IDE: Optimization problems and population dynamics

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XI Congress GAFEVOL 2017

October 23-26, University of Brasilia, Brazil

INTRODUCTION

We present in this paper the results contained in the article “Optimization problems for one-impulsive models from population dynamics” (see [1]) studied in the end of our project.

“Impulsive differential equations (IDE) are suitable mathematical models for a large class of real processes. These processes are submitted to short temporary perturbations, that are negligible compared to the process duration. Thus the perturbations occur immediately as impulses”.

Most models from population dynamics are represented by the ODE

$$\frac{dN(t)}{dt} = f(t, N), \tag{1}$$

where $N = N(t) > 0$ is the population biomass (size) in time $t \geq 0$ and $f(t, N)$ is the total growth rate of population biomass. Besides that, in many cases we have $f(t, N) = f(N)$, then we say that there is a temporal constancy of the environment and the ODE (1) is said to be autonomous.

Depending on the choice of f and the characteristics of the environment and of the studied population, we obtain different models. Then, the *Logistic* and *Gompertz* equations are considered and they are represented, respectively, by the equations

$$\frac{dN}{dt} = \frac{r}{K} N(K - N) \quad \text{e} \quad \frac{dN}{dt} = N(r - \gamma \ln N)\,.$$

Definition. Given a continuous application $f : \mathbb{R}^+ \times \Omega \to \mathbb{R}^n$, with $\Omega \subseteq \mathbb{R}^n$ open, an application $\mathrm{I}_k : \Omega \to \Omega$, $k \in \mathbb{N}$, and a sequence of instants $(t_k)_{k\in\mathbb{N}}$ with $\lim\limits_{k\to\infty} t_k = +\infty$. An Initial Value Impulsive Problem with *pre-fixed impulse instants* is given by

$$\begin{cases} \dfrac{d\eta(t)}{dt} = f(t, \eta(t)),\ t \neq t_k,\ k \in \mathbb{N}, \\ \Delta\eta(t) = \mathrm{I}_k(\eta(t)), \quad t = t_k, \\ \eta(t_0) = \eta_0, \end{cases} \tag{2}$$

where $\Delta\eta(t_k) = \eta(t_k^+) - \eta(t_k^-) = \mathrm{I}_k(\phi(t_k))$ is the *impulse condition*, with $\eta(t_k^-) = \eta(t_k)$ and $\eta(t_k^+) = \eta(t_k) + \mathrm{I}_k(t_k)$.

Definition (Solution)**.** A function $\phi : [t_0, t_0 + \alpha) \to \mathbb{R}^n$, with $t_0 \geq 0$, is a solution of (2) if

i) $\phi(t_0) = \eta_0$;

ii) $\dfrac{d\phi(t)}{dt} = f(t, \phi(t))$ for all $t \in (t_k, t_{k+1}]$, for each $k \in \mathbb{N}$;

iii) $\Delta\phi(t_k) = \mathrm{I}_k(\phi(t_k))$, for all $k \in \mathbb{N}$.

OBJECTIVES

We intend to construct an impulsive problem as follows

$$\begin{cases} \eta'(t) = f(t, \eta),\ t \neq \tau,\ t \in [0, T]\,, \\ \Delta\eta(\tau) = \eta(\tau^+) - \eta(\tau^-) = -I\,, \\ \eta(0) = x_0\,, \end{cases} \tag{3}$$

with $I > 0$ and unique moment of impulse $0 < \tau \leq T$, and study the optimization problems. This construction is done with the auxiliary results, which ones are important to the main results.

HYPOTHESES

We introduce the following hypotheses (H):

H1) $\varphi_1, \varphi_2 : [0, T] \to \mathbb{R}$, $\varphi_1, \varphi_2 \in \mathcal{C}([0, T], \mathbb{R})$, $T \in (0, \infty)$;

H2) There exists a constant $I > 0$ such that

$$\varphi_1(t) + 2I < \varphi_2(t)\,,\ \forall t \in [0, T]\,;$$

H3) $f : \overline{D} \to \mathbb{R}$, where $D = \{(t, x) \in \mathbb{R}^2 \,|\, t \in [0, T],\ x \in (\varphi_1(t), \varphi_2(t))\}$;

H4) $f(t, \varphi_1(t)) = f(t, \varphi_2(t)) = 0$, for all $t \in [0, T]$;

H5) $f \in \mathcal{C}(\overline{D}, \mathbb{R})$;

H6) For each $t \in [0, T]$, the function $F(x) = f(t, x)$ has a unique maximum in the point $M(t) \in (\varphi_1(t), \varphi_2(t))$ and $\varphi_1(t) + I \leq M(t) \leq \varphi_2(t) - I$;

H7) $f \in \mathcal{C}^1(D, \mathbb{R})$;

H8) $\forall (t, x) \in [0, T] \times (M(t), M(t) + I)$,

$$f(t, x) + \frac{f_t(t, x) - f_t(t, x - I)}{f_x(t, x) - f_x(t, x - I)} > 0\,,$$

where $f_t(t, x) = \dfrac{\partial f}{\partial t}(t, x)$ e $f_x(t, x) = \dfrac{\partial f}{\partial x}(t, x)$;

H9) $\varphi_1(t)$ is a decreasing function in $[0, T]$;

H10) $\varphi_2(t)$ is a increasing function in $[0, T]$;

H11) $\varphi_1(0) + I \leq x_0 \leq \varphi_2(0)$.

AUXILIARY RESULTS

Lemma 1. Let conditions (H1)-(H6) hold. Then for $t \in [0, T]$ and for the equation

$$g(x) = f(t, x) - f(t, x - I) = 0, \tag{4}$$

the next statements are valid:

i) This equation possesses a solution $\psi(t)$;

ii) $\psi(t) \in (M(t), M(t) + I)$;

iii) The solution $\psi(t)$ is unique;

iv) $\psi \in \mathcal{C}\left([0, T], \mathbb{R}^+\right)$.

We consider now the ordinary problem

$$\begin{cases} \dfrac{dx}{dt} = f(t, x), \\ x(0) = x_0, \end{cases} \tag{5}$$

and the solution is denoted by $x(t; 0, x_0)$ or just $x(t; x_0)$.

We will also consider, besides (3), the following impulsive problem:

$$\begin{cases} \dfrac{d\mu}{dt} = f(t, \mu),\ t \neq \hat{\tau}, \\ \Delta\mu(\hat{\tau}) = \mu(\hat{\tau}^+) - \mu(\hat{\tau}^-) = -I, \\ \mu(0) = x_0, \end{cases} \tag{6}$$

See that we take (6) changing only the instant of impulse from (3), since $\hat{\tau} \in [0, T]$. Therefore, the solutions of (3) e (6) will be denoted by $\eta(t)$ and $\mu(t)$, respectively, and are given by

$$\eta(t) = \begin{cases} x(t; x_0), & t \in [0, \tau], \\ x(t; \tau, x(\tau; x_0) - I), & t \in (\tau, T], \end{cases}$$

$$\mu(t) = \begin{cases} x(t; x_0), & t \in [0, \hat{\tau}], \\ x(t; \hat{\tau}, x(\hat{\tau}; x_0) - I), & t \in (\hat{\tau}, T]. \end{cases}$$

Lemma 2. Let the hypotheses (H1)-(H5) and (H9)-(H11) be valid. Then the solution of problem (6) is continuable to T, in other words, the integral curve of this problem remains in $\overline{D}$ for $t \in [0, T]$.

Lemma 3. Let the hypothesis (H) hold. Then the set

$$\Delta = \{t \in [0, T] \,|\, x(t; x_0) = \psi(t)\}$$

consist of, no more, than on point.

Due to the last result, the impulse instant in (3) will be the element of Δ, thus the problem will have one impulse or none, if $\Delta = \varnothing$.

Therefore, the solution of our problem satisfies, at moment of impulse τ,

$$f(\tau, \eta(\tau)) = f(\tau, \eta(\tau) - I)\,.$$

MAIN RESULTS

Theorem 1. Let the following conditions hold:

i) The hypotheses (H) are valid;

ii) $\Delta = \{t \in [0, T] \,|\, x(t; x_0) = \psi(t)\} = \{\tau\}$;

iii) $\hat{\tau} \neq \tau$, $\hat{\tau} \in [0, T]$.

Then $\eta(T) > \mu(T)$.

Theorem 2. Let the following conditions hold:

i) The hypotheses (H) are valid;

ii) $\Delta = \{t \in [0, T] \,|\, x(t; x_0) = \psi(t)\} = \varnothing$;

iii) $0 \leq \hat{\tau} < T$.

Then $\eta(T) - I = x(T; x_0) - I > \mu(T)$.

APPLICATIONS

Now, we apply the results to the population dynamic models finding the conditions to satisfy the hypothesis (H).

Logistic Model

Let us consider the logistic equation supposing that there is not a temporal constancy of the environment, then

$$\frac{dN}{dt} = \frac{r}{K(t)} N(K(t) - N),\ N(0) = N_0,$$

where $r > 0$.

Since $f(t, N) = rN(K(t) - N)/K(t)$, we will choose $\varphi_1 \equiv 0$ and $\varphi_2 = K(t)$, since $K \in \mathcal{C}^1([0, T], \mathbb{R}^+)$ increasing such that

$$K(t) < \sqrt{(K(0)^2 - I^2)e^{rt} + I^2}\,.$$

Thus, we have $D = \{(t, N) \,|\, t \in [0, T],\ 0 < N < K(t)\}$. And we need to take $0 < I < \min\{K(t)/2 \,|\, t \in [0, T]\}$ and $I \leq N_0 < K(0)$. Then, the hypothesis (H) are valid. Therefore, the results can be applied to the model. Besides that, solving equation (4), we get $\psi(t) = (K(t) + I)/2$.

To exemplify, consider the ordinary problem

$$\frac{dN}{dt} = \frac{r}{K} N(K - N),\ N(0) = N_0\,,$$

with $K > 0$ constant, which solution is given by

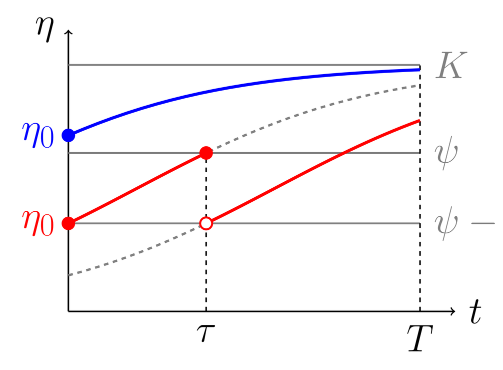
$$N(t; N_0) = \frac{KN_0}{N_0 + (K - N_0)e^{-rt}}\,.$$

So, to find the impulse moment of the impulsive problem

$$\begin{cases} \dfrac{d\eta}{dt} = \dfrac{r}{K}\eta(K - \eta),\ t \neq \tau \\ \Delta\eta(\tau) = -I, \\ \eta(t_0) = \eta_0, \end{cases}$$

we must solve the equation $N(\tau; N_0) = \psi = (K + I)/2$, getting

$$\tau = \frac{1}{r} \ln\left(\frac{K - N_0}{N_0} \cdot \frac{K + I}{K - I}\right)\,.$$



Gompertz Model

Consider the Gompertz equation

$$\frac{dN}{dt} = N(r - \gamma \ln N),\ N(0) = N_0,$$

where $r, \gamma > 0$.

As $f(N) = N(r - \gamma \ln N)$, it follows that f is not defined to $N = 0$. However, using the fact that $\lim\limits_{N\to 0} N \ln N = 0$, we will define $f(0) = 0$. Then, taking $\varphi_1 \equiv 0$, $\varphi_2 \equiv e^{r/\gamma}$, $D = \{(t, N) \,|\, t \in [0, T],\ 0 < N < e^{r/\gamma}\}$, $0 < I < e^{r/\gamma}/e$ and $I < N_0 < e^{r/\gamma}$, we guarantee that the hypotheses are satisfied. Hence, the results are able to be applied.

But this model has a bigger difficulty than the previous one to find ψ and the instant of impulse τ. It happens because, when we try to solve the equation $f(\psi) - f(\psi - I) = 0$, we get

$$e^{r/\gamma} = \frac{\psi^{\psi/I}}{(\psi - I)^{(\psi - I)/I}}\,.$$

Then,we can only find τ satisfying $N(\tau; N_0) = \psi$ calculating numerically.

CONCLUSIONS

The first optimization result (Theorem 1) says that, in the case in which $\Delta \neq \varnothing$, the solution of (3) with impulse at $\tau \in \Delta$ will have a higher value at T than the solution (6) with impulse at $\hat{\tau} \neq \tau$, $\hat{\tau} \in [0, T]$.

On the other hand, the Theorem 2 says that, in the case with $\Delta = \varnothing$, the solution with no impulses of (3) will assume a value at T such that, even subtracting I at this instant, is still higher than the assumed by the solution with impulse moment of (6).

THANKS

I am grateful for the opportunity to participate in the XI Congress GAFEVOL. I thank IGCE for the financial assistance. And I also thank Professor Marta for all teachings and orientations.

FINANCING

The project was partially financed by PIBIC/UNESP.

Referências

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