Branching random walk in random environment on trees

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Abstract

We study a supercritical branching random walk on a rooted tree with random environment. We are interested in the case where both the branching and the step transition parameters are random quantities. Criteria of (strong) recurrence and (strong) transience are presented for this model.

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Introduction

Branching random walk with discrete time is a process that can be described as follows. We start from one particle in some given site of a graph $\mathcal{G}$. If at some moment a particle is in site $x \in \mathcal{G}$, then, independently of everything, it generates new particles according to some probability distribution (which may depend on x). Those new particles then jump, independently of each other, to other sites of $\mathcal{G}$ according to some given transition probabilities. Let us underline that there is no "one-particle-per-site" rule.

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In this paper we study the above model in random environment. This means that first of all, for each site x the collection of transition probabilities and branching parameters is sampled from the set of all possible parameters according to a given distribution (or distributions). We present conditions for the discrete time branching random walk in random environment to be recurrent or transient (almost surely with respect to the random environment).

Several authors have recently considered branching random walks in random environment. Examples are [1, 5, 6, 8, 9, 10, 15, 20] just to mention a few. Branching random walks in *many-dimensional* random environment are considered in [1, 10, 20]. Questions related to the local and total particle populations are studied in [1, 20], while the main question of [10] is recurrence/transience of the process as seen from the origin, that is, whether the cloud of particles touches the origin almost surely. In [1, 5, 8, 9, 10, 20], the random environment is related *only* to the branching mechanism. One-dimensional branching random walk in i.i.d. random environment which includes both the branching mechanism and the step transition is considered in [6]. The paper [15] generalizes the results of [6] to the case of non-i.i.d. random environment. In the present paper we deal with branching random walk on a rooted homogeneous tree in a full random environment, which means that both the transition and the branching probabilities are sampled from a given distribution(s); in addition, the random environment is not i.i.d.

To the best of our knowledge, branching random walks in random environment on trees have not been yet considered in the literature (for ordinary random walks in random environment on trees, see e.g. [13, 17]). On the other hand, homogeneous branching random walks on trees were studied in a number of papers; see e.g. [18] and references therein. In the paper [3], among other things, the so-called frog model (with *random* initial configuration) on a homogeneous tree was considered. The frog model is a model which is in some sense in the frontier between random walk and branching random walk (there is branching only when a particle gets to a site which was not visited before); other results for the frog model can be found in [2, 4, 19, 21].

The main tool that we use in this paper is the method of Lyapunov (test) functions. For a general introduction to the subject see [7]. Examples of successful applications of this method to branching random walks can be found in [6, 11, 12, 14, 15, 16]. It is important to note that usually it is possible to use this method either in dimension 1, or in problems with spatially homogeneous transition probabilities (i.e., without random environment). For the authors, it was surprising to realize that the method of Lyapunov functions works for branching random walks in random environment on trees as well (however, the strategy of the proof here is considerably different from that of [15]).

The paper is organized as follows. In Section 1 we begin by introducing some notation and defining the branching random walk on trees with random environment. The results on recurrence and transience for the process as seen from the root are stated in Section 2. In Section 3 we give the proofs.

1 Definitions and construction of random environment

Fix a positive integer d. Let $\mathbb{T}_d$ be the the set of all finite words $x = x^{(1)} \dots x^{(k)}$, where $x^{(i)} \in \{1, \dots, d\}$ for $i = 1, \dots, k$, including the empty word $\mathcal{O}$. For any $x = x^{(1)} \dots x^{(n)}, y = y^{(1)} \dots y^{(m)} \in \mathbb{T}_d$ define the concatenation of x and y by $xy = x^{(1)} \dots x^{(n)} y^{(1)} \dots y^{(m)}$. If $x = x^{(1)} \dots x^{(n-1)} x^{(n)} \in \mathbb{T}_d$, define $\mathfrak{u}_i(x) = x^{(1)} \dots x^{(n)} i$ for all $i = 1, \dots, d$, and, for all $x \neq \mathcal{O}$ define $\mathfrak{d}(x) = x^{(1)} \dots x^{(n-1)}$. To give a graph structure to $\mathbb{T}_d$, we declare $x, y \in \mathbb{T}_d$ adjacent iff $x = \mathfrak{d}(y)$ or $y = \mathfrak{d}(x)$; clearly, proceeding in this way we obtain what is usually called a rooted d-ary tree (with $\mathcal{O}$ being the root). Futhermore, define $\|x\|$ to be the length of the word x (equivalently, $\|x\|$ is the distance from x to $\mathcal{O}$). Let $\tau_x : \mathbb{T}_d \mapsto \mathbb{T}_d$ be a map defined by $\tau_x(y) = xy$; denote

$$\begin{aligned} V_n &= \{x \in \mathbb{T}_d : \|x\| = n\}, \\ \mathbb{T}_{d,n} &= \{x \in \mathbb{T}_d : \|x\| \leq n\}, \\ \mathbb{T}_d(x) &= \tau_x(\mathbb{T}_d), \\ \mathbb{T}_{d,n}(x) &= \tau_x(\mathbb{T}_{d,n}). \end{aligned}$$

In order to facilitate introducing the notions of transience and recurrence, let us add also a "special" vertex $\mathfrak{S}$, which is connected only to the root $\mathcal{O}$ (write formally $\mathfrak{S} = \mathfrak{d}(\mathcal{O})$).

A discrete time branching random walk on $\mathbb{T}_d \cup \{\mathfrak{S}\}$ with fixed environment evolves according to the following rules. Initially a particle is placed at a fixed site x_0 of $\mathbb{T}_d$. At any time over the evolution of the process, any particle which is at a site x is replaced by a random number $k \geq 1$ of offspring, independently of everything else, with probability $r_k(x)$. That means that every single particle is replaced by at least one offspring. Each one of the offspring of a given particle which is at site $x \in \mathbb{T}_d$, may either jump to $\mathfrak{d}(x)$ with probability $q(x)$, stay at site x with probability $s(x)$, or go to some of the vertices $\mathfrak{u}_i(x), i = 1, \ldots, d$, with probabilities $p_i(x)$, $i = 1, \ldots, d$ respectively. Also, let

$$r(x) = \sum_{n=1}^{\infty} n r_n(x)$$

be the mean number of offspring at x. Aiming toward a definition for recurrence/transience, we declare the vertex $\mathfrak{S}$ to be an *absorbing site* in the sense that, once a particle comes to that site, it stays there forever without having any further offspring, which is equivalent to putting $s(\mathfrak{S}) = 1$ and $r_1(\mathfrak{S}) = 1$.

Let us consider a sequence of random variables $\mathcal{N}(t)$, $t = 0, 1, 2, \ldots$, where $\mathcal{N}(t)$ is the number of particles at site $\mathfrak{S}$ at time t. As this sequence is non-decreasing, the random variable

$$\nu = \lim_{t \to \infty} \mathcal{N}(t)$$

exists for all realizations of the process and may assume the value $+\infty$.

In this paper, we denote by $\mathbf{P}$ and $\mathbf{E}$ the probability and the expectation that concern the random environment, while $\mathbb{P}$ and $\mathbb{E}$ stand for the probability and the expectation concerning the evolution of the process when the environment is fixed.

Following [6, 15], consider

Definition 1.1 *We say that the branching random walk on $\mathbb{T}_d$ is*

- *recurrent, if $\mathbb{P}[\nu \geq 1] = 1$;*
- *strongly recurrent, if it is recurrent and $\mathbb{E}\nu < \infty$;*
- *transient, if $\mathbb{P}[\nu \geq 1] < 1$;*
- *strongly transient, if it is transient and $\mathbb{E}\nu < \infty$.*

Provided that for every pair x and y of vertices of $\mathfrak{S}$, x is reachable from y and y is reachable from x, one can get that Definition 1.1 does not depend on the position of the initial particle at time 0.

Now, we construct the random environment. First, define $\mathcal{M}$ to be the set of all infinite sequences $\xi = (q^\xi, s^\xi, p_1^\xi, \dots p_d^\xi, r_1^\xi, r_2^\xi, \dots)$ of non-negative numbers such that

$$q^\xi + s^\xi + \sum_{i=1}^{d} p_i^\xi = 1, \quad \sum_{n=1}^{\infty} r_n^\xi = 1, \quad \text{and} \quad r^\xi := \sum_{n=1}^{\infty} n r_n^\xi < \infty.$$

Let $\mathcal{A}$ be some finite set. Let $Q_i, i \in \mathcal{A}$, be a collection of probability distributions on $\mathcal{M}$ which satisfy the following two conditions:

$$Q_j[\xi \in \mathcal{M} : q^\xi > 0, p_1^\xi > 0, \dots, p_d^\xi > 0] = 1 \text{ for all } j \in \mathcal{A}, \tag{1.1}$$

and

$$Q_{j_0}[\xi \in \mathcal{M} : r^\xi > 1] > 0 \text{ for at least one } j_0 \in \mathcal{A}. \tag{1.2}$$

The condition (1.1) is to guarantee that, within $\mathbb{T}_d$, any site is reachable from any other site, while the condition (1.2) ensures that there is branching. To define the random environment for the branching random walk on tree we proceed as follows. Let $(P_{i,(j_1,\dots,j_d)}, i \in \mathcal{A}, (j_1, \dots, j_d) \in \mathcal{A}^d)$, be a collection of nonnegative numbers with the following properties:

(P1) $\displaystyle\sum_{(j_1,\dots,j_d)\in\mathcal{A}^d} P_{i,(j_1,\dots,j_d)} = 1$ for all $i \in \mathcal{A}$, and

(P2) for every $i, i' \in \mathcal{A}$, there exists a finite sequence $i = i_0, i_1, i_2, \dots, i_n = i'$ such that for all $k = 1, \dots, n$ there exist $m_1^{(k)}, \dots, m_d^{(k)} \in \mathcal{A}$ such that $P_{i_{k-1},(m_1^{(k)},\dots,m_d^{(k)})} > 0$ and $i_k \in \{m_1^{(k)}, \dots, m_d^{(k)}\}$ (this property is referred to as irreducibility).

Then, to each $x \in \mathbb{T}_d$ we assign a number $\eta_x \in \mathcal{A}$ by using a realization of a multitype Galton-Watson branching process governed by the collection $(P_{i,(j_1,\dots,j_d)})$. Specifically, first we choose $\eta_{\mathcal{O}}$ uniformly from $\mathcal{A}$. Then,

$$\mathbf{P}[\eta_{\mathfrak{u}_1(x)} = j_1, \dots, \eta_{\mathfrak{u}_d(x)} = j_d \mid \eta_x = i] = P_{i,(j_1,\dots,j_d)},$$

where $i \in \mathcal{A}$ and $(j_1, \dots, j_d) \in \mathcal{A}^d$. In this way, we obtain a realization of the multitype Galton-Watson process $\boldsymbol{\eta} = (\eta_x, x \in \mathbb{T}_d)$. Consider a realization of the collection of random variables $\boldsymbol{\xi} = (\xi_x, x \in \mathbb{T}_d)$, $\xi_x \in \mathcal{M}$ with the following properties:

- the random variables $(\xi_x, x \in \mathbb{T}_d)$ are conditionally independent given the realization of $\boldsymbol{\eta}$;
- for every $x \in \mathbb{T}_d$, given $\eta_x = i$, the distribution of ξ_x is Q_i.

The pair $(\boldsymbol{\eta}, \boldsymbol{\xi}) = ((\eta_x, \xi_x), x \in \mathbb{T}_d)$ is called *random environment*. Then, for all $x \in \mathbb{T}_d$ we define $q(x) = q^{\xi_x}, s(x) = s^{\xi_x}, p_i(x) = p_i^{\xi_x}$ for $i = 1, \ldots, d$ and $r_i(x) = r_i^{\xi_x}$ for $i = 1, 2, \ldots$ Once again, we stress that those parameters are chosen once and do not change during the subsequent evolution of the branching random walk.

2 Results

For every $\xi \in \mathcal{M}$ let us define the function

$$h^{\xi}(\lambda_1, \ldots, \lambda_d) = q^{\xi} \mathcal{L}\Big(\frac{1}{r^{\xi}} - s^{\xi} - \sum_{j=1}^{d} p_j^{\xi} \lambda_j\Big) \tag{2.1}$$

for $0 \le \lambda_i \le +\infty$, where

$$\mathcal{L}(x) = \begin{cases} x^{-1}, & \text{if } x > 0, \\ +\infty, & \text{if } x = 0, \\ -\infty, & \text{if } x < 0 \text{ or } x = -\infty. \end{cases}$$

Moreover, for each $i \in \mathcal{A}$ and $\lambda_1, \ldots, \lambda_d \in [0, +\infty) \cup \{+\infty, -\infty\}$ put

$$H_i(\lambda_1, \ldots, \lambda_d) = \begin{cases} -\infty, & \text{if } \lambda_i = -\infty \\ & \text{for at least one } i \in \{1, \ldots, d\}, \\ & \text{or there exists } \xi \in \operatorname{supp} Q_i \\ & \text{such that } h^{\xi}(\lambda_1, \ldots, \lambda_d) = -\infty, \\ \sup\limits_{\xi \in \operatorname{supp} Q_i} h^{\xi}(\lambda_1, \ldots, \lambda_d), & \text{otherwise.} \end{cases} \tag{2.2}$$

For every $i \in \mathcal{A}$ define

$$C(i) = \{(j_1, \ldots, j_d) \in \mathcal{A}^d : P_{i,(j_1,\ldots,j_d)} > 0\}.$$

Also, for every $(j_1, \ldots, j_d) \in \mathcal{A}^d$ let

$$\Psi(j_1, \ldots, j_d) = \{i \in \mathcal{A} : P_{i,(j_1,\ldots,j_d)} > 0\}.$$

The main result of this paper is the following

Theorem 2.1 *For $\mathbb{E}\nu$ to be finite $\mathbf{P}$-a.s., it is necessary and sufficient that there exists a collection $(\lambda_i \in (0,+\infty), i \in \mathcal{A})$ such that for every $i \in \mathcal{A}$*

$$\lambda_i \geq H_i(\lambda_{j_1}, \dots, \lambda_{j_d}) > 0 \quad \textit{for all } (j_1, \dots, j_d) \in C(i). \tag{2.3}$$

Moreover, if there exists $i \in \mathcal{A}$ such that $\lambda_i < 1$, then the process is strongly transient $\mathbf{P}$-a.s. If $\lambda_i \geq 1$ for all $i \in \mathcal{A}$, then the process is strongly recurrent $\mathbf{P}$-a.s.

To complete the classification, we state the following

Theorem 2.2 *If there is no positive solution to (2.3) (i.e., if $\mathbb{E}\nu = \infty$ $\mathbf{P}$-a.s.), then the branching random walk is recurrent (not strongly) $\mathbf{P}$-a.s.*

It is interesting to note that for the case $|\mathcal{A}| = 1$ the branching random walk in random environment on $\mathbb{T}_d$ is equivalent (from the point of view of Definition 1.1) to the corresponding branching random walk in random environment on $\mathbb{R}_+$ (q^{ξ_x} is the probability of jumping to the left and $\sum_{i=1}^d p_i^{\xi_x}$ is the probability of jumping to the right; the classification for the one-dimensional branching random walk in random environment was given in [6]). The heuristic explanation for this is the following: as shown in [6], to guarantee the nonexistence of $\mathbb{E}\nu$, it is sufficient to find x and N such that the random environment in the points of the interval $[x, x+N]$ has some given properties. From the proof there it is possible to deduce that if the environment on that interval has those properties, then with positive probability the number of particles in the process will go to infinity even if we kill all particles which leave the interval $[x, x+N]$. Passing to the process on the tree $\mathbb{T}_d$, it is then possible to imagine a finite subtree $\mathbb{T}_{d,N}(x)$ such that the environment is roughly the same in all $y \in \mathbb{T}_{d,i}(x) \setminus \mathbb{T}_{d,i-1}(x)$, $i = 1, \dots, N$. By keeping track of only the number of particles in the sets $\mathbb{T}_{d,i}(x) \setminus \mathbb{T}_{d,i-1}(x)$, we can compare the branching random walk in random environment on tree with the one-dimensional one (we do not give all the details).

When $|\mathcal{A}| \geq 2$, to apply Theorem 2.1, one must check if the intersection of $\sum_{i \in \mathcal{A}} |C(i)|$ subsets of $\mathbb{R}_+^{|\mathcal{A}|}$ (defined by (2.3)) is nonempty. Consider the following simple example:

Example. Suppose that $|\mathcal{A}| = 2$, $d = 2$, $C(1) = \{(1,1),(1,2)\}$, $C(2) = \{(1,1)\}$, $Q_1[q^\xi = p_1^\xi = p_2^\xi = \frac{1}{3}, r^\xi = 1] = 1$, and for some fixed $\alpha > 1$,

$Q_2[q^\xi = \frac{1}{13}, p_1^\xi = \frac{5}{13}, p_2^\xi = \frac{7}{13}, r^\xi = \alpha] = 1$. Then, conditions (2.3) become:

$$\lambda_1 \geq (3 - 2\lambda_1)^{-1} > 0, \tag{2.4}$$

$$\lambda_1 \geq (3 - \lambda_1 - \lambda_2)^{-1} > 0, \tag{2.5}$$

$$\lambda_2 \geq \Big(\frac{13}{\alpha} - 12\lambda_1\Big)^{-1} > 0. \tag{2.6}$$

It is elementary to obtain that the subsets of $\mathbb{R}_+^2$ defined by (2.4)–(2.6) have nonempty intersection if and only of $\alpha \leq \frac{13}{8}$. Moreover, in that case the intersection will be inside the unit square $\{(\lambda_1, \lambda_2) : 0 < \lambda_{1,2} < 1\}$. So, Theorems 2.1 and 2.2 show that if $\alpha \leq \frac{13}{8}$, then the process is strongly transient, and if $\alpha > \frac{13}{8}$, then the process is recurrent (not strongly).

3 Proofs

To prove Theorem 2.1, we need the following fact:

Lemma 3.1 *For $\mathbb{E}\nu$ to be finite, it is necessary that there exist a positive function $f(x)$, $x \in \mathbb{T}_d \cup \{\mathfrak{S}\}$, such that for all $x \in \mathbb{T}_d$*

$$q(x)f(\mathfrak{d}(x)) + s(x)f(x) + \sum_{i=1}^{d} p_i(x)f(\mathfrak{u}_i(x)) = \frac{f(x)}{r(x)}. \tag{3.1}$$

It is sufficient for $\mathbb{E}\nu$ to be finite that there exist a positive function $f(x)$, $x \in \mathbb{T}_d \cup \{\mathfrak{S}\}$, such that for all $x \in \mathbb{T}_d$

$$q(x)f(\mathfrak{d}(x)) + s(x)f(x) + \sum_{i=1}^{d} p_i(x)f(\mathfrak{u}_i(x)) \leq \frac{f(x)}{r(x)}. \tag{3.2}$$

Moreover, if f satisfies (3.2) and there exists a sequence $x_1, x_2, x_3, \ldots$ such that $f(x_n) \to 0$ as $n \to \infty$, then the branching random walk is strongly transient. If $f(x) \to \infty$ as $\|x\| \to \infty$, then the branching random walk is strongly recurrent.

Lemma 3.1 is a composition of Theorems 2.1 and 2.2 of [6] which in turn are inspired by corresponding results in [11, 12, 16].

Proof of Theorem 2.1. Let us prove the sufficient condition for $\mathbb{E}\nu$ to be finite. To this end we show that the existence of solution of (2.3) implies the existence of function $f(x)$ satisfying (3.2). For each $x \in \mathbb{T}_d$ let $x_0 = \mathcal{O}, x_1 \dots, x_{\|x\|} = x \in \mathbb{T}_d$ be the unique self-avoiding path that connects $\mathcal{O}$ to x. Now we define the function f on $\mathbb{T}_d$ as follows:

$$f(x) = \prod_{i=1}^{\|x\|} \lambda_{\eta_{x_i}}, \tag{3.3}$$

and put $f(\mathfrak{S}) = 1$. It is straightforward to get from (2.3) that for this function the condition (3.2) is satisfied, thus proving the sufficiency in Theorem 2.1.

Let us deal with the strong transience and the strong recurrence. Denote by $\mathbf{1}$ the vector $(1, \dots, 1) \in \mathbb{R}^d$. Observe that it is true that $h^\xi(\mathbf{1}) \geq 1$ which in turn implies that $H_i(\mathbf{1}) \geq 1$ for all $i \in \mathcal{A}$ (provided, of course, that $h^\xi(\mathbf{1}) \neq \pm\infty$, $H_i(\mathbf{1}) \neq \pm\infty$). Observe also that if $j \in \mathcal{A}$ is such that $Q_j\{\xi \in M : r^\xi > 1\} > 0$ (the existence of such j is guaranteed by condition (1.2)), then $H_j(\mathbf{1}) > 1$ (again, in the case when $H_j(\mathbf{1}) \neq \pm\infty$).

Now, suppose that for some $i \in \mathcal{A}$ it holds that $\lambda_i < 1$. In this case, it is true that for any $(j_1, \dots, j_d) \in C(i)$ there exists k such that $\lambda_{j_k} < 1$, because, if it is not the case, then we would have $H_i(\lambda_{j_1}, \dots, \lambda_{j_d}) \geq H_i(\mathbf{1}) \geq 1 > \lambda_i$, which contradicts (2.3). From this observation, since $\mathcal{A}$ is finite, we easily get that $\mathbf{P}$-a.s. there exists a sequence $x_1, x_2, x_3, \ldots \in \mathbb{T}_d$ such that $f(x_n) \to 0$ as $n \to \infty$. Applying Lemma 3.1, we get the proof of the strong transience.

To get the proof for the strong recurrence, let us first prove the following fact: if $\lambda_i \geq 1$ for all $i \in \mathcal{A}$, then $\lambda_i > 1$ for all $i \in \mathcal{A}$. Indeed, first, the case $\lambda_i = 1$ for all $i \in \mathcal{A}$ was already ruled out. Then, suppose that there exist $i_0, j_0 \in \mathcal{A}$ such that $\lambda_{i_0} = 1, \lambda_{j_0} > 1$. Due to the property (P2), in this case we can find $i_1, j_1 \in \mathcal{A}$ such that $\lambda_{i_1} = 1, \lambda_{j_1} > 1$ and there exists $(k_1, \dots, k_d) \in C(i_1)$ such that $j_1 \in \{k_1, \dots, k_d\}$. So we have $H_{i_1}(\lambda_{k_1}, \dots, \lambda_{k_d}) > H_{i_1}(\mathbf{1}) \geq 1 = \lambda_{i_1}$, which contradicts (2.3) as well. Then, since $\lambda_i > 1$ for all $i \in \mathcal{A}$ and $\mathcal{A}$ is finite, it is straightforward to see that $f(x) \to \infty$ as $x \to \infty$ for f defined by (3.3). The proof of strong recurrence now follows from Lemma 3.1.

Now we prove the necessary condition for $\mathbb{E}\nu$ to be finite. From now on we assume that $\mathbb{E}\nu < \infty$ and that there is no solution of (2.3), in order to reach a contradiction. So, if $\mathbb{E}\nu < \infty$ then, by Lemma 3.1, there exists a positive function satisfying (3.1). That function can be specified uniquely through the following collection of quotients

$$\rho(x) = \frac{f(x)}{f(\mathfrak{d}(x))}, \quad x \in \mathbb{T}_d,$$

together with the value of $f(\mathfrak{S})$.

Recalling how the quantities $p_i(x), s(x), q(x), r(x)$ relate to the random environment $(\boldsymbol{\eta}, \boldsymbol{\xi})$, we get from (2.1) and (3.1) that

$$\rho(x) = h^{\xi_i}(\rho(\mathfrak{u}_1(x)), \dots, \rho(\mathfrak{u}_d(x))), \quad x \in \mathbb{T}_d. \tag{3.4}$$

Let $\mathcal{D} = \mathcal{A} \times (\mathbb{R}_+ \cup \{-\infty, +\infty\})$. Define

$$\Lambda(b_1, \dots, b_d) = \bigcup_{j \in \Psi(i_1, \dots, i_d)} \{j\} \times \{H_j(\beta_1, \dots, \beta_d)\} \subset \mathcal{D},$$

where $b_1, \dots, b_d \in \mathcal{D}$, $b_j = (i_j, \beta_j)$. Consider a sequence of sets $B_0, B_1, \dots \subset \mathcal{D}$ constructed in the following way: $B_0 = \mathcal{A} \times \{0\}$ and, inductively, if $B_n = \{b_1, \dots, b_m\}$ where $b_j = (i_j, \beta_j)$ for $j = 1, \dots, m$, then

$$B_{n+1} = \bigcup_{\substack{b_{k_i} \in B_n \\ i=1,\dots,d}} \Lambda(b_{k_1}, \dots, b_{k_d}).$$

Let $B = \cup_{i=1}^{\infty} B_i \subset \mathcal{D}$. For all $i \in \mathcal{A}$ define

$$\lambda_i = \begin{cases} -\infty, & \text{if } (i, -\infty) \in B, \\ \sup\{\beta : (i, \beta) \in B\}, & \text{otherwise.} \end{cases}$$

Note that, due to (2.2) and the property (P2), if $\lambda_i \in \{+\infty, -\infty\}$ for at least one $i \in \mathcal{A}$, then $\lambda_i = -\infty$ for all $i \in \mathcal{A}$. Now, from (2.1) and (2.2) it is straightforward to get that for any $a > 0$ and any $i, i_1, \dots, i_d \in \mathcal{A}$ the set $\{x_{i_k} \geq 0, k = 1, \dots, d : 0 < H_i(x_{i_1}, \dots, x_{i_d}) \leq a\}$ is a closed subset of $\mathbb{R}_+^{|\mathcal{A}|}$. From that and the monotonicity properties of the function H we get that if $\lambda_i \in \mathbb{R}_+$ for all $i \in \mathcal{A}$, then $(\lambda_i, i \in \mathcal{A})$ provide the solution for (2.3).

For the case when there exists i such that $\lambda_i = -\infty$, define

$$N = \min\{j : B_j \text{ contains an element with second coordinate } (-\infty)\}.$$

Now, we construct two functions $\ell : \mathbb{T}_{d,N} \mapsto \mathcal{A}$ and $\gamma : \mathbb{T}_{d,N} \mapsto (\mathbb{R}_+ \cup \{+\infty, -\infty\})$. Let $b^{(N)} = (i^{(N)}, -\infty) \in B_N$. Put $\ell(\mathcal{O}) = i^{(N)}$ and $\gamma(\mathcal{O}) = -\infty$. Inductively, suppose that $\ell(x)$ and $\gamma(x)$ were constructed for some $x \in \mathbb{T}_{d,N-1}$. Denote by $\Phi^{(x)} \subset \mathcal{D}^d$ the set of all $(b''_1, \dots, b''_d) \in B^d_{N-\|x\|-1}$ such that $(\ell(x), \gamma(x)) \in \Lambda(b''_1, \dots, b''_d)$. Take any $(b'_1, \dots, b'_d)$ from the set $\Phi^{(x)}$, where $b'_k = (i'_k, \beta'_k)$, $k = 1, \dots, d$. Put $\ell(\mathfrak{u}_k(x)) = i'_k$ and $\gamma(\mathfrak{u}_k(x)) = \beta'_k$. So, the pair $(\ell(x), \gamma(x))$ was constructed for all $x \in \mathbb{T}_{d,N}$ (note that $\gamma(x) = 0$ for all $x \in V_N$). By construction, it is a fact that

$$\mathbf{P}[\eta_x = \ell(x) \text{ for all } x \in \mathbb{T}_{d,N} \mid \eta_{\mathcal{O}} = \ell(\mathcal{O})] > 0. \tag{3.5}$$

For all $x \in V_{N-1}$ define a function $\mathcal{H}_x$ of d arguments by $\mathcal{H}_x = H_{\ell(x)}$. Then, for all $x \in V_{N-2}$ define $\mathcal{H}_x = H_{\ell(x)}(\mathcal{H}_{\mathfrak{u}_1(x)}, \dots, \mathcal{H}_{\mathfrak{u}_d(x)})$. Proceeding in this way, we finally define $\mathcal{H}_{\mathcal{O}}$, which is a function of d^N arguments. Clearly, it holds that

$$\mathcal{H}_{\mathcal{O}}(0, \dots, 0) = -\infty. \tag{3.6}$$

For $x \in \mathbb{T}_{d,N}$ let $\Xi_x^{(\varepsilon)} \subset \mathcal{M}$ be the set (recall (2.1)–(2.2))

$$\Big\{\xi \in \mathcal{M} : \frac{1}{q^\xi}\Big(\frac{1}{r^\xi} - s^\xi - \sum_{j=1}^{d} p_j^\xi \gamma(\mathfrak{u}_j(x))\Big) \\ \leq \inf_{\xi \in \operatorname{supp} Q_{\ell(x)}} \frac{1}{q^\xi}\Big(\frac{1}{r^\xi} - s^\xi - \sum_{j=1}^{d} p_j^\xi \gamma(\mathfrak{u}_j(x))\Big) + \varepsilon\Big\}.$$

Due to (3.5), for every $\varepsilon > 0$ it is true that

$$\mathbf{P}\Big[\bigcap_{x \in \mathbb{T}_{d,N}} \{\xi_x \in \Xi_x^{(\varepsilon)}\} \;\Big|\; \eta_{\mathcal{O}} = \ell(\mathcal{O})\Big] > 0.$$

This means that $\mathbf{P}$-almost surely there exists $z \in \mathbb{T}_d$ such that $\xi_{\tau_z(y)} \in \Xi_y^{(\varepsilon)}$ for all $y \in \mathbb{T}_{d,N}$.

Now we go back to the collection $(\rho(\cdot))$ obtained by (3.4) and suppose for the moment that $\rho(x) = 0$ for all $x \in \mathbb{T}_{d,N}(z) \setminus \mathbb{T}_{d,N-1}(z)$. From (3.6) and continuity one gets that for ε small enough, if $\xi_{\tau_z(x)} \in \Xi_x^{(\varepsilon)}$ for all $x \in \mathbb{T}_{d,N}$, then $\rho(z) \notin (0, +\infty)$. Using the monotonicity of h (by (3.4), it shows that the case $\rho(x) = 0$ for all $x \in \mathbb{T}_{d,N}(z) \setminus \mathbb{T}_{d,N-1}(z)$ is the most favorable for the existence of nonnegative solution of (3.1) on $\mathbb{T}_{d,N}(z)$), we conclude the proof of Theorem 2.1. $\square$

Proof of Theorem 2.2. From (1.1) it follows that for all $i \in \mathcal{A}$ there exists $\varepsilon > 0$ such that

$$Q_i[\xi : p_1^\xi > \varepsilon] > 0. \tag{3.7}$$

Note the following fact: if $\mathbb{E}\nu = \infty$ $\mathbf{P}$-a.s., then for any $x \in \mathbb{T}_d$, if we start from one particle in any site of $\mathbb{T}_d(x) \setminus \{x\}$ and make x an absorbing state, then the mean number of particles absorbed in x will be infinite. Fix $x_0 \in \mathbb{T}_d \setminus \{\mathcal{O}\}$ and suppose that $\eta_{x_0} = i$ and $p_1(x_0) > \varepsilon$. Consider the model which starts from one particle at $x_0 1$ and with x_0 being an absorbing site. Then, as noted above, we have $\mathbb{E}\nu_{x_0} = \infty$, where ν_{x_0} is the number of particles absorbed in x_0. Define

$$\mathcal{K}_{x_0,i} = \mathcal{K}_{x_0,i}(\boldsymbol{\eta}, \boldsymbol{\xi}) = \min\Big\{n \geq 1 : \mathbb{E}\nu_{x_0}^{(n)} > \frac{1}{\varepsilon}\Big\},$$

where $\nu_{x_0}^{(n)}$ is the number of particles absorbed in x_0 during the first n steps. Clearly, $\mathcal{K}_{x_0,i} < \infty$ $\mathbf{P}$-a.s.

Now, let us choose $K \geq 1$ in such a way that

$$\min_{i\in\mathcal{A}} \mathbf{P}[\mathcal{K}_{x_0,i} \leq K] > 0. \tag{3.8}$$

Call a site x *good*, if $p_1(x) > \varepsilon$, and $\mathcal{K}_{x,\eta_x} \leq K$. If x is good, then analogously to the proof of Theorem 4.3 of [6], it is possible to construct an embedded supercritical Galton-Watson branching process in x. This is done in the following way: Start with one particle in x. At the next moment, consider the offspring of this particle which have jumped to $x1$; make x absorbing, and consider the descendants of the particle(s) from $x1$ which are stuck in x by time $K+1$. Clearly, since x is good, their expected number is strictly greater than 1. Iterating this construction, we obtain a supercritical branching process in x (we do not present here all the details, cf. [6]). Note that this process does not depend on particles that step outside of $\mathbb{T}_{d,K}(x)$. Note also that the event $\{x \text{ is good}\}$ depends only on the values of the environment on $\mathbb{T}_{d,K}(x)$.

For good x, let $\zeta(x)$ be the indicator of the event that the above embedded branching process does not become extinct; if x is not good, put $\zeta(x) \equiv 0$. By construction, it is true that if x is good, then $\mathbb{P}[\zeta(x) = 1] > 0$. Moreover, there exists $\varepsilon' > 0$ such that

$$\inf_{i\in\mathcal{A}} \mathbf{E}[\mathbb{P}[\zeta(x) = 1] \mid \eta_x = i] \geq \varepsilon'. \tag{3.9}$$

Consider the set $U = V_K \cup V_{2K} \cup V_{3K} \cup \ldots$, and suppose without loss of generality that the initial particle starts from $\mathcal{O}$. Let $y_1, y_2, y_3, \ldots$ be the sequence of *distinct* sites from U, according to the order they were visited by the descendants of the initial particle (which means that if some site from U was revisited, it is not included in that sequence). By construction and (3.9), provided that y_i is good, $\zeta(y_i)$ dominates a Bernoulli random variable with parameter ε' (with respect to the *annealed* law $\int \mathbb{P}[\,\cdot\,]d\mathbf{P}$). Also, the fact whether y_i is good (given η_{y_i}) is independent of the history of the branching random walk prior to the moment of hitting y_i (since the first particle that visits y_i comes there from below, and, as noted above, the event $\{y_i \text{ is good}\}$ depends only on the values of the environment on $\mathbb{T}_{d,K}(y_i)$). As, by (3.7)–(3.8), $\mathbf{P}[x \text{ is good}] > 0$, this shows that for $\mathbf{P}$-almost all environments $\mathbb{P}$-almost surely at least one of the embedded branching processes will survive, and so $\mathfrak{S}$ will be visited with $\mathbb{P}$-probability 1. $\square$

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