A note on percolation of Poisson sticks

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September 28, 2002

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Abstract

We study the existence of percolation in the model constructed by a superposition of a countable number of so-called Poisson sticks models. We prove that if there is no percolation in initial model and the rescaling parameter is large enough then there is no percolation in this multiscale model.

Keywords: Poisson sticks, multiscale percolation, critical probability

*Partially supported by CNPq (300676/00–0)
†Partially supported by FAPESP (00/11462–5)

1 Introduction and result

In the recent years many authors have studied probabilistic models of the following type. First, some random subset of $\mathbb{R}^d$ is constructed. Then, an independent copy of this set is rescaled and is in some sense superpositioned with the original random set; this procedure is performed a countable number of times. The first model of this type, known as Mandelbrot's model or random Cantor set, was introduced by Mandelbrot (1974) and subsequently extensively studied by many authors, see Chayes et al. (1988), Chayes, Chayes (1989), Chayes et al. (1991), Chayes et al. (1997), Falconer, Grimmett (1992), Menshikov et al. (2001a, 2002) and references therein. A continuous model of this type, namely multiscale Poisson Boolean model, was considered first in Section 8.1 of Meester, Roy (1996) and then studied by Menshikov et al. (2001a, 2002). In this note we study the so-called multiscale Poisson sticks model, which can be viewed as a continuous analogue of multiscale bond percolation introduced in Menshikov et al. (2001b).

The Poisson sticks model is described as follows. Consider a Poisson field $\mathcal{X}$ on $\mathbb{R}^2$ with rate λ. At each point of this field we place a segment (stick) of random length l and of random direction θ centered at this point, where l and θ are independent (see Fig. 1). The lengths and directions of different sticks are independent and identically distributed (i.i.d.). We say that percolation occurs if the set of the sticks contains an infinite connected component (*cluster*), i.e. there is an infinite sequence of the distinct points of the Poisson field $\{x_1, x_2, \ldots\}$ such that the stick placed at x_i intersects the

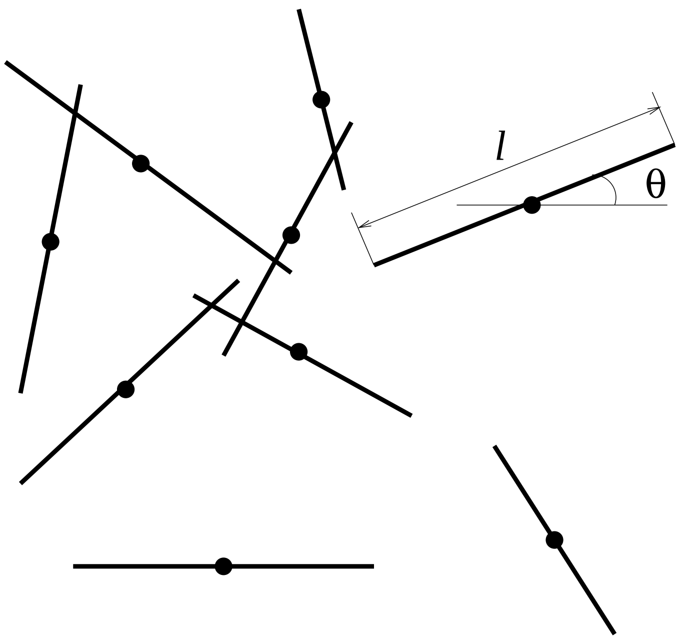

Figure 1: Poisson sticks

stick placed in x_{i+1}. By definition, $\theta \in [0, \pi)$ a.s. and we suppose also that the support of θ has at least two points (otherwise, if all the sticks have the same orientation, the percolation never occurs).

Throughout this paper we assume that there exists $L < \infty$ such that $l \leq L$ a.s. In this case (cf. Roy (1991)) there exists λ_{cr} such that there is no percolation (i.e., all clusters are finite a.s.) for $\lambda < \lambda_{cr}$, and for $\lambda > \lambda_{cr}$ there exists an infinite cluster a.s.

It is also possible to consider this problem in d-dimensional space, where sticks are substituted by pieces of hyperplanes, but here we will study only 2-dimensional case.

The multiscale model is defined in the following way. According to what was written in the first paragraph of this section, we are going to consider a superposition of a countable number of percolation models, where each of those is a rescaling of an independent copy of the model described above. For $i = 1, 2, \ldots$ we call the i-th model from this construction by level-i model, and it is described as follows. Fix $R > 1$. In the level-i model the rate of the Poisson field is λR^{2i}, the lengths of the sticks are distributed as lR^{-i}. The distribution of random directions θ is the same for all levels. The models of different levels are independent. In other words, the level-i model can be obtained in the following way. Take an independent copy of the initial model and apply the homothetic transformation $x \mapsto R^{-i}x$. This shows that if the initial model is in the subcritical regime, then the level-i model is also in it.

In the sequel we will refer to the initial model as level-0 model. Let $U^{(i)}$ be the union of all level-i sticks. The objective is to study if there is percolation in the set $U = \cup_{i=0}^{\infty} U^{(i)}$, that is, if U contains a continuous path $\gamma : \mathbb{R} \mapsto U$, such that γ is not contained in any finite box.

The main result of the paper is the following

Theorem 1 *If $\lambda < \lambda_{cr}$ then there exists R_0 such that for any $R > R_0$ there is no percolation in the set U (recall that λ_{cr} is the critical intensity for the level-0 model).*

2 Proof of Theorem 1

Choose the origin as the center of one of the sticks. Note that the Poisson field conditioned to $0 \in \mathcal{X}$ apart from the point in the origin is also the Poisson field with the same rate. Denote by $U^{(0)}(0)$ the cluster which contains 0.

Let $U_n = \cup_{i=0}^n U^{(i)}$. To prove Theorem 1, it is sufficient to show that for sufficiently large m the probability of the existence of a path from 0 to a sphere with radius m in U_n is uniformly small in n (see Menshikov et al. (2001a)). So we fix n and consider percolation in U_n.

For $\lambda < \lambda_{cr}$ we have (cf. Roy (1991)) $\mathbf{E}|U^{(0)}(0)| < \infty$, where $|U^{(0)}(0)|$ denotes the number of sticks in $U^{(0)}(0)$, that is, the expected size of the cluster is finite. Moreover, from this it is not hard to get that for any bounded set $A \subset \mathbb{R}^2$, the expected size of all clusters which intersect A is finite as well.

Definition 1 *Let $A \subset \mathbb{R}^2$. For fixed $\varepsilon > 0$, the ε-expansion of A is defined as*

$$\mathfrak{E}_\varepsilon(A) = \{y: \text{ there exists } x \in A \text{ such that } \|x-y\| \le \varepsilon\},$$

where $\|\cdot\|$ denotes the Euclidean norm.

Consider the modification of the level-0 model in which the sticks are ε-expanded. Let λ_ε be the critical rate in this model. Clearly, $\lambda_\varepsilon \le \lambda_{cr}$.

The key to the proof of Theorem 1 is the following

Theorem 2 *If $\lambda < \lambda_{cr}$ then there exists ε such that $\lambda < \lambda_\varepsilon$.*

Proof. We construct the cluster using the generation method (cf. Hall (1985), Meester, Roy (1996), Menshikov et al. (1986)). Let $\mathcal{G}_0$ be a one-element set which contains the stick centered at the origin, we call it the zero generation. The first generation $\mathcal{G}_1$ is the set of all sticks (except that of $\mathcal{G}_0$) which intersect the ball with radius L centered at the origin (i.e., the idea is to "majorize" the initial stick, so that the construction below does not depend on its orientation). Now, suppose that the first $k \geq 1$ generations $\mathcal{G}_1, \ldots, \mathcal{G}_k$ are already constructed. If $\mathcal{G}_k = \emptyset$, then by definition $\mathcal{G}_{k'} = \emptyset$ for all $k' > k$. Otherwise, $(k+1)$-th generation $\mathcal{G}_{k+1}$ is formed by those sticks which intersect the sticks from $\mathcal{G}_k$ and do not belong to $\cup_{i=1}^{k} \mathcal{G}_i$.

Since $\lambda < \lambda_{cr}$, then, using the fact that

$$\sum_{i=1}^{\infty} \mathbf{E}|\mathcal{G}_i| < \infty,$$

it is clear that there exist k_0 and δ, $0 < \delta < 1$, such that $\mathbf{E}|\mathcal{G}_{k_0}| \leq 1 - \delta$.

Note that $\mathcal{G}_{k_0}$ contains only sticks which are totally inside the sphere centered at 0 with radius $2(k_0 + 1)L$; denote this sphere by $\mathcal{S}_{k_0}$. Let ξ be the total number of sticks inside $\mathcal{S}_{k_0}$. Clearly, $\xi < \infty$ a.s. and also $\mathbf{E}\xi < \infty$. Denote by η the minimal distance between clusters in $\mathcal{S}_{k_0}$ in the model where the initial stick is substituted by the ball with radius L centered at the origin, with the convention that if there is only one cluster in $\mathcal{S}_{k_0}$ then $\eta = \infty$.

Let $\mathcal{H}_{k_0}^{\varepsilon}$ be the set of ε-expanded sticks in k_0-th generation of the ε-expanded model, constructed in the same way.

It can be easily seen that

$$|\mathcal{H}_{k_0}^{\varepsilon}| \leq \begin{cases} |\mathcal{G}_{k_0}|, & \text{if} \quad \eta > 2\varepsilon, \\ \xi, & \text{if} \quad \eta \leq 2\varepsilon. \end{cases} \tag{1}$$

Using that

$$\mathbf{E}|\mathcal{G}_{k_0}| = \sum_{i=1}^{\infty} \mathbf{E}(|\mathcal{G}_{k_0}| \,|\, \xi = i)\mathbf{P}\{\xi = i\} \leq 1 - \delta,$$

and by (1), we obtain

$$\mathbf{E}(|\mathcal{H}^{\varepsilon}_{k_0}| \,|\, \xi = i) \le \mathbf{E}(|\mathcal{G}_{k_0}| \,|\, \xi = i) + i\mathbf{P}\{\eta \le 2\varepsilon \mid \xi = i\}.$$

Thus we have

$$\begin{aligned}
\mathbf{E}|\mathcal{H}^{\varepsilon}_{k_0}| &\le \mathbf{E}|\mathcal{G}_{k_0}| + \sum_{i=1}^{\infty} i\mathbf{P}\{\eta \le 2\varepsilon \mid \xi = i\}\mathbf{P}\{\xi = i\} \\
&\le \mathbf{E}|\mathcal{G}_{k_0}| + \sum_{i=1}^{M} i\mathbf{P}\{\eta \le 2\varepsilon \mid \xi = i\}\mathbf{P}\{\xi = i\} \\
&\quad + \sum_{i=M+1}^{\infty} i\mathbf{P}\{\eta \le 2\varepsilon \mid \xi = i\}\mathbf{P}\{\xi = i\} \\
&\le \mathbf{E}|\mathcal{G}_{k_0}| + M\mathbf{P}\{\eta \le 2\varepsilon\} + \sum_{i=M+1}^{\infty} i\mathbf{P}\{\xi = i\}.
\end{aligned}$$

First, choose M so that $\sum_{i=M+1}^{\infty} i\mathbf{P}\{\xi = i\} < \delta/3$, then, choose ε such that $M\mathbf{P}\{\eta \le 2\varepsilon\} < \delta/3$. Therefore, one gets that there exists β, $0 < \beta < 1$, such that $\mathbf{E}|\mathcal{H}^{\varepsilon}_{k_0}| < \beta$. Then, as in [6], we remark that the Poisson process has "lack of memory" property, which means that if $\mathcal{X}$ is homogeneous Poisson process and $x_1, \dots, x_m$ are arbitrary fixed points, then the conditional distribution of $\mathcal{X} \setminus \{x_1, \dots, x_m\}$ given that points of $\mathcal{X}$ occur at $x_1, \dots, x_m$ is the same as the unconditional distribution of $\mathcal{X}$. Since, by construction, the inequality $\mathbf{E}|\mathcal{H}^{\varepsilon}_{k_0}| < \beta$ holds for any orientation of the initial stick, we can use the lack of the memory property to get that the expectation of the total number of sticks in generations $k_0, 2k_0, \dots$ is less than $\beta + \beta^2 + \beta^3 + \dots < \infty$, so there is no percolation in the ε-expanded model, that is, $\lambda < \lambda_\varepsilon$. ∎

Now, choose ε such that $\lambda < \lambda_\varepsilon < \lambda_{cr}$. From this moment on, we basically follow the proof of Theorems 1.1 and 1.2 in Menshikov et al. (2001a). For

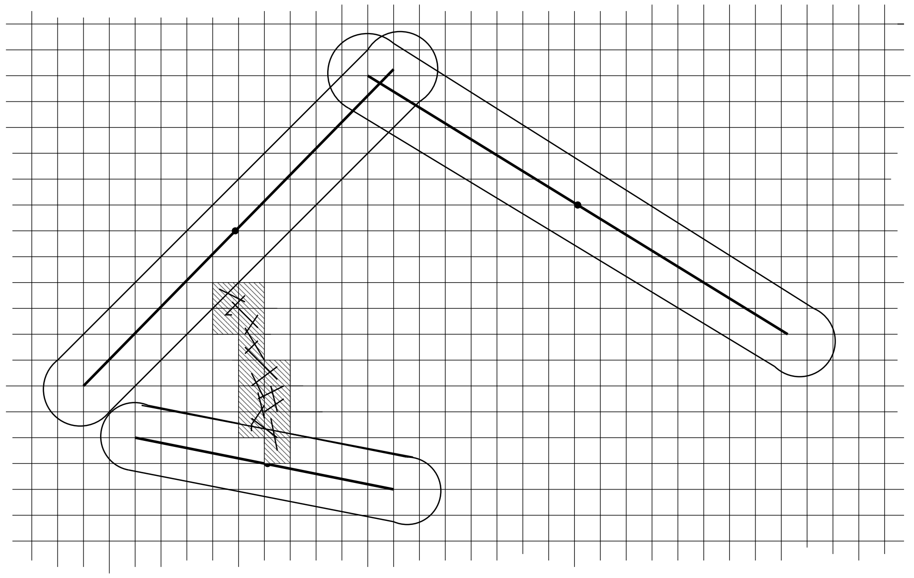

Figure 2: Good and passable sets

arbitrary $n \geq 1$, let $V^{(i)} = \mathfrak{E}_\varepsilon(U^{(i)})$, $i = 0, \ldots, n$. Consider the partition of $\mathbb{R}^2$ into the squares with side $\varepsilon R^{-i}/\sqrt{2}$ which we call level-i squares. We now define passable sets $P_0, \ldots, P_n$ and good sets $G_0, \ldots, G_n$.

Definition 2 *Let $G_n := U^{(n)}$. For $i < n$ level-i square is passable iff it intersects a connected component of diameter greater than $2\varepsilon R^{-i}$ from G_{i+1}. The set P_i is defined to be the set of all passable level-i squares. The set G_i is defined as $G_i := P_i \cup V^{(i)}$ (see Fig. 2, the dashed squares are passable).*

Lemma 1 *Percolation in U_n implies percolation in G_0.*

Proof. It is enough to prove that

$$\text{percolation in } \cup_{i=0}^{n} U^{(i)} \text{ implies percolation in } G_k \cup \bigcup_{i=0}^{k-1} V^{(i)}, \qquad (2)$$

$k = n, n-1, \dots, 0$.

For $k = n$ we get $G_n = U^{(n)}$, so (2) holds. Suppose that (2) holds for $k+1$ and let us prove it for k. We need to show that percolation in $G_{k+1} \cup \bigcup_{i=0}^{k} V^{(i)}$ implies percolation in $G_k \cup \bigcup_{i=0}^{k-1} V^{(i)}$. Consider an infinite continuous path γ in $G_{k+1} \cup V^{(0)} \dots \cup V^{(k)}$. The intersection

$$\Big(G_{k+1} \setminus \Big(\bigcup_{i=0}^{k} U^{(i)}\Big)\Big) \cap \gamma$$

can be decomposed into at most countable number of continuous pieces $\gamma_1, \gamma_2, \dots$ Take any γ' from this sequence and suppose that its extremal points belong to $U^{(i_1)}$ and $U^{(i_2)}$, $i_1, i_2 \le k$. Then, two cases are possible, either $\gamma' \subset V^{(i_1)} \cup V^{(i_2)}$, or $\gamma' \subset P_k$. In both cases we get

$$\gamma' \subset P_k \cup \bigcup_{i=0}^{k} V^{(i)} = G_k \cup \bigcup_{i=0}^{k-1} V^{(i)},$$

which proves the induction step, and so Lemma 1 holds. ∎

Define also the models $W^{(i)}$ which use the Poisson point process with density $\lambda' R^{2i}$, $\lambda < \lambda' < \lambda_\varepsilon$ and ε-expanded sticks with length lR^{-i} and direction θ independent of all other models. When $\lambda' < \lambda_\varepsilon$, there is no percolation in $W^{(i)}$ for any $i = 0, \dots, n$.

Lemma 2 *For R large enough, G_i can be dominated by $W^{(i)}$ (i.e., it is possible to construct a coupling in such a way that $G_i \subset W^{(i)}$ a.s.).*

Proof. We prove this lemma by induction. By definition, G_n is dominated by $W^{(n)}$. Suppose that the lemma holds for $k+1$ and let us prove it for k. Fix some $K \in P_k$. Using the fact that the size of the cluster has exponential tail in the subcritical phase, analogously to formulas (3) and (5) in Menshikov et al. (2001a) one can show that

$$\mathbf{P}\{K \text{ is passable}\} \to 0, \quad R \to \infty \tag{3}$$

uniformly in k. The idea is that if K is passable, then there exists a path of length at least cR in G_{k+1}, where c is a positive constant and the length of the path is the number of squares in it. Since, by the induction assumption, G_{k+1} is dominated by $W^{(k+1)}$ (which is in the subcritical phase), (3) follows.

Let identify the level-k squares with the points of $\mathbb{Z}^2$ and consider the random field $\{\eta_k(x)\}_{x\in\mathbb{Z}^2}$, where

$$\eta_k(x) = \mathbf{1}\{\text{the square corresponding to } x \text{ is passable}\}$$

The state of a square depends only on the states of the squares inside some finite region, so the result of Liggett et al. (1997) can be used. Thus, the random field $\{\eta_k(x)\}_{x\in\mathbb{Z}^2}$ can be dominated by Bernoulli (i.e. independent) random field $\{\xi_k(x)\}_{x\in\mathbb{Z}^2}$ with parameter $\sigma(R)$. That is, it is possible to couple the two random fields in such a way that $\eta_k(x) \le \xi_k(x)$ a.s. for all $x \in \mathbb{Z}^2$. Since the probability that a square is passable is small for enough large R, the parameter $\sigma(R)$ can be made arbitrarily close to 0 choosing R again large enough. The choice of R depends only on ε and λ', but not on n.

Note that there exists $\alpha > 0$ such that the Bernoulli random field of level-k squares (i.e., the field of squares such that $\xi_k(x) = 1$ for corresponding x) can be dominated by εR^{-k}-extended Poisson sticks with density αR^{2k}, length lR^{-k} and direction θ. Indeed, we can claim that the square is selected if it contains a center of a stick (note that the square's side is chosen in a way that the square is completely covered by the expanded stick). Since $\sigma(R)$ is close to 0, one concludes that α can be chosen to be close to 0 as well. Take R such that $\lambda + \alpha < \lambda'$. So, the good level-$k$ set is dominated by $W^{(k)}$. ∎

Since G_0 is dominated by $W^{(0)}$, and $W^{(0)}$ is in subcritical phase, there is no percolation in G_0. So, by Lemma 1 there is no percolation in U_n, and thus, since the choice of R does not depend on n, there is no percolation in U. Theorem 1 is completely proved. ∎

Acknowledgements

We thank the anonymous referee for the valuable comments and suggestions.

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