the system in polar coordinates

$$\begin{aligned} \dot{x} &= \dot{r}\cos\theta - r\sin\theta\cdot\dot{\theta} \\ \dot{y} &= \dot{r}\sin\theta + r\cos\theta\cdot\dot{\theta} \end{aligned} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}\cdot\begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix},$$

then inverting the coordinate change matrix:

$$\begin{aligned} \begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix} &= \frac{1}{r}\begin{pmatrix} r\cos\theta & r\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} r\sin\theta \\ -r\cos\theta - \varepsilon\cdot(f(r\cos\theta)\cdot r\sin\theta + \mathrm{sgn}(r\sin\theta)\cdot g(r\cos\theta)) \end{pmatrix} \\ &= \begin{pmatrix} -\varepsilon\cdot\sin\theta\cdot(f(r\cos\theta)\cdot r\sin\theta + \mathrm{sgn}(r\sin\theta)\cdot g(r\cos\theta)) \\ -1 - \frac{\varepsilon}{r}\cdot(f(r\cos\theta)\cdot r\sin\theta + \mathrm{sgn}(r\sin\theta)\cdot g(r\cos\theta)) \end{pmatrix}. \end{aligned}$$

To ease our calculations, put $a = (f(r\cos\theta) \cdot r\sin\theta + \operatorname{sgn}(r\sin\theta) \cdot g(r\cos\theta))$. Then, admitting θ as the new independent variable, we have:

$$\frac{dr}{d\theta} = \frac{-\varepsilon \cdot \sin\theta \cdot a}{-1 - \frac{\varepsilon}{r} \cdot \cos\theta \cdot a} = \phi(\varepsilon).$$

Expanding the series of ϕ around $\varepsilon = 0$ we get

$$\phi(\varepsilon) = \phi(0) + \phi'(0) \cdot \varepsilon + \mathcal{O}(\varepsilon^2),$$

but clearly $\phi(0) = 0$, and

$$\phi'(0) = \left.\frac{-\sin\theta \cdot a(-1 - \frac{\xi}{r} \cdot \cos\theta \cdot a) - (-\xi\sin\theta \cdot a \cdot \frac{\cos\theta \cdot a}{r})}{(-1 - \frac{\xi}{r} \cdot \cos\theta \cdot a)^2}\right|_{\xi=0} = \sin\theta \cdot a.$$

Hence

$$\frac{dr}{d\theta} = \sin\theta \cdot (f(r\cos\theta) \cdot r\sin\theta + \operatorname{sgn}(r\sin\theta) \cdot g(r\cos\theta)) \cdot \varepsilon + \mathcal{O}(\varepsilon^2).$$

Writing $F(r,\theta) = \sin\theta \cdot (f(r\cos\theta) \cdot r\sin\theta + \operatorname{sgn}(r\sin\theta) \cdot g(r\cos\theta))$ and $\mathcal{O}(\varepsilon^2) = R(r,\theta) \cdot \varepsilon^2$; define the averaged function $F_0(r)$ as the following integral:

$$F_0(r) = \frac{1}{2\pi} \int_0^{2\pi} (f(r\cos\theta) \cdot r\sin^2\theta + \operatorname{sgn}(r\sin\theta) \cdot \sin\theta \cdot g(r\cos\theta))d\theta.$$

Note that $\operatorname{sgn}(r\sin\theta) = \operatorname{sgn}(\sin\theta)$, since $r > 0$, which yields the value $+1$ when $\theta \in (0,\pi)$ and -1 when $\theta \in (\pi, 2\pi)$, then:

$$F_0(r) = \frac{1}{2\pi}\left[\int_0^{2\pi} (f(r\cos\theta) \cdot r\sin^2\theta d\theta + \int_0^{\pi} \sin\theta \cdot g(r\cos\theta))d\theta - \int_\pi^{2\pi} \sin\theta \cdot g(r\cos\theta))d\theta\right],$$

and since

$$\int_\pi^{2\pi} \sin\theta \cdot g(r\cos\theta))d\theta = \int_\pi^{0} \sin\theta \cdot g(r\cos\theta))d\theta = -\int_0^{\pi} \sin\theta \cdot g(r\cos\theta))d\theta,$$

the averaged function can be written as:

$$F_0(r) = \frac{1}{2\pi}\left[\int_0^{2\pi} (f(r\cos\theta) \cdot r\sin^2\theta d\theta + 2 \cdot \int_0^{\pi} \sin\theta \cdot g(r\cos\theta))d\theta\right].$$

Let I and J denote the following integrals:

$$I = \int_0^{2\pi} (f(r\cos\theta) \cdot r\sin^2\theta,$$
$$J = \int_0^{\pi} \sin\theta \cdot g(r\cos\theta))d\theta,$$

then for the first integral we have:

$$I = \int_0^{2\pi} \sum_{i=0}^{n} a_i r^i \cos^i\theta \cdot r\sin^2\theta d\theta = \sum_{i=0}^{n} a_i r^{i+1} \cdot \int_0^{2\pi} \cos^i\theta \sin^2\theta d\theta.$$

As in (MARTINS; MEREU, 2014), we use the following formulas:

$$\int_0^{2\pi} \cos^{2k+1}\theta \cdot \sin^2\theta d\theta = 0, k = 0, 1, 2...$$
$$\int_0^{2\pi} \cos^{2k}\theta \cdot \sin^2\theta d\theta = \pi\alpha_k \neq 0, k = 0, 1, 2...$$

thus

$$I = \sum_{i=0}^{\left[\frac{n}{2}\right]} \pi a_{2i}\alpha_{2i} r^{2i+1}.$$

Hence I is a polynomial formed exclusively by odd exponents of r, and with $\left[\dfrac{n}{2}\right] + 1$ terms.

For the second integral, notice that

$$J = \int_0^{\pi} \sum_{j=0}^{m} b_j r^j \cos^j\theta \cdot \sin\theta d\theta = \sum_{j=0}^{m} b_j r^j \int_0^{\pi} \cos^j\theta \cdot \sin\theta d\theta.$$

In order to evaluate this integral, consider the change of variables $u = \cos\theta$, $du = \sin\theta d\theta$ then:

$$\int_0^{\pi} \cos^j\theta \cdot \sin\theta d\theta = \int_1^{-1} u^j du = \frac{(-1)^{j+1} - 1}{j+1}.$$

If j is odd, then $j+1$ is even and $\dfrac{(-1)^{j+1} - 1}{j+1} = 0$; but, when j is even, i.e. when $j+1$ is odd, then $\dfrac{(-1)^{j+1} - 1}{j+1} = -\dfrac{2}{j+1} \neq 0$. Hence

$$J = \sum_{j=0}^{\left[\frac{m}{2}\right]} \frac{-2b_{2j} r^{2j}}{2j+1},$$

which means that J is a polynomial with $\left[\dfrac{m}{2}\right] + 1$ monomials, formed by even exponents of r.

Since $F_0(r) = \dfrac{1}{2\pi}(I + 2J)$ and I and J don't have any powers of r in common, it follows that $F_0(r)$ is a polynomial with $\left[\dfrac{n}{2}\right] + \left[\dfrac{m}{2}\right] + 2$ terms. Therefore, by the Descartes' Theorem, $F_0(r)$ can have at most $\left[\dfrac{n}{2}\right] + \left[\dfrac{m}{2}\right] + 1$ roots; then, if we could apply Theorem 11, it would follow that the system (4.6) can have $\left[\dfrac{n}{2}\right] + \left[\dfrac{m}{2}\right] + 1$ limit cycles bifurcating from the linear center. Moreover, since we can choose the coefficients from $F_0(r)$ such that it has exactly $\left[\dfrac{n}{2}\right] + \left[\dfrac{m}{2}\right] + 1$ roots, this maximum number of limit cycles can indeed be achieved. Thus, to complete this proof, all we need to do is verify that the hypotheses from Theorem 11 are fulfilled.

First of all, for a given bounded set D, it is easy to see that $F^j(r,\theta)$ and $R^j(r,\theta,\varepsilon)$, $j=1,2$, are Lipschitz with respect to the variable r since they are polynomials in r; moreover, they are also periodic on θ with period 2π, hence the hypothesis *H2* is satisfied. We are left to show that so are the hypotheses *H1* and *H3* — this will be done by proving the following propositions:

Proposition 1. *For $|\varepsilon| \neq 0$ sufficiently small, there exists an open bounded set C such that every solution of system (4.6) reaches Σ on Σ^c.*

Proof of Proposition 1: Since the original system is autonomous, it's sufficient to analyze under which conditions the set Σ^c exists.

First, notice that we can write $\Sigma = h^{-1}(0)$, where $h(x,y)=y$. Let X and Y denote the smooth pieces of the system, i.e.:

$$X(x,y) = \begin{pmatrix} y \\ -x-\varepsilon(f(x)\cdot y + g(x)) \end{pmatrix} \text{ and } Y(x,y) = \begin{pmatrix} y \\ -x-\varepsilon(f(x)\cdot y - g(x)) \end{pmatrix}.$$

Let $p \in \Sigma$ be the point where the solution crosses the discontinuity, then $p=(x,0)$; computing the Lie derivatives on p:

$$\begin{aligned} Xh(p) &= X(p)\cdot\nabla h(p) = (0,-x+\varepsilon\cdot g(x))\cdot(0,1) = -x+\varepsilon\cdot g(x), \\ Yh(p) &= Y(p)\cdot\nabla h(p) = (0,-x-\varepsilon\cdot g(x))\cdot(0,1) = -x-\varepsilon\cdot g(x), \end{aligned}$$

thus $Xh(p)\cdot Yh(p) = x^2-\varepsilon^2(g(x))^2$. Therefore, for a sufficiently small $|\varepsilon|$ and $x\neq 0$, $p\in\Sigma^c$, hence we can find a bounded set such that every solution passing through it reaches Σ at a crossing point. $\square$

Proposition 2. *The coefficients of the polynomials f and g can be chosen in a way that, for every $a\in\mathbb{R}$ with $F_0(a)=0$, there exists a neighborhood U of a such that $F_0(z)\neq 0$ for all $z\in\overline{U}\backslash\{a\}$ and $d_B(F_0,U,0)\neq 0$*

Proof of Proposition 2: Let $Z=\{r\in\mathbb{R}^+ : F_0(r)=0\}$ be the set of positive zeros of the averaged function. The Brouwer degree of a C^1-function f with respect to a neighborhood V of zero is given by (see (BUICĂ; LLIBRE, 2004)):

$$d_B(f,V,0) = \sum_{a\in f^{-1}(0)\cap V} \text{sign}(\det Df(a))$$

and, since F_0 is a polynomial on $r\in\mathbb{R}$, then

$$d_B(F_0,V,0) = \sum_{a\in Z\cap V} \text{sign}(F_0'(a)).$$

Let $a\in Z$, then there exists a neighborhood U such that $F_0(z)\neq 0\ \forall z\in\overline{U}\backslash\{a\}$ (e.g. consider an open interval with radius half the distance to the next zero of F_0); then

$$d_B(F_0,U,0) = \text{sign}(F_0'(a)).$$

Thus what is left to prove is that we can choose the coefficients of F_0 such that $F_0'(a) \neq 0$ for every $a \in Z$.

Recall that, if $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{j=0}^{m} b_j x^j$, then

$$F_0(r) = \frac{1}{2\pi}\left[\sum_{i=0}^{\left[\frac{n}{2}\right]} \pi a_{2i}\alpha_{2i} r^{2i+1} + 2\cdot\sum_{j=0}^{\left[\frac{m}{2}\right]} \frac{-2b_{2j}r^{2j}}{2j+1}\right],$$

or simply

$$F_0(r) = \sum_{i=0}^{\left[\frac{n}{2}\right]} \tilde{a}_{2i} r^{2i+1} + \sum_{j=0}^{\left[\frac{m}{2}\right]} \tilde{b}_{2j} r^{2j}.$$

Since $m \geqslant 1$, we can choose $b_0 \neq 0$, hence $\tilde{b}_0 \neq 0$. We will show that this is sufficient to prove that, in order to choose the coefficients of a polynomial with a maximum number of positive roots, we can't have the derivatives vanishing at the roots.

Consider $p(x) = c_0 + \sum_{j=1}^{n} c_j x^{l_j}$ a polynomial with $n+1$ terms for which we have chosen the coefficients such that it has n positive roots. Notice that Theorem 12 guarantees such choice and this number is maximal. Since $p(x)$ is a C^∞-function, then so is $p'(x)$; hence, the Mean Value Theorem implies that between each zero of $p(x)$ there should exist a zero of $p'(x)$. However, $p'(x) = \sum_{j=1}^{n} l_j \cdot c_j x^{l_j - 1}$ is a polynomial with n terms; thus, by Theorem 12, it can have at most $n-1$ positive roots, which means that all the positive real roots of $p'(x)$ lie between the positive roots of $p(x)$, therefore p can't have it's derivative vanishing at its positive roots.

Since F_0 fulfills the above conditions, we can choose $\tilde{a}_{2i}$ and $\tilde{b}_{2j}$ so that $F_0(r)$ has $\left[\frac{n}{2}\right] + \left[\frac{m}{2}\right] + 1$ roots and for every a with $F_0(a) = 0$ there's a neighborhood $U \ni a$ in which $F_0(z) \neq 0$ for all $z \in U$ and

$$d_B(F_0, U, 0) = \text{sign}(F_0'(a)) \neq 0.$$

Furthermore, since $\tilde{a}_{2i} = \pi a_{2i}\alpha_{2i}$ with $\alpha_{2i} \neq 0$ and $\tilde{b}_{2j} = \frac{-4b_{2j}}{2j+1}$, we can choose a_i and b_j, $i = 1, ..., n$ and $j = 1, ..., m$, such that we obtain the desired coefficients for F_0. □

Example 9. Let's find a system like (4.6) with $n = 4$ and $m = 2$ such that the maximum number of predicted limit cycles is achieved, which is, in this case, 4 limit cycles. In order to do so, consider the following system:

$$\begin{aligned}
\dot{x} &= y,\\
\dot{y} &= -x - \varepsilon[(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4)\cdot y + \text{sgn}(y)(b_0 + b_1x + b_2x^2)].
\end{aligned}$$

Our first step is to make a change of variables to write the original system in polar coordinates, which yields the system:

$$\begin{aligned}\dot{r} =& -\varepsilon\sin(\theta)(r\sin(\theta)(a_0 + r\cos(\theta)(a_1 + r\cos(\theta)(a_2 + r\cos(\theta)(a_3 + a_4 r\cos(\theta)))))+\\ &+\operatorname{sgn}(r)\operatorname{sgn}(\sin(\theta))(b_0 + r\cos(\theta)(b_1 + b_2 r\cos(\theta)))),\\ \dot{\theta} =& -\sin(\theta)(\varepsilon\cos(\theta)(a_0 + r\cos(\theta)(a_1 + r\cos(\theta)(a_2 + r\cos(\theta)(a_3 + a_4 r\cos(\theta))))) + \sin(\theta))+\\ &-\frac{\varepsilon\operatorname{sgn}(r)\cos(\theta)\operatorname{sgn}(\sin(\theta))(b_0 + r\cos(\theta)(b_1 + b_2 r\cos(\theta)))}{r} - \cos^2(\theta).\end{aligned}$$

Taking θ as the new independent variable and applying the Taylor expansion series until order 2, we get the following differential equation:

$$\begin{aligned}\frac{dr}{d\theta} =& \varepsilon(r\sin^2(\theta)(\ a_0 + r\cos(\theta)(\ a_1 + r\cos(\theta)(\ a_2 + r\cos(\theta)(\ a_3 +\ \ a_4 r\cos(\theta)))))+\\ &+\operatorname{sgn}(\sin(\theta))(\ b_0 + r\cos(\theta)(\ b_1 +\ \ b_2 r\cos(\theta)))) + \mathcal{O}(\varepsilon^2).\end{aligned}$$

Then we can compute the averaged function $F_0(r) = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} F(r,\theta)d\theta$, where $F(r,\theta)$ is the expression multiplying ε in the above equation:

$$F_0(r) = \frac{\varepsilon\left(r\left(3\pi\left(8a_0 + 2a_2 r^2 + a_4 r^4\right) + 32b_2 r\right) + 96b_0\right)}{48\pi}.$$

Having found the averaged function, our next step is to force the existence of four positive roots; i.e. we build a system on the coefficients of $F_0(r)$ by replacing r by four positive values, namely $r = 1, 2, 3$ and 4.

$$\begin{cases} 3\pi(8a_0 + 2a_2 + a_4) + 32b_2 + 96b_0 = 0,\\ 2(3\pi(8a_0 + 8a_2 + 16a_4) + 64b_2) + 96b_0 = 0,\\ 3(3\pi(8a_0 + 18a_2 + 81a_4) + 96b_2) + 96b_0 = 0,\\ 4(3\pi(8a_0 + 32a_2 + 256a_4) + 128b_2) + 96b_0 = 0.\end{cases} \tag{4.7}$$

Solving the system (4.7) on a_0, a_2, a_4 and b_0, we get

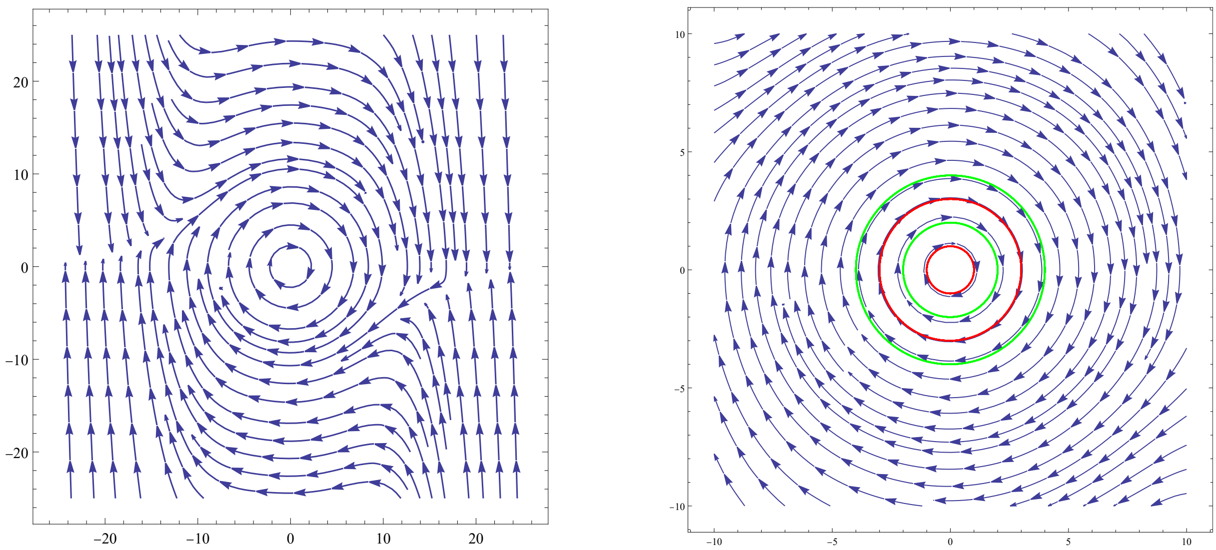
$$\left\{a_0 = -\frac{476b_2}{225\pi}, a_2 = -\frac{52b_2}{45\pi}, a_4 = \frac{8b_2}{225\pi}, b_0 = \frac{4b_2}{15}\right\}.$$

If we set $b_2 = 1$ and $a_1 = a_3 = b_1 = 0$, the original system will be as following:

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= -x - \varepsilon\left(\left(\frac{8x^4}{225\pi} - \frac{52x^2}{45\pi} - \frac{476}{225\pi}\right)y + \left(x^2 + \frac{4}{15}\right)\operatorname{sgn}(y)\right).\end{aligned} \tag{4.8}$$

If we do the inverse calculations, the averaged system derived from system (4.8) will have the exact 4 roots that we forced in our calculations, which implies by Theorem 11 that, for a sufficiently small ε, this system should have 4 limit cycles. Indeed, using

Wolfram Mathematica we are able to detect these limit cycles for $\varepsilon = \dfrac{1}{100}$ studying the behavior of the numeric solution of the system near the points $(1, 0)$, $(2, 0)$, $(3, 0)$ and $(4, 0)$. We notice that between the origin and the point $(1, 0)$ the solutions are spiraling towards the origin, while between $(1, 0)$ and $(2, 0)$ they are repelled from the direction of the origin. This behavior will invert itself between $(2, 0)$ and $(3, 0)$, then again between $(3, 0)$ and $(4, 0)$ and then, finally, after $(4, 0)$ the solutions are all attracted towards the origin. Therefore we will have an unstable limit cycle near $(1, 0)$, a stable limit cycle near $(2, 0)$, another unstable one near $(3, 0)$ and another stable one near $(4, 0)$. Figure 21 illustrates the limit cycles in the phase portrait.



(a) Global phase portrait of system (4.8)

(b) The green cycles are stable and the red cycles are unstable

Figure 21 – Illustration from the general behavior of system (4.8)

Remark 3. *We call figure 21 an illustration because the precise location of the limit cycles are not computed by the averaging method. Notice that in this example we need a* $\varepsilon = \dfrac{1}{100}$ *in order to make all the cycles appear, which is considerably smaller then the* ε *in example 8, hence the variations of the spirals near the cycles are also smaller, making it more difficult to locate the cycle.*

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