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Markov Processes and Related Fields

Exact Power Estimates For Countable Markov Chains

M.V. Menshikov * and S.Yu. Popov *

Laboratory of Large Random Systems, Faculty of Mathematics and Mechanics, Moscow State University, 119899, Moscow, Russia.

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Abstract. We develop new methods for studying non-exponential asymptotics of the stationary probabilities and the convergence rates of countable Markov chains. For some well-known cases like zero drift random walks on the quadrant we get coinciding lower and upper bounds.

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1. Introduction

In this paper we investigate the behaviour of the stationary measure for several classes of countable Markov chains. We obtain the exact asymptotics for several large deviations problems for these classes. [1] and [3] established conditions for finiteness or non-finiteness of the p^{th} moments of passage-times (by passage-time we mean the time to hit some finite set). In this paper we prove that the stationary measure decreases polynomially with increasing states. Furthermore we show that the rate of convergence to stationarity is polynomially quickly in time and we determine the exact asymptotics.

The technique for studying this problem is based on the so-called method of Lyapunov functions (or test functions). This martingale approach is quite natural for such problems.

In Section 2.1 we consider a one dimensional version of Lamperti's problem [8]. For this problem we find the exact asymptotics for the stationary probabilities (Theorems 2.1 and 2.2). The martingale approach to similar problems can be found in [11]. However, applying the results of [11] to our case

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only yields one-sided bounds. Moreover, these bounds are different from the real asymptotic behaviour.

In Section 2.2 we again study the asymptotic behaviour of the stationary probabilities for one dimensional random walks with asymptotically zero drifts. As opposed to the previous section the mean drift is of order $x^{-\alpha}$, where $0 < \alpha < 1$ (in Lamperti's problem $\alpha = 1$). In this case the stationary measure is of the form $\pi_n \simeq \exp\{-\gamma n^\beta\}$, where $\gamma > 0$, $0 < \beta < 1$.

In Section 3 we prove that the rate of convergence to stationarity in Lamperti's problem is polynomial. To this end we use the results of B. Rogozin. In [10] he showed that in general the rate of convergence depends on the tails of the distribution of the first entrance time for some finite set. The analysis of the tails of the distribution of the passage-time moments for finite sets can be found in [1] and [3]. Let us note that closely related results to [10] can be found in [9], but for our purposes [10] was more convenient.

Section 4 generalises the results of Sections 2, 3 to arbitrary countable Markov chains by means of Lyapunov functions.

Section 5 applies these results to reflected random walks in $\mathbf{Z}_+^2$ with zero drifts in the interior, and we obtain the precise polynomial asymptotics of the stationary measure as well as the rate of convergence to this measure. Let us note that in [2] the upper bound for the rate of convergence for the same problem can be found.

2. One-dimensional random walks with asymptotically zero drifts

2.1. Lamperti's problem

Let us consider a homogeneous, irreducible, aperiodic Markov chain $\{\xi_t\}$, $t = 0, 1, 2, \ldots$ in discrete time; $\xi_t \in \mathbf{Z}_+$, where $\mathbf{Z}_+$ is the set of all non-negative integers. Denote by $P_{ij} = \mathsf{P}\{\xi_n = j \,|\, \xi_{n-1} = i\}$ the transition probabilities of Markov chain $\{\xi_t\}$. We assume the following condition to hold.

Condition D. There exists $d \in \mathbf{Z}_+$ such that $P_{ij} = 0$ if $|i-j| > d$.

We will call d the *maximal value of jumps.* Let us write

$$m(x) = \mathsf{E}(\xi_{n+1} - \xi_n \,|\, \xi_n = x)\ ,\ x \in \mathbf{Z}_+, \tag{2.1}$$

and

$$b(x) = \mathsf{E}((\xi_{n+1} - \xi_n)^2 \,|\, \xi_n = x)\ ,\ x \in \mathbf{Z}_+. \tag{2.2}$$

Suppose, that there exists $\varepsilon_0 > 0$ such that $b(x) > \varepsilon_0$ for any x. Let the following inequality hold for some $\theta_1 > \theta_2 > 1$ and for all sufficiently large x:

$$-\frac{\theta_1 b(x)}{2x} \le m(x) \le -\frac{\theta_2 b(x)}{2x}. \tag{2.3}$$

The ergodicity of the Markov chain ξ_n follows from conditions (2.3) and D (for the proof see [8] and also [4]). Write

$$\pi_j = \lim_{n\to\infty} P^n_{0j}.$$

The main purpose of this section is to obtain the asymptotics of π_j, as $j\to\infty$. Next we formulate the two main results of this section.

Theorem 2.1. *For any $\varepsilon > 0$ there exists $C = C(\varepsilon) \in \mathbf{R}_+$, such that the following inequality holds for all n:*

$$\pi_n \le Cn^{-\theta_2+\varepsilon}. \tag{2.4}$$

To prove the reverse inequality, we need an additional condition.

Condition A (local aperiodicity). There exists $\delta > 0$ such that for all i, j: $|i-j| < d$, there exists $n = n(i,j)$ such that $P^n_{ij} > \delta$.

Theorem 2.2. *Assume Condition A. Then for any $\varepsilon > 0$ there exists $C \in \mathbf{R}_+$, such that the following inequality holds for all n:*

$$\pi_n \ge Cn^{-\theta_1-\varepsilon}. \tag{2.5}$$

We will first formulate and prove two useful lemmas. The proof method is based on the martingale technique that was developed by G. Fayolle, V. Malyshev and M. Menshikov in [6].

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space equipped with a filtration $\{\mathcal{F}_n\}_{n\ge 0}$. Let $\{S_i, i\ge 0\}$ be a sequence of $\{\mathcal{F}_n\}$-adapted real non-negative random variables; $S_0 \equiv \text{const}$. Denote by τ the first entrance time of $[0, C]$, i.e.

$$\tau(\omega) = \min\{n \ge 1 : S_n \le C\} \ .$$

Let us introduce the corresponding stopped sequence $\tilde{S}_n = S_{n\wedge\tau}$, where

$$n\wedge\tau = \begin{cases} n, & \text{if } n \le \tau, \\ \tau, & \text{if } n > \tau. \end{cases} \tag{2.6}$$

Lemma 2.1. *Assume that $S_0 > C$ and that for some $\delta > 0$ and any $n \ge 0$*

$$\mathsf{E}(\tilde{S}_{n+1} - \tilde{S}_n \mid \mathcal{F}_n) \le -\delta I_{(\tau>n)} \quad \text{a.s.} \tag{2.7}$$

Then

$$\mathsf{E}\tau \le \frac{S_0}{\delta} < \infty \ . \tag{2.8}$$

For the proof see [6]. □

We will also need the reverse inequality (2.8).

Lemma 2.2. *Let $\{\eta_k\}$, $k = 1, 2, \dots$, be an arbitrary Markov chain on the finite state space $M = \mathbf{Z} \cap [-l, m]$, with l, m positive integers. For $x \in M$, $x > 0$ let*

$$\tau_x = \min\{n \geq 1 : \eta_n \leq 0 \,|\, \eta_0 = x\}$$

Let $f(n)$ be a function with $f(n) > 0$ for all $n > 0$, $f(n) \leq 0$ for $n \leq 0$, satisfying the following inequality:

$$\mathsf{E}(f(\eta_{n+1}) - f(\eta_n) \,|\, \eta_n = x) \geq -\delta \tag{2.9}$$

for $x = 1, 2, \dots m$. Then

$$\mathsf{E}\tau_x \geq \frac{f(x)}{\delta}. \tag{2.10}$$

Proof. Let $\tilde{\eta}_n = \eta_{n\wedge\tau}$ and $\tilde{S}_n = f(\tilde{\eta}_n)$. For sake of brevity we suppress the dependence on x in our notation, and hence we write $\tau = \tau_x$. We have

$$\mathsf{E}(\tilde{S}_{n+1} - \tilde{S}_n \,|\, \tilde{S}_n) \geq -\delta I_{(\tau>n)} \quad \text{a.s.} \tag{2.11}$$

Taking the expectation in (2.11) we have

$$\mathsf{E}(\tilde{S}_{n+1} - \tilde{S}_n) \geq -\delta \mathsf{P}\{\tau > n\}$$

and iterating this inequality we find

$$\begin{aligned} \mathsf{E}\tilde{S}_{n+1} &= \mathsf{E}(\tilde{S}_{n+1} - \tilde{S}_n + \tilde{S}_n - \ldots - \tilde{S}_0 + \tilde{S}_0) \\ &\geq -\delta \sum_{i=0}^{n} \mathsf{P}\{\tau > i\} + \tilde{S}_0 \,. \end{aligned}$$

Consequently,

$$\sum_{i=0}^{n} \mathsf{P}\{\tau > i\} \geq \frac{S_0}{\delta} - \frac{\mathsf{E}\tilde{S}_{n+1}}{\delta}\,.$$

Since the Markov chain is finite and $\tau < \infty$ a.s. (if the latter were not true, we would have $\mathsf{E}\tau = \infty$ and (2.10)), we have that

$$\tilde{S}_n \xrightarrow{L_1} S_\tau\,, \quad \text{as } n \to \infty$$

and, since $\mathsf{E}S_\tau \leq 0$,

$$\sum_{i=0}^{\infty} \mathsf{P}\{\tau > i\} = \mathsf{E}\tau_x \geq \frac{S_0}{\delta}\,.$$

This completes the proof of Lemma 2.2. □

Next we shall study the mean passage times for finite segments. For obtaining some useful bounds, we will use the above two lemmas. We introduce some notation. For $x \in \mathbf{Z}_+$, $B \subset \mathbf{Z}_+$ let

$$\tau_{x,B} = \min\{n \geq 1 : \xi_n \in B \,|\, \xi_0 = x\}\ , \tag{2.12}$$

$$\overline{\tau}_{x,B} = \min\{n \geq 0 : \xi_n \in B \,|\, \xi_0 = x\} \,, \tag{2.13}$$

and

$$\mathsf{E}\tau_{x,B} = m_{x,B}, \qquad \mathsf{E}\overline{\tau}_{x,B} = \overline{m}_{x,B}. \tag{2.14}$$

Let $B_n = \{n, \ldots n+d-1\}$. Let us formulate and prove the following important lemma.

Lemma 2.3. *For any $\varepsilon > 0$ there exist C_1, C_2, $C_3 \in \mathbf{R}_+$ such that for all n and for all x with $n-d < x < n$ we have*

$$m_{x,B_n} \geq C_1 n^{\theta_2 - \varepsilon}, \tag{2.15}$$

$$m_{x,B_n} \leq C_2 n^{\theta_1 + \varepsilon}, \tag{2.16}$$

and for all x with $n+d-1 < x \leq n+2d-1$ we have

$$m_{x,B_n} \leq C_3 n. \tag{2.17}$$

Proof. First we will prove inequality (2.17). Consider the function $f(n) = n^2$. Since $b(x)$ is bounded away from zero and because of (2.3), we have for x large enough that

$$\begin{aligned}
\mathsf{E}(f(\xi_{n+1}) - f(\xi_n) \,|\, \xi_n = x) &= \mathsf{E}(\xi_{n+1}^2 - \xi_n^2 \,|\, \xi_n = x) \\
&= \mathsf{E}(\xi_{n+1}^2 - 2\xi_{n+1}\xi_n + \xi_n^2 \\
&\quad + 2\xi_n(\xi_{n+1} - \xi_n) \,|\, \xi_n = x) \\
&= b(x) + 2xm(x) \leq b(x)(1-\theta_1) < -\delta_0,
\end{aligned}$$

where δ_0 is positive and independent of x. Next consider x with $n+d \leq x \leq n+2d-1$ i.e. $x = n+d+l$ for l with $0 \leq l \leq d-1$. Let $f_1(y) = f(y) - n^2$. Suppose, that $\xi_0 = x$. Let $\tau = \tau_x$ be the first entrance time of B_n. Applying Lemma 2.1 to the sequence $S_n = f_1(\xi_n)$ we obtain that

$$\mathsf{E}\tau < \frac{S_0}{\delta} < C_3 n,$$

as

$$S_0 = f_1(\xi_0) = f_1(x) = (n+d+l)^2 - n^2 < Cn$$

for some $C > 0$. This proves (2.17).

We prove (2.15). Let $\xi_0 = x_1$ for $n-d < x_1 \leq n-1$, and let τ be the first entrance time of B_n. Consider the function $f(l) = n^{\theta_2+1-\varepsilon} - l^{\theta_2+1-\varepsilon}$ on the set $\{0, \ldots n+d-1\}$ and set $S_k = f(\xi_k)$. For ε sufficiently small we will check the conditions of Lemma 2.2. Let $x \in \mathbf{Z}_+$ and $0 < x \leq n-1$. Taking into account the boundedness of jumps and using a Taylor expansion, we have:

$$\mathsf{E}(f(\xi_{k+1}) - f(\xi_k) \,|\, \xi_k = x) =$$

$$
\begin{aligned}
&= \mathsf{E}(\xi_k^{\theta_2+1-\varepsilon} - \xi_{k+1}^{\theta_2+1-\varepsilon} \mid \xi_k = x)\\
&= \mathsf{E}\big(x^{\theta_2+1-\varepsilon}\big(1 - (\xi_{k+1}/x)^{\theta_2+1-\varepsilon}\big) \,\big|\, \xi_k = x\big)\\
&= x^{\theta_2+1-\varepsilon}\mathsf{E}\big(1 - (1 + (\xi_{k+1} - \xi_k)/x)^{\theta_2+1-\varepsilon} \,\big|\, \xi_k = x\big)\\
&= x^{\theta_2+1-\varepsilon}\mathsf{E}\big(1 - (1 + (\theta_2 + 1 - \varepsilon)\frac{\xi_{k+1} - \xi_k}{x} + \\
&\qquad + \mathrm{o}(x^{-2})) \,\big|\, \xi_k = x\big)\\
&= -x^{\theta_2+1-\varepsilon}\Big(\frac{(\theta_2 + 1 - \varepsilon)m(x)}{x} + \frac{(\theta_2 - \varepsilon)(\theta_2 + 1 - \varepsilon)b(x)}{2x^2} + \mathrm{o}(x^{-2})\Big)\\
&\geq -(\theta_2 + 1 - \varepsilon)x^{\theta_2-1-\varepsilon}\big(-\theta_2 b(x) + (\theta_2 - \varepsilon)b(x)\big) + \mathrm{o}(x^{-\theta_2-1-\varepsilon})\\
&= (\theta_2 + 1 - \varepsilon)x^{\theta_2-1-\varepsilon}\varepsilon b(x) + \mathrm{o}(x^{-\theta_2-1-\varepsilon}).
\end{aligned}
$$

Consequently there exists x_0 (independent of n), such that for $x_0 \leq x \leq n-1$

$$\mathsf{E}(f(\xi_{k+1}) - f(\xi_k) \mid \xi_k = x) \geq 0.$$

Next choosing

$$\delta_1 = \max_{x \leq x_0} \mathsf{E}(|f(\xi_{k+1}) - f(\xi_k)| \mid \xi_k = x)$$

we find

$$\mathsf{E}(f(\xi_{k+1}) - f(\xi_k) \mid \xi_k = x) \geq -\delta_1,$$

and so the conditions of Lemma 2.2 are valid. Hence for $n - d < x_1 \leq n-1$

$$m_{x_1,B_n} = \mathsf{E}\tau_{x_1,B_n} \geq C(n^{\theta_2+1-\varepsilon} - {x_1}^{\theta_2+1-\varepsilon}) \geq C_1 n^{\theta_2-\varepsilon}$$

for some $C_1 > 0$, thus proving (2.15).

Next we turn to inequality (2.16). Analogously, if we consider the function $f(k) = (n+d-1)^{\theta_1+1+\varepsilon} - k^{\theta_1+1+\varepsilon}$ on the set $\{0, \dots n+d-1\}$, it is not difficult to prove that there exists x_0 (independent of n), such that $f(\xi_k)$ is a strict supermartingale on $\{x_0 < \xi_k < n\}$, i.e. for some $\delta_2 > 0$

$$\mathsf{E}(f(\xi_{k+1}) - f(\xi_k) \mid \xi_k = x) \leq -\delta_2$$

for all $n > x > x_0$ (assuming that n is large enough). Since the Markov chain ξ_t is irreducible aperiodic, there exists some positive integer q_0 such that for some $\delta_3 > 0$

$$\mathsf{P}\{\xi_{q_0} > x_0 \mid \xi_0 = x\} > \delta_3$$

for all $x \leq x_0$. Let $\delta = \min\{\delta_2, \delta_3\}$.

Consider the following sequence of stopping times N_i.

1. $N_0 = 0$.

2. if $\xi_{N_k} > x_0$, then $N_{k+1} = N_k + 1$.

3. if $\xi_{N_k} \le x_0$, then $N_{k+1} = N_k + q_0$.

Consider the function $f_1(k) = f(k) \wedge f(x_0)$ and introduce the sequence of random variables $\{S_k\}$, $k = 0, 1, 2, \dots$ defined by

$$S_k = f_1(\xi_{N_k}),$$

where $\xi_0 = \text{const}$ and $n - d \le \xi_0 < n$.

Let for fixed n

$$\tau^* = \{\min k > 0 : \xi_{N_k} \in B_n\},$$

and $\tilde{S}_k = S_{k \wedge \tau^*}$. It is not difficult to see that

$$\mathsf{E}(\tilde{S}_{k+1} - \tilde{S}_k \mid \tilde{S}_k) \le -\delta I_{\{\tau^* > k\}} \quad \text{a.s.},$$

so that the sequence $\{S_k\}$ satisfies the conditions of Lemma 2.1. Consequently, there exists a constant $C_4 > 0$, such that

$$\mathsf{E}\tau^* \le \frac{S_0}{\delta} \le C_4 n^{\theta_1 + \varepsilon}.$$

Since the sequence of stopping times N_i is such that for all i

$$N_{i+1} - N_i \le q_0 \quad \text{a.s.},$$

we immediately get (2.16). This completes the proof of Lemma 2.3. $\square$

Now we are in the position to complete the proof of the main results of this section.

Proof of Theorems 2.1 *and* 2.2. First of all, let us formulate the following simple, but useful lemma.

Lemma 2.4. *If Condition* A *holds, then there exists* $C \in \mathbf{R}_+$ *such that for all* i, j: $|i - j| \le d$

$$\pi_i \le C \pi_j. \tag{2.18}$$

Proof. This is almost evident. It is easily obtained that $P_{ji}^k > \delta$ for some k implies $\pi_i \ge \delta \pi_j$. Inequality (2.18) immediately follows. $\square$

We continue the proof of Theorems 2.1 and 2.2. Write

$$D_n^1 = \{n - d, \dots n - 1\},$$

and

$$D_n^2 = \{n + d, \dots n + 2d - 1\}.$$

Use the notation introduced earlier in this section for the two following relations (cf. [5], [7]):

$$\sum_{i \in B} \pi_i m_{i,B} = 1 \tag{2.19}$$

and

$$m_{i,B} = \sum_j P_{ij}\overline{m}_{j,B} + 1, \tag{2.20}$$

where $B \subset \mathbf{Z}_+$. Substituting (2.20) into (2.19) and taking into account the boundedness of jumps, we get:

$$\sum_{i\in B_n} \pi_i \Big(\sum_{j\in D_n^1} P_{ij} m_{j,B} + \sum_{j\in D_n^2} P_{ij} m_{j,B} + 1 \Big) = 1 \tag{2.21}$$

where we have used the fact, that $\overline{m}_{i,B} = m_{i,B}$ for $i \notin B$.

Next we shall prove Theorem 2.1. Using (2.21) we have

$$\pi_n \sum_{j\in D_n^1} P_{nj} m_{j,B} \le 1$$

and

$$\pi_n \sum_{j\in D_n^1} P_{nj} \min_{j\in D_n^1} m_{j,B} \le 1.$$

As $b(x)$ is bounded away from zero and as $m(x)$ is asymptotically zero, there exists $v > 0$ such that

$$\sum_{j\in D_n^1} P_{nj} \ge v$$

for all sufficiently large n. Consequently,

$$v\pi_n \min_{j\in D_n^1} m_{j,B} \le 1.$$

Using (2.15) (see Lemma 2.3), we get (2.4), thus establishing the proof of Theorem 2.1. □

Let us now prove Theorem 2.2. Using Lemma 2.4 and (2.21) we obtain the existence of $F_1 > 0$, such that for all n

$$F_1\pi_n \sum_{i\in B_n} \Big(\sum_{j\in D_n^1} P_{ij} m_{j,B} + \sum_{j\in D_n^2} P_{ij} m_{j,B} + 1 \Big) \ge 1$$

and

$$F_1\pi_n d \Big(\sum_{j\in D_n^1} m_{j,B} + \sum_{j\in D_n^2} m_{j,B} + 1 \Big) \ge 1.$$

Using the bounds (2.16) and (2.17) we obtain (2.5). This completes the proof of Theorem 2.2. □

Remark 1. *In general, Condition* A *is crucial. If* A *fails it is possible to construct an example of a Markov chain* ξ_t*, for which*

$$m(x) = -\frac{b(x)}{x},$$

(i.e. $\theta = 2$*), and*

$$\pi_{2n} > Cn^{-2}\ ,\ \pi_{2n+1} < Cn^{-4}.$$

2.2. Slowly decaying drifts

This section will be very brief and schematic. The goal of our paper is the critical case, in which the stationary measure decreases polynomially with the state; but here we touch upon the non-critical case. Since the analysis closely resembles the critical case, we will only give the formulation of the results and the main ideas.

In this section we shall study one-dimensional random walks satisfying Conditions A and D, but with condition (2.3) is substituted by another condition. Again we are interested in the asymptotics of the stationary measure π_n, and we shall show that our methods are also applicable in this case.

Instead of Condition (2.3), let the following inequality be satisfied

$$-\frac{\theta_1 b(x)}{x^\alpha} \le m(x) \le -\frac{\theta_2 b(x)}{x^\alpha} \tag{2.22}$$

for some $\alpha < 1$, some $\theta_1 > \theta_2 > 0$ and for all sufficiently large x, where $m(x)$ and $b(x)$ have been introduced in (2.1) and (2.2). Sometimes Condition (2.22) appears to be too strong, so instead of (2.22) we consider a generalised version. Let the following inequality

$$-\frac{\theta_1 b(x)}{x^\alpha} \le m(x) \le -\frac{\theta_2 b(x)}{x^\beta} \tag{2.23}$$

hold for some $\alpha \le \beta < 1$, some $\theta_1, \theta_2 > 0$ all sufficiently large x.

The main result of this section is the following theorem.

Theorem 2.3. *Assume condition* (2.23). *Then for any $\varepsilon > 0$ there exists N large, such that for all $n \ge N$ the following inequality is valid:*

$$\exp\{-\left(\frac{2\theta_1}{1-\alpha} + \varepsilon\right) n^{1-\alpha}\} \le \pi_n \le \exp\{-\left(\frac{2\theta_2}{1-\beta} - \varepsilon\right) n^{1-\beta}\}. \tag{2.24}$$

Proof. The proof of the above two inequalities is similar to the proofs of Theorems 2.1 and 2.2, except for one detail: the Lyapunov functions in this case differ from the previous sections. Instead of the functions $f(n) = n^\gamma$ that were used to construct the corresponding Lyapunov functions in Lemma 2.3, we have to use functions $f(n) = \exp\{Cn^{1-\alpha}\}$ for different C. As an example let us show that, depending on the value C, $f(\xi_n)$ becomes a super- or a submartingale on $\{\xi_n > x_0\}$ for x_0 sufficiently large.

Let

$$f(n) = \exp\{\left(\frac{2\theta_2}{1-\alpha} - \varepsilon\right) n^{1-\alpha}\} ,$$

and suppose that (2.22) holds. Let us prove that for large enough x

$$\mathsf{E}(f(\xi_{n+1}) - f(\xi_n) \,|\, \xi_n = x) \le -\delta f(x) x^{-2\alpha} \tag{2.25}$$

for some $\delta > 0$.

Indeed, writing $K = (2\theta_2)/(1-\alpha) - \varepsilon$, we get

$$
\begin{aligned}
&\mathsf{E}(f(\xi_{n+1}) - f(\xi_n) \mid \xi_n = x) \\
&= f(x)\mathsf{E}(\exp\{K(\xi_{n+1}^{1-\alpha} - \xi_n^{1-\alpha})\} - 1 \mid \xi_n = x) \\
&= f(x)\mathsf{E}\big(\exp\{Kx^{1-\alpha}\left((1 + (\xi_{n+1} - \xi_n)/x)^{1-\alpha} - 1\right)\} - 1 \mid \xi_n = x\big) \\
&= f(x)\mathsf{E}\big(\exp\{(1-\alpha)Kx^{-\alpha}(\xi_{n+1} - \xi_n) + \mathrm{O}(x^{-(\alpha+1)})\} - 1 \mid \xi_n = x\big) \\
&= f(x)\mathsf{E}\big((1-\alpha)Kx^{-\alpha}(\xi_{n+1} - \xi_n) + \\
&\qquad + \frac{((1-\alpha)K)^2}{2}x^{-2\alpha}(\xi_{n+1} - \xi_n)^2 + \mathrm{o}(x^{-2\alpha}) \mid \xi_n = x\big) \\
&= f(x)x^{-\alpha}K(1-\alpha)\big(m(x) + \frac{(1-\alpha)K}{2}b(x)x^{-\alpha} + \mathrm{o}(x^{-\alpha})\big) \\
&\le f(x)x^{-2\alpha}K(1-\alpha)\big(-\theta_2 b(x) + \frac{(1-\alpha)K}{2}b(x) + \mathrm{o}(1)\big) \\
&\le -\delta x^{-2\alpha} f(x)
\end{aligned}
$$

for some $\delta > 0$, and so (2.25) is valid.

As has been mentioned before, the remainder of the proof resembles Theorems 2.1 and 2.2, and so we will omit further details. □

3. Rate of convergence

The previous section established bounds on the stationary measure π_n as $n \to \infty$. But we still have no information on rate at which the n-step probabilities P_{ii}^n converge to π_i. There are many examples of Markov chains with an exponential rate of convergence to stationarity. But we shall show that in our case the rate of convergence is not exponential, but polynomial.

Let us introduce some notation. Write $R_n = P_{00}^n - \pi_0$ for the quantity to be estimated. Write $\tau = \tau_{0,\{0\}}$ and $m = m_{0,\{0\}}$, where $\tau_{0,\{0\}}$ and $m_{0,\{0\}}$ have been defined in (2.12) and (2.14). Then $\pi_0 = m^{-1}$ (cf. [5]). We can prove the following theorem.

Theorem 3.1. *If (2.3) holds, then for any $\varepsilon > 0$ there exist positive reals C_1, C_2 such that for all large enough n the following inequality holds:*

$$C_1 n^{-(\theta_1 - 1 + \varepsilon)/2} \le |R_n| \le C_2 n^{-(\theta_2 - 1 - \varepsilon)/2}. \tag{3.1}$$

Proof. Let

$$H_n = m^{-2} \sum_{k=n+1}^{\infty} \mathsf{P}\{\tau \ge k+1\} .$$

Theorem 1 of [10] implies (in our notations) that R_n can be represented as follows:

$$R_n = H_n + G_n,$$

where G_n is of order less than H_n, provided that $\beta > \alpha > 1$, $\mathsf{E}\tau^\alpha < \infty$ and $\mathsf{E}\tau^\beta = \infty$. The only thing left to derive are find upper and lower bounds of $\mathsf{P}\{\tau > k\}$.

From the proof of Lemma 2.3 it follows that (2.3) implies for any $\varepsilon > 0$ the existence of x_0 large, such that $\xi_t^{\theta_2+1-\varepsilon}$ is a supermartingale on $\{\xi_t > x_0\}$, i.e. for $x > x_0$

$$\mathsf{E}(\xi_{t+1}^{\theta_2+1-\varepsilon} - \xi_t^{\theta_2+1-\varepsilon} \,|\, \xi_t = x) \leq 0,$$

and $\xi_t^{\theta_1+1+\varepsilon}$ is a submartingale on $\{\xi_t > x_0\}$. We use this fact to formulate and prove the following lemma.

Lemma 3.1. *For any $\varepsilon > 0$ there exists a constant $C_1 > 0$ such that for all large enough k*

$$\mathsf{P}\{\tau > k\} \geq C_1 k^{-(\theta_1+1+\varepsilon)/2}. \tag{3.2}$$

Proof. We need to introduce some additional notation. For states i, j and for set B we define the n-step taboo transition probabilities

$${}_BP_{ij}^n = \mathsf{P}\{\xi_n = j, \xi_k \notin B \text{ for } k = 1, \dots n-1 \,|\, \xi_0 = i\} \tag{3.3}$$

with taboo set B. For $B = \{k\}$ we will evidently write ${}_{\{k\}}P_{ij}^n \equiv {}_kP_{ij}^n$.

Let further

$$f(k) = k^{-(\theta_1+1+\varepsilon)/2}.$$

Using Corollary 1 [1] we obtain in our case for some set

$$A = \{0, 1, 2, \dots a_0\}$$

and for some $x_0 > a_0$ the existence of a constant $C_2 > 0$, such that

$$\mathsf{P}\{\tau_{x_0,A} > k\} \geq C_2 f(k).$$

Since our Markov chain is irreducible and aperiodic, there exists $\delta > 0$ and n_0 such that

$${}_0P_{0x_0}^{n_0} > \delta.$$

As $0 \in A$

$$\begin{aligned} \mathsf{P}\{\tau_{0,\{0\}} > s\} &\geq \delta \mathsf{P}\{\tau_{x_0,\{0\}} > s - n_0\} \\ &\geq \delta \mathsf{P}\{\tau_{x_0,A} > s - n_0\} \geq \delta f(s-n_0) \geq C_1 f(s) \end{aligned}$$

for some $C_1 > 0$ and $s > n_0$. This shows Lemma 3.1. □

Taking the summation in (3.2) over all $k > n$ we obtain the first inequality in (3.1).

As Theorem 1 of [3] shows, in our case for some x_0, some set

$$A = \{0, 1, 2, \dots a_0\}$$

and for any $x > x_0 > a_0$ for any $\varepsilon > 0$ there exists the expectation $\mathsf{E}\tau_{x,A}^{(\theta_2+1-\varepsilon)/2}$. But we are interested in the finiteness of moments of $\tau \equiv \tau_{0,\{0\}}$. To finally complete the proof of the theorem we will therefore need one more lemma.

Lemma 3.2. *Let $\{\xi_t\}$ be an irreducible aperiodic Markov chain on a countable state space X with the following properties.*

1. *for all i it holds that $P_{ij} \neq 0$ only for a finite number of states j.*
2. *for some $\alpha > 1$, some set I and some finite set B*

$$\mathsf{E}\tau_{i,B}^{\alpha} < \infty \tag{3.4}$$

   for all $i \in I$.

3. *the set $X \setminus I$ is finite.*

Then for any set C and any state $l \in X$

$$\mathsf{E}\tau_{l,C}^{\alpha} < \infty. \tag{3.5}$$

Proof. This is a direct consequence of Theorem A.1 in the Appendix. □

Applying this lemma to our situation, we immediately get the finiteness of the expectation $\mathsf{E}\tau^{(\theta_2+1-\varepsilon)/2}$. We shall prove the following simple proposition to take advantage of this fact.

Lemma 3.3. *Let τ be a positive integer-valued random variable. Let $\mathsf{E}\tau^\alpha < \infty$ for some $\alpha > 1$. Then there exists a positive constant L such that for all $k > 0$*

$$\mathsf{P}\{\tau > k\} \leq Lk^{-\alpha}. \tag{3.6}$$

Proof. Indeed, using Chebyshev's inequality, we get

$$\mathsf{P}\{\tau > k\} = \mathsf{P}\{\tau^\alpha > k^\alpha\} \leq \frac{\mathsf{E}\tau^\alpha}{k^\alpha},$$

thus proving Lemma 3.3. □

Taking the summation in (3.6) over all $k > n$ we finally obtain the second inequality in (3.1), and so the proof of Theorem 3.1 is completed. □

4. Investigating the behaviour of the stationary measures of countable Markov chains through Lyapunov functions

In the previous section we treated one-dimensional Markov chains. The main goal of this section is to show, that our ideas are applicable to more general situations.

First we prove a general result. Let $\{\xi_n\}$ be a discrete time irreducible, aperiodic Markov chain on a countable state space X. Suppose that the Markov chain is ergodic and denote by $\{\pi_k\}$ its stationary measure. Let $f : X \to \mathbf{R}_+$ be a function satisfying the three following conditions:

Condition L. There exists a constant $L > 0$, such that

$$|f(\xi_{n+1}) - f(\xi_n)| < L \quad \text{a.s.}$$

Condition V. There exists real numbers $p_0 > 0$ and $v > 0$, such that

$$\mathsf{P}\{f(\xi_{n+1}) \leq f(\xi_n) - v \mid \xi_n = x\} \geq p_0$$

for all $x \in X$ but a finite number.

Condition F. If $A \in \mathbf{R}_+$ is a bounded set, then the set $f^{-1}(A)$ is finite.

Remark 2. *It can be easily seen that Conditions L and F imply that for all $x \in X$ there exists only a finite number of states $y \in X$ with $P_{xy} \neq 0$.*

Before formulating the general result, let us introduce some additional notation. Let

$$A_n = [n, n+L] \ , \ B_n = f^{-1}(A_n) \ ,$$
$$C_n = [n, n+v] \ , \ D_n = f^{-1}(C_n) \ ,$$

where the quantities L and v are those from Conditions L and V. For any $B \subset X$ let

$$\pi_B = \sum_{i \in B} \pi_i .$$

Write further $Z_n = f(\xi_n)$.

Theorem 4.1. *Assume Conditions* L, V *and* F. *Suppose that for some $a_0 > 0$, $2 < \beta_1 < \beta_2$ there exist $\delta_1, \delta_2 > 0$ such that*

$$\mathsf{E}(Z_{n+1}^{\beta_1} - Z_n^{\beta_1} \mid Z_n = x) \quad \leq \quad -\delta_1 x^{\beta_1 - 2} \tag{4.1}$$
$$\mathsf{E}(Z_{n+1}^{\beta_2} - Z_n^{\beta_2} \mid Z_n = x) \quad \geq \quad \delta_2 x^{\beta_2 - 2} \tag{4.2}$$

for all $x \in f(X)$ such that $x \geq a_0$. Then there exist positive constants C_1, C_2 such that for all n sufficiently large

$$\pi_{B_n} \geq \frac{C_1}{n^{\beta_2 - 1}} \tag{4.3}$$

$$\pi_{D_n} \leq \frac{C_2}{n^{\beta_1 - 1}}. \tag{4.4}$$

Corollary 4.1. *Let $E_n = f^{-1}([n, +\infty))$. Then there exists positive constants C_3 and C_4, such that*

$$\frac{C_3}{n^{\beta_2 - 2}} \leq \pi_{E_n} \leq \frac{C_4}{n^{\beta_1 - 2}} \quad .$$

Proof of Theorem 4.1. We introduce more notation. For any set C let

$$\bar{C} = \{j \in X \mid \exists i \in C : P_{ij} \neq 0\} \cup C$$

and

$$\tilde{C} = \bar{C} \setminus C.$$

Furthermore, let

$$\begin{aligned} B_n^- &= \{j \in \tilde{B}_n \mid f(j) < n\} \\ B_n^+ &= \{j \in \tilde{B}_n \mid f(j) > n + L\}. \end{aligned}$$

As in the proof of Lemma 2.3 and using Lemmas 2.1 and 2.2, we can prove the three following inequalities (for all sufficiently large n):

for $j \in B_n^-$ there exist $K_1, K_2 > 0$ such that

$$m_{j,B_n} \geq K_1 n^{\beta_1 - 1}, \tag{4.5}$$

$$m_{j,B_n} \leq K_2 n^{\beta_2 - 1}, \tag{4.6}$$

and for $j \in B_n^+$ there exists $K_3 > 0$ such that

$$m_{j,B_n} \leq K_3 n. \tag{4.7}$$

By virtue of Conditions L and V, it is sufficient to consider only states j, such that $f(j) \in [n - L, n - v]$ or $f(j) \in [n + L + v, n + 2L]$.

So let us rewrite formula (2.21):

$$\sum_{i \in B_n} \pi_i \Big(\sum_{j \in \tilde{B}_n} P_{ij} m_{j,B_n} + 1 \Big) = 1. \tag{4.8}$$

We prove inequality (4.3). Indeed, using (4.6) and (4.7) we get

$$\sum_{j \in \tilde{B}_n} P_{ij} m_{j,B_n} \leq \sum_{j \in \tilde{B}_n} P_{ij} K_4 n^{\beta_2 - 1} \leq K_4 n^{\beta_2 - 1}$$

and substituting this into (4.8) we obtain

$$\pi_{B_n} = \sum_{i \in B_n} \pi_i \geq \frac{1}{K_4 n^{\beta_2 - 1} + 1} \geq \frac{C_1}{n^{\beta_2 - 1}}$$

for some $C_1 > 0$, and so (4.3) is proved.

Let us turn to inequality (4.4). Using (4.5) and Condition V we can write

$$\begin{aligned} 1 = \sum_{i \in B_n} \pi_i \Big(\sum_{j \in \tilde{B}_n} P_{ij} m_{j,B_n} + 1 \Big) &\geq \sum_{i \in D_n} \pi_i \sum_{j \in \tilde{B}_n} P_{ij} m_{j,B_n} \\ &\geq p_0 K_1 n^{\beta_1 - 1} \sum_{i \in D_n} \pi_i = p_0 K_1 n^{\beta_1 - 1} \pi_{D_n}. \end{aligned}$$

Consequently

$$\pi_{D_n} \leq \frac{C_2}{n^{\beta_1 - 1}}$$

for some $C_2 > 0$ and so (4.4) is proved. This establishes Theorem 4.1. □

The proof of Corollary 4.1 is evident and so we omit it. □

Next we obtain a result on the rate of convergence in this case.

Theorem 4.2. *Let $i \in X$ be an arbitrary state. Suppose that the conditions of Theorem 4.1 are valid. Then there exist two positive real numbers $C_1 = C_1(i)$ and $C_2 = C_2(i)$, not depending on n, such that*

$$C_1 n^{-(\beta_2 - 1)/2} \leq |\pi_i - P_{ii}^n| \leq C_2 n^{-(\beta_1 - 1)/2}$$

for n sufficiently large.

Proof. Note that the results in [1] and [3] that were used to prove Theorem 3.1, are based on inequalities like (4.1) and (4.2). Consequently, the results in [1] and [3] are all valid in this situation. So the proof of the present theorem is similar to Theorem 3.1. We omit details. □

5. Applications: Reflected random walks in $\mathbf{Z}_+^2$ with zero drifts

We will apply the general results in the previous section to random walks on the quadrant with zero drift in the interior. Let us describe this random walk (our description follows [3], [4]).

We consider a discrete time homogeneous Markov chain $\mathcal{L} = \{\xi_n,\ n \geq 0\}$ on the lattice in the positive quarter plane

$$\mathbf{Z}_+^2 = \{(x, y) : x, y \geq 0 \text{ are integers }\}.$$

We assume it to be irreducible and aperiodic. We suppose that the one-step transition probabilities only depend on the position of ξ_n in the following way. For inside points $(x, y \geq 1)$

$$\mathsf{P}\{\xi_{n+1} = (x + k, y + m) \mid \xi_n = (x, y)\} = r_{k,m},$$

for points on the axes ($x = 0,\ y \geq 1$ or $y = 0,\ x \geq 1$)

$$\mathsf{P}\left\{\xi_{n+1} = (x+k, m) \mid \xi_n = (x, 0)\right\} = q_{k,m},$$

$$\mathsf{P}\left\{\xi_{n+1} = (k, y+m) \mid \xi_n = (0, y)\right\} = p_{k,m}$$

and

$$\mathsf{P}\left\{\xi_{n+1} = (k, m) \mid \xi_n = (0, 0)\right\} = r^0_{k,m}.$$

Moreover, let us require the following conditions to be satisfied:

1. (Lower boundedness)

$$\begin{array}{rcll} r_{k,m} & = & 0, & \text{if } k < -1 \text{ or } m < -1 \\ q_{k,m} & = & 0, & \text{if } k < -1 \text{ or } m < 0 \\ p_{k,m} & = & 0, & \text{if } k < 0 \text{ or } m < -1 \\ r^0_{k,m} & = & 0, & \text{if } k, m < 0. \end{array}$$

2. (Upper boundedness)

$$r^0_{k,m} = r_{k,m} = q_{k,m} = p_{k,m} = 0 \ \text{ if } \max\{k, m\} > N_0$$

for $N_0 \in \mathbf{Z}_+$ fixed.

We assume that

$$\sum_{i,j} i r_{i,j} = \sum_{i,j} j r_{i,j} = 0.$$

Let us define λ_x, λ_y and R by

$$\lambda_x = \sum_{i,j} i^2 r_{i,j}\ ,\ \lambda_y = \sum_{i,j} j^2 r_{i,j}\ ,\ R = \sum_{i,j} ij r_{i,j}.$$

Furthermore, write

$$u_1 = \sum_{i,j} i q_{i,j}\ ,\ u_2 = \sum_{i,j} j q_{i,j}$$

and

$$w_1 = \sum_{i,j} i p_{i,j}\ ,\ u_2 = \sum_{i,j} j p_{i,j};$$

let

$$b = \sqrt{\frac{\lambda_y}{\lambda_x}},\ \ r = \frac{R}{\sqrt{\lambda_x \lambda_y}},\ \ a = \frac{1}{\sqrt{\lambda_y(1-r^2)}}$$

and let

$$\gamma = \arccos(-r),\quad \gamma \in (0, \pi)\ .$$

We introduce the two angles $\alpha_1, \alpha_2 \in (-\pi/2, \pi/2)$ (see [3] for a geometrical interpretation):

$$\alpha_1 = \arctan\left\{\frac{1}{\sqrt{1-r^2}}\left(r - b\frac{u_1}{u_2}\right)\right\},$$

$$\alpha_2 = \arctan\left\{\frac{1}{\sqrt{1-r^2}}\left(r - \frac{w_2}{bw_1}\right)\right\}.$$

Next we define a characteristic of our Markov chain that plays an important role in investigating its properties. It is

$$\alpha = \frac{\alpha_1 + \alpha_2}{\gamma}.$$

[3] shows that the Markov chain ξ_n is ergodic, if $\alpha > 2$. Denote also

$$W_n = \{(i,j) \in \mathbf{Z}^2 \mid \sqrt{i^2 + j^2} > n\}.$$

We are finally in the position to formulate the main result of this section.

Theorem 5.1. *Let ξ_n be the above defined Markov chain and let $\alpha > 2$. Then for any $\varepsilon > 0$ there exist constants $C_1, C_2 > 0$ such that*

$$\frac{C_1}{n^{\alpha-2+\varepsilon}} \leq \pi_{W_n} \leq \frac{C_2}{n^{\alpha-2-\varepsilon}} \tag{5.1}$$

for all n.

Proof. Write $\beta_1 = \alpha - \varepsilon$, $\beta_2 = \alpha + \varepsilon$. [3] explicitly construct a function $\psi : \mathbf{Z}_+^2 \to \mathbf{R}_+$ satisfying (4.1) and (4.2). Using simple geometric arguments, it is not difficult to show that this function also satisfies Conditions L, V and F. We can apply Corollary 4.1 and noticing that for some t_1, t_2 independent of n

$$E_{t_1 n} \subset W_n \subset E_{t_2 n}$$

(indeed, this easily follows from geometric arguments) we immediately get (5.1). This completes the proof of Theorem 5.1. □

Finally we briefly touch upon the question of the rate of convergence in this case. Let us formulate the following theorem.

Theorem 5.2. *For any $\varepsilon > 0$ and any state $\tilde{s} = (i,j)$ there exist two positive reals $C_1 = C_1(\tilde{s})$ and $C_2 = C_2(\tilde{s})$ such that*

$$C_1 n^{-\alpha+1-\varepsilon} \leq |\pi_{\tilde{s}} - P_{\tilde{s}\tilde{s}}^n| \leq C_2 n^{-\alpha+1+\varepsilon}$$

for all n.

Proof. A direct consequence of Theorem 4.2. □

Appendix.

We formulate and prove the technical result necessary in Section 3.

Theorem A.1. *Let $\{\xi_t\}$ be an irreducible Markov chain on a countable state space X with the following properties.*

1. *for all i $P_{ij} \neq 0$ only for finite number of states j.*
2. *for some $\alpha > 0$, some set I and some finite set B*
$$\mathsf{E}\tau_{i,B}^{\alpha} < \infty \tag{A.1}$$
for all $i \in I$.
3. *there exists a finite set D such that either the set $X \setminus (B \cup D \cup I)$ is empty, or for all $j \in X \setminus (B \cup D \cup I)$ there exists $i \in I$ and n_0 such that ${}_BP_{ij}^{n_0} > 0$.*

Then for any set C and any state $l \in X$
$$\mathsf{E}\tau_{l,C}^{\alpha} < \infty\ . \tag{A.2}$$

Proof. Clearly, without loss of generality we can suppose $B \cap D = \emptyset$. First we prove that $\mathsf{E}\tau_{j,B}^{\alpha} < \infty$, for all j. If $j \in I$, then this is a direct consequence of the conditions of the theorem, so there are two possible cases: $j \in X \setminus (B \cup D \cup I)$ and $j \in D$ (the case $j \in B$ easily follows from these two).

Case 1: $j \in X \setminus (B \cup D \cup I)$. Suppose that $\mathsf{E}\tau_{j,B}^{\alpha} = \infty$. Then there exists $i \in I$ and n_0 such that ${}_BP_{ij}^{n_0} > 0$. Then
$$\mathsf{E}\tau_{i,B}^{\alpha} \geq {}_BP_{ij}^{n_0}\mathsf{E}(n_0 + \tau_{j,B})^{\alpha}. \tag{A.3}$$
But the right-hand side of (A.3) is equal to ∞ and so $\mathsf{E}\tau_{i,B}^{\alpha} = \infty$. But this contradicts (A.1).

Case 2: $j \in D$. Let $\sigma \equiv \tau_{j,X\setminus D}$ be the first moment that ξ_t leaves the set D. Since D is finite and $\mathsf{P}\{\sigma < \infty\} = 1$, there exists $\gamma > 0$ such that
$$\mathsf{P}\{\sigma = k\} \leq e^{-\gamma k}.$$
Letting
$$s = \max_{k \in \tilde{D}} \mathsf{E}\overline{\tau}_{k,B}^{\alpha},$$
it follows that
$$\mathsf{E}\tau_{j,B}^{\alpha} = \sum_{k=1}^{\infty} \mathsf{P}\{\sigma = k\}\mathsf{E}(\tau_{j,B}^{\alpha} \mid \sigma = k).$$
But, as $\xi_\sigma \in \tilde{D}$ (for the notation cf. the beginning of the proof of Theorem 4.1) we have
$$\begin{aligned} \mathsf{E}(\tau_{j,B}^{\alpha} \mid \sigma = k) &= \mathsf{E}((\overline{\tau}_{\xi_\sigma,B} + k)^{\alpha} \mid \sigma = k) \\ &\leq 2^{\alpha}(\mathsf{E}(\overline{\tau}_{\xi_\sigma,B}^{\alpha} \mid \sigma = k) + k^{\alpha}) \leq 2^{\alpha}(s + k^{\alpha}) \end{aligned}$$

(here we have used the simple inequality

$$(a_1 + \ldots + a_n)^\alpha \le n^\alpha (a_1^\alpha + \ldots + a_n^\alpha) \tag{A.4}$$

for $\alpha > 0$ and $a_i \ge 0$.)

We conclude that

$$\mathsf{E}\tau_{j,B}^\alpha \le \sum_{k=1}^{\infty} e^{-\gamma k} 2^\alpha (s + k^\alpha) < \infty$$

and so we have proved that for any $j \in X$

$$\mathsf{E}\tau_{j,B}^\alpha < \infty. \tag{A.5}$$

We continue with the proof of Theorem A.1. We want to prove that for any $l, q \in X$

$$\mathsf{E}\tau_{l,\{q\}}^\alpha < \infty.$$

Clearly, without loss of generality we may assume that $q \in B$. Indeed, if this is not true, then we consider the set $B' = B \cup \{q\}$, and it is not difficult to see that $\tau_{j,B'} \le \tau_{j,B}$ for all j.

Let $\sigma_1 \equiv \tau_{l,B}$, and

$$\sigma_{k+1} = \{\min n : n > \sigma_k\,,\, \xi_n \in B\}, \quad k \ge 1$$

where σ_k is the k-th first passage time of B. Clearly, from (A.5) it follows that our Markov chain is recurrent. Hence $\sigma_k < \infty$ a.s., for all k. We consider a new Markov chain $\{\eta_k\}$ on the state space B, which is obtained from $\{\xi_k\}$ in the following way: $\eta_k = \xi_{\sigma_k}$. Denote

$$w = \{\min k > 0 : \eta_k = q \,|\, \xi_0 = l\}.$$

Clearly, $\mathsf{E}\tau_{l,\{q\}}^\alpha = \mathsf{E}\sigma_w^\alpha$ and we have

$$\mathsf{E}\sigma_w^\alpha = \sum_{k=1}^{\infty} \mathsf{P}\{w = k\} \mathsf{E}(\sigma_k^\alpha \,|\, w = k). \tag{A.6}$$

Since the Markov chain $\{\xi_k\}$ is irreducible, $\{\eta_k\}$ is also irreducible, and as the state space of $\{\eta_k\}$ is finite, there exist constants $K_1 > 0$ and $\delta > 0$ such that

$$\mathsf{P}\{w = k\} \le K_1 e^{-\delta k}. \tag{A.7}$$

Let us estimate $\mathsf{E}(\sigma_k^\alpha \,|\, w = k)$. For $m, n \in B$ denote by Q_{mn} the transition probabilities of the Markov chain $\{\eta_k\}$. Let

$$r_1 = \min_{\substack{m,n \in B:\\ Q_{mn} \neq 0}} Q_{mn}.$$

Denote $u_k = \sigma_{k+1} - \sigma_k$. From (A.5) it follows that for all $m \in B$

$$\mathsf{E}(u_k^\alpha \mid \eta_k = m) < \infty. \tag{A.8}$$

Let us prove that for all $m, n \in B$, such that $Q_{mn} \neq 0$

$$\mathsf{E}(u_k^\alpha \mid \eta_k = m, \eta_{k+1} = n) < \infty.$$

Indeed,

$$\mathsf{E}(u_k^\alpha \mid \eta_k = m) \geq Q_{mn} \mathsf{E}(u_k^\alpha \mid \eta_k = m, \eta_{k+1} = n),$$

and so

$$\mathsf{E}(u_k^\alpha \mid \eta_k = m, \eta_{k+1} = n) \leq \frac{1}{r_1} \mathsf{E}(u_k^\alpha \mid \eta_k = m) < \infty.$$

Consequently, since B is finite, there exists $r_2 > 0$ such that

$$\mathsf{E}(u_k^\alpha \mid \eta_k, \eta_{k+1}) \leq r_2 \quad \text{a.s.}$$

Then, using (A.4) again we have

$$\begin{aligned}
&\mathsf{E}(\sigma_k^\alpha \mid w = k) = \\
&\quad = \quad \mathsf{E}((\sigma_1 + u_1 + \ldots + u_{k-1})^\alpha \mid w = k) \\
&\quad \leq \quad k^\alpha \mathsf{E}\sigma_1 + k^\alpha \mathsf{E}(u_1^\alpha + \ldots + u_{k-1}^\alpha \mid \eta_1 \neq q, \ldots \eta_{k-1} \neq q, \eta_k = q) \\
&\quad \leq \quad k^\alpha \mathsf{E}\sigma_1 + k^\alpha r_2 (k-1) \leq r_3 k^{\alpha+1},
\end{aligned}$$

for all k and some $r_3 > 0$.

So, using (A.6) and (A.7) we have

$$\mathsf{E}\tau_{l,\{q\}}^\alpha \leq \sum_{k=1}^{\infty} K_1 e^{-\delta k} r_3 k^{\alpha+1} < \infty.$$

To complete the proof of Theorem A.1, we only have to note, that then $\tau_{l,\{q\}} \geq \tau_{l,C}$, if $q \in C$, so that

$$\mathsf{E}\tau_{l,C}^\alpha \leq \mathsf{E}\tau_{l,\{q\}}^\alpha < \infty.$$

Thus, we have proved Theorem A.1. □

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