On a many-dimensional random walk in a rarefied random environment

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Abstract

We consider a modification of the Simple Random Walk (SRW), which can be described as follows. Initially, any $x \in \mathbb{Z}^d$ becomes "special" with probability $p(x)$; then, in all special sites we modify the transition probabilities in order to create a drift which is directed outwards the origin (in the case of one- or two-dimensional SRW) or towards the origin (for higher dimensions), thus constructing a random environment. Then, based on the asymptotic behaviour of the function $p(x)$, we give some sufficient conditions for transience and recurrence.

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1 Introduction and results

It is a well known fact that SRW in $\mathbb{Z}^d$ is recurrent for $d \le 2$ and transient for $d \ge 3$. Now, suppose that SRW is perturbed in the following way. Initially, we perform a random procedure to split $\mathbb{Z}^d$ into the set of "ordinary" sites and the set of "special" sites. This is done in such a way that the density of the special sites decreases as the distance from the origin increases. Then, in ordinary sites the process behaves like SRW, but in special sites the rules are different. For the processes of this type, transience/recurrence was studied in the papers [2, 3, 4]. More specifically, the subject of [3] is the model where the special sites are traps (this is equivalent to always sending the particle to the origin from any special site). In [2] there is branching in special sites, i.e. any particle which enters a special site generates more particles there. In [4] the so-called frog model was studied. That model can be briefly described as follows: into every special site we place a "sleeping" particle, which begins to move only when some other particle hits it.

In the present paper we continue the line of research of [2, 3, 4]. Let us describe the model which is of interest here.

For any $x \in \mathbb{Z}^d$, some number $p(x)$, $0 \le p(x) \le 1$, is given. First, each site $x \in \mathbb{Z}^d$ becomes special (or "green") with probability $p(x)$ and does not change its status in the sequel. Denote by G the set of all the green sites. We refer to the set G as "the environment". The non-special sites are called "ordinary", and they do not change their status either. Then we start a random walk ξ_n in $\mathbb{Z}^d$ with the following transition probabilities. From ordinary sites, the particle jumps to one of its nearest neighbors with probability $1/2d$. Now, if x is green, then the particle jumps from x to y with probability P_{xy}. Denote by

$$M(x) = \sum_{y \in \mathbb{Z}^d} (y - x) P_{xy}$$

the vector of mean drift from x and by $\mathrm{Proj} M(x)$ its projection to the radius vector x. Assume that P_{xy} are such that

(A) $\mathrm{Proj} M(x)$ is directed towards the origin if $d \ge 3$, and outwards if $d \le 2$;

(B) there exists $D > 0$ such that if $\|x - y\| > D$, then $P_{xy} = 0$.

It is important to note that for each $x \in G$ there are a lot of possible choices of the transition probabilities P_{xy} which satisfy **(A)** and **(B)**. Moreover, unless in one-dimensional case, it seems to be difficult to specify a generic rule of placing the concrete values of the transition probabilities into the special points. So, from now on we shall just assume that, in the case $d \geq 2$, for the "green" points the outgoing transition probabilities are somehow chosen in such a way that conditions **(A)** and **(B)** are satisfied (case $d = 1$ will be discussed separately). We stress once again that the rule of placing transition probabilities to "green" points is fixed, in the sense that the transition probabilities in x only depend on the fact whether it is "green" or not, and do not depend on the location of other "green" points or transition probabilities there. We remark also that the results concerning the case $d \geq 2$ (Theorem 1.1) are valid for any particular choice of the transition probabilities in green points satisfying **(A)** and **(B)**.

Thus, we have constructed a random walk in a *quenched* random environment. We are interested in the classification of this random walk from the point of view of recurrence and transience. Note that we say that the random walk is recurrent if it is recurrent for almost all environments, and the same for transience.

Our main results are the following theorems.

Theorem 1.1 *If there exists $\varepsilon > 0$ such that for all x large enough*

$$p(x) \geq \frac{1}{\|x\|^{1-\varepsilon}}, \tag{1.1}$$

and for some $a > 0$ it holds that $\|\mathrm{Proj} M(x)\| \geq a$ for all $x \in G$, then

(a) if $d = 2$, then the random walk is transient;

(b) if $d \geq 3$, then the random walk is recurrent;

If there exists $\varepsilon > 0$ such that for all x large enough

$$p(x) \leq \frac{1}{\|x\|^{1+\varepsilon}}, \tag{1.2}$$

then

(c) if $d = 2$, then the random walk is recurrent ;

(d) if $d \geq 3$, then the random walk is transient.

In dimension 1 it is possible to define the model in a more constructive way. First of all, suppose that for some integer $D > 0$ (which, as before, stands for the maximal jump from green points) some probability mass function $P(x)$, $x \in \{-D, \dots, D\}$ is given. Denote

$$\begin{aligned} a_+ &= \sum_{x=1}^{D} xP(x), \\ a_- &= -\sum_{x=-D}^{-1} xP(x), \end{aligned}$$

and suppose that $a_+ > a_- > 0$. Now, we define the one-dimensional random walk in random environment as follows:

- to guarantee that the site 0 is reachable from all other sites, we declare all $x \in \{-D, \dots, D\}$ to be ordinary;
- if $x \in G$ (remember that G is the set of "green" points) and $|x| > D$, put
$$P_{xy} = \begin{cases} P(y-x), & \text{if} \quad x > D, \\ P(x-y), & \text{if} \quad x < -D. \end{cases} \tag{1.3}$$

For this one-dimensional model we are able to give a more precise classification.

Theorem 1.2 *Let* $\beta_0 = \log \frac{a_+}{a_-}$.

(i) If
$$\limsup_{|x|\to\infty} |x|p(x) < \beta_0^{-1},$$
then the random walk is recurrent a.s.;

(ii) If
$$\liminf_{|x|\to\infty} |x|p(x) > \beta_0^{-1},$$
then the random walk is transient a.s.

Note that it is possible to define the model in less constructive way, analogously to what was done in higher dimensions, that is, only requiring that the drift in green points is directed outwards the origin and bounded away from 0 by some constant. In this case the constant β_0^{-1} in Theorem 1.2 will be substituted by two other constants corresponding to the cases (i) and (ii).

2 Proofs

2.1 Basic facts

Here we formulate some known facts, which will be useful in the course of the proofs of Theorems 1.1 and 1.2.

Let us consider a time homogeneous irreducible Markov chain

$$\mathcal{L} = \{\eta_n\}_{n\in\mathbb{N}}$$

with a countable state space $X = \{x_1, x_2, \ldots\}$.

Theorem 2.1 (Theorem 2.2.2, [5]) *Suppose that there exist a set $A \subset X$ and a function $f : X \to \mathbb{R}$, $f \geq 0$, with the following properties:*

- *there exists $x \in X \setminus A$ such that $f(x) < \inf_{y\in A} f(y)$;*
- *for all $x \notin A$ we have*

$$\mathbf{E}(f(\eta_{t+1}) - f(\eta_t) \mid \eta_t = x) \leq 0.$$

Then the Markov chain $\mathcal{L}$ is transient.

Theorem 2.2 (Theorem 2.2.1, [5]) *Suppose that there exists a finite set $A \subset X$ and a function $f : X \to \mathbb{R}$, $f \geq 0$, such that $f(x_j) \to +\infty$ as $j \to \infty$ and for all $x \notin A$ we have*

$$\mathbf{E}(f(\eta_{t+1}) - f(\eta_t) \mid \eta_t = x) \leq 0.$$

Then the Markov chain $\mathcal{L}$ is recurrent.

Theorem 2.3 (Kolmogorov, see e.g. [7], Chapter IV, §3) *Consider a strictly increasing sequence of positive numbers $\{b_i\}_{i=1,2,\ldots}$ such that $b_i \to \infty$ as $i \to \infty$. Let $\{\zeta_i\}_{i=1,2,\ldots}$ be a sequence of independent random variables with the following properties:*

- $\mathbf{E}\zeta_i^2 < \infty, i = 1, \ldots$
- $\sum\limits_{i=1}^{\infty} \dfrac{\mathrm{Var}\,\zeta_i}{b_i^2} < \infty.$

Then

$$\frac{S_n - \mathbf{E}S_n}{b_n} \to 0 \text{ a.s. as } n \to \infty, \text{ where } S_n = \sum_{i=1}^{n} \zeta_i.$$

Throughout this paper ξ'_n will stand for the simple random walk in $\mathbb{Z}^d$ (starting from 0, unless otherwise stated; let us remind that ξ_n is the original random walk in random environment). The following result can be found in [1] (Theorem 2.2).

Lemma 2.1 *We have*

- *if $d = 2, x \neq 0$ and $n \geq \|x\|^2$, then there exists $c_1 > 0$ such that*

$$\mathbf{P}[\xi'_t = x \text{ for at least one } t = 1, \ldots, n\ ] \geq \frac{c_1}{\log \|x\|}; \tag{2.1}$$

- *if $d \geq 3, x \neq 0$ and $n \geq \|x\|^2$, then there exists $c_2 = c_2(d) > 0$ such that*

$$\mathbf{P}[\xi'_t = x \text{ for at least one } t = 1, \ldots, n\ ] \geq \frac{c_2}{\|x\|^{d-2}}. \tag{2.2}$$

We use also the following elementary fact (see e.g. [6], Lemma 5.1):

Lemma 2.2 *If $d \geq 2$, then as $x \to \infty$*

$$\mathbf{E}(\|\xi'_k\| - \|\xi'_{k-1}\| \mid \xi'_{k-1} = x) = \frac{d-1}{2d}\frac{1}{\|x\|} + O(\|x\|^{-3}) \tag{2.3}$$

and

$$\mathbf{E}((\|\xi'_k\| - \|\xi'_{k-1}\|)^2 \mid \xi'_{k-1} = x) = \frac{1}{d} + O(\|x\|^{-2}). \tag{2.4}$$

We need also the following result concerning SRW:

Lemma 2.3 *Let ξ'_n be the two-dimensional SRW. Let $S(r) = \{x \in \mathbb{Z}^2 : r \leq \|x\| < r+1\}$. Then there exists $c_3 > 0$ such that for any $r_1, r_2 > 0$ such that $r_1 + 1 < r_2$ and x such that $r_1 + 1 \leq \|x\| < r_2$ it holds*

$$p := \mathbf{P}[\xi'_n \text{ hits } S(r_2) \text{ before hitting } S(r_1) \mid \xi'_0 = x] \geq c_3 \frac{\log^2 \|x\| - \log^2 r_1}{\log^2 r_2 - \log^2 r_1}.$$

Proof. It is straightforward to get that there exists K such that $\log^2(\|\xi'_n\|+K)$ is a submartingale, that is,

$$\mathbf{E}(\log^2(\|\xi'_{n+1}\|+K)-\log^2(\|\xi'_n\|+K)\mid \xi'_n=y)\geq 0$$

for all $y\in\mathbb{Z}^2$. Thus,

$$p\log^2(r_2+1+K)+(1-p)\log^2(r_1+K)\geq \log^2(\|x\|+K)$$

and so

$$p\geq \frac{\log^2(\|x\|+K)-\log^2(r_1+K)}{\log^2(r_2+1+K)-\log^2(r_1+K)}\geq c_3\frac{\log^2\|x\|-\log^2 r_1}{\log^2 r_2-\log^2 r_1}$$

for some $c_3>0$. ∎

2.2 Some auxiliary lemmas

In this section we formulate and prove some auxiliary results which are specifically related to the construction of the random set G.

The next two results easily follow from Borel-Cantelli lemma and will not be proved here.

Lemma 2.4 *If $d\geq 2$ and for some $\varepsilon>0$ and all x large enough*

$$p(x)\geq \frac{1}{\|x\|^{1-\varepsilon}},$$

then for almost all G there exists $\varepsilon_1>0$ such that a.s. for all large enough $y\in\mathbb{Z}^d$ there exists $z\in G$ which verifies $\|y-z\|<\|y\|^{1/d-\varepsilon_1}$.

Lemma 2.5 *If $d\geq 2$ and for some $\varepsilon>0$, $p(x)\leq\frac{1}{\|x\|^{1+\varepsilon}}$ for all x large enough, then for almost all G there exists $\varepsilon_2>0$ such that for all large enough x the ball with center in x and radius $\|x\|^{\varepsilon_2}$ contains at most $d-1$ green sites a.s.*

The following fact will also be of use:

Lemma 2.6 *Let $d\geq 3$. Fix r sites in $\mathbb{Z}^d\setminus\{0\}$, where $r\leq 2d-1$. Let $\tilde{p}$ be the probability that simple random walk (starting from 0) never hits any of those sites. Then, regardless of the positions of these r sites, there exists a positive number $c_4=c_4(d)<1$, such that $1-\tilde{p}<c_4$.*

Proof. Let $y_1, \dots, y_r$ be the chosen sites. For $A \subset \mathbb{Z}^d$ let

$$q(x, A) = \mathbf{P}[\text{SRW starting from } x \text{ hits } A].$$

We prove this lemma by induction in r.

If $r = 1$, it is straightforward to get that there exists h_0 such that if $\|x - y\| > h_0$, then

$$\mathbf{P}[\text{SRW starting from } x \text{ ever hits } y] < \frac{1}{2}.$$

As

$$p' = \sup_{y : \|x-y\| \le h_0} q(x, \{y\}) < 1,$$

we get that $1 - \tilde{p} \le \max\{p', \frac{1}{2}\}$.

Suppose that the lemma was proved for $r < 2d - 1$ and let us prove it for $r + 1$. Let c_r be the minimal probability that SRW starting from x ever hits $\{y_1, \dots, y_r\}$. By the induction assumption, $c_r > 0$. Choose now y_{r+1}. Note that there exists h_r such that if $\|x - y_{r+1}\| > h_r$, then

$$\mathbf{P}[\text{SRW starting from } x \text{ hits } \{y_1, \dots, y_{r+1}\}] < 1 - \frac{c_r}{2}.$$

We have also that

$$p' = \sup_{\substack{y_1, \dots, y_r : \\ \|x - y_i\| \le h_r}} q(x, \{y_1, \dots, y_r\}) < 1,$$

as the number of different sets $\{y_1, \dots, y_r\}$ in the ball with center in x and radius h_r is finite, and $r < 2d$, and thus $\tilde{p} \ge \min\{1 - p', \frac{c_r}{2}\}$. Lemma 2.6 is proved. ∎

Lemma 2.7 *(i) Let $d \ge 3$, and there exists $\varepsilon > 0$ such that $p(x) \le \frac{1}{\|x\|^{1+\varepsilon}}$ for all x large enough. Furthermore, take any $\varepsilon_3 < \frac{2}{d}\varepsilon$ and put $m := \|x\|^{1+\varepsilon_3}$, $U = \{y \in \mathbb{Z}^d : \|y - x\| \le \|x\|^{\frac{1}{2} + \frac{\varepsilon_3}{2} + \varepsilon_4}\}$ for any $\varepsilon_4 > 0$. Then for the random walk in random environment ξ_n it holds that*

(i1) for almost all G (i.e., for almost all the realizations of the random environment) there exist $c_5, c_6 > 0$ not depending on G, such that for all x large enough (this is G-dependent, i.e., we mean in fact that there exists $\kappa_0 = \kappa_0(G) > 0$ such that the expression below holds for all x such that $\|x\| \ge \kappa_0$)

$$\mathbf{P}[\xi_n \in U \text{ for all } n \le m \mid \xi_0 = x] \ge 1 - e^{-c_5 \|x\|^{c_6}};$$

(i2) for almost all G there exists $c_7 > 0$ not depending on G, such that for all x large enough

$$\sum_{k=1}^{m} \mathbf{P}[\xi_k \in G \mid \xi_0 = x] \le c_7.$$

(ii) Let $d = 2$, and there exists $\varepsilon > 0$ such that $p(x) \le \frac{1}{\|x\|^{1+\varepsilon}}$ for all x large enough. Furthermore, take any $\varepsilon_3 < \varepsilon$ and put $m := \|x\|^{1+\varepsilon_3}$, $U = \{y \in \mathbb{Z}^d : \|y - x\| \le \|x\|^{\frac{1}{2}+\frac{\varepsilon_3}{2}+\varepsilon_4}\}$ for any $\varepsilon_4 > 0$. Then

(ii1) for almost all G there exist $c_8, c_9 > 0$ not depending on G, such that for all x large enough

$$\mathbf{P}[\xi_n \in U \text{ for all } n \le m \mid \xi_0 = x] \ge 1 - e^{-c_8\|x\|^{c_9}};$$

(ii2) for almost all G there exists $c_{10} > 0$ not depending on G, such that for all x large enough

$$\sum_{k=1}^{m} \mathbf{P}[\xi_k \in G \mid \xi_0 = x] \le c_{10} \log^2 \|x\|.$$

Proof. (i) Let $G' = G \cap U$. Define random variables τ_i and σ_i by:

$$\begin{aligned} \tau_0 &= \sigma_0 = 0, \\ \tau_i &= \min\{n > \sigma_{i-1} : \xi_n \in G'\}, \\ \sigma_i &= \min\{n > \tau_i : \xi_n \notin G'\}, \end{aligned}$$

$i = 1, 2 \ldots$, and define the random variable N by $N = \max\{n : \tau_n \le m\}$. Consider the following events:

$$\begin{aligned} A &= \{\xi_n \in U \text{ for all } n \le m\}, \\ C_n &= \{\tau_{n+1} - \tau_n > m\}, \\ D_n &= \Big\{ \sup_{i \in [0,(\tau_{n+1}-\tau_n)\wedge m]} \|\xi_{\sigma_n + i} - \xi_{\sigma_n}\| \le \|x\|^{\frac{1+\varepsilon_3}{2}+\frac{\varepsilon_4}{3}} \Big\}, \\ E &= \{N \le \|x\|^{\frac{\varepsilon_4}{3}}\}. \end{aligned}$$

We have

$$\mathbf{P}[A] \ge \mathbf{P}\Big[E \cap \Big(\bigcap_{i=1}^{\|x\|^{\frac{\varepsilon_4}{3}}} D_n\Big)\Big] \ge \mathbf{P}[E] - \sum_{i=1}^{\|x\|^{\frac{\varepsilon_4}{3}}} \mathbf{P}[\overline{D}_n]. \tag{2.5}$$

Note that, by coupling with SRW,

$$\sum_{i=1}^{\|x\|^{\frac{\varepsilon_4}{3}}} \mathbf{P}[\overline{D}_n] \le e^{-c_{11}\|x\|^{\varepsilon_5}} \tag{2.6}$$

for some $c_{11}, \varepsilon_5 > 0$. One can write

$$\begin{aligned} \mathbf{P}[\overline{E}] &\le \mathbf{P}[\overline{C}_1 \dots \overline{C}_{\|x\|^{\frac{\varepsilon_4}{3}}}] \\ &= \mathbf{P}[\overline{C}_1]\mathbf{P}[\overline{C}_2 \mid \overline{C}_1] \cdots \mathbf{P}[\overline{C}_{\|x\|^{\frac{\varepsilon_4}{3}}} \mid \overline{C}_1\overline{C}_2 \dots \overline{C}_{\|x\|^{\frac{\varepsilon_4}{3}}-1}] \end{aligned} \tag{2.7}$$

and

$$\mathbf{P}[\overline{C}_i \mid \overline{C}_1 \dots \overline{C}_{i-1}] \le \mathbf{P}[\overline{C}_i \mid \overline{C}_1 \dots \overline{C}_{i-1}, D_1 \dots D_{i-1}] + \mathbf{P}[\overline{D_1 \dots D_{i-1}}]. \tag{2.8}$$

The term $\mathbf{P}[\overline{D_1 \dots D_{i-1}}]$ can be estimated by using (2.6), so we need to prove only that $\mathbf{P}[\overline{C}_i \mid \overline{C}_1 \dots \overline{C}_{i-1}, D_1 \dots D_{i-1}] \le c_{12}$ for some $c_{12} < 1$.

Let $B_k = \{\xi_{\tau_i+k} \in G'\}$. Without loosing generality we can take ε_4 such that $2\varepsilon_4 < \frac{2}{d}\varepsilon - \varepsilon_3$.

Consider a partition of $\mathbb{Z}^d$ by the cubes

$$\begin{aligned} K_{i_1\dots i_d} = &\{z = (z_1, \dots, z_d) \in \mathbb{Z}^d : \\ &(i_k - 1)\|x\|^{\frac{1}{d}+\varepsilon_6} \le z_k < i_k\|x\|^{\frac{1}{d}+\varepsilon_6}, \quad k = 1, \dots, d\} \end{aligned} \tag{2.9}$$

where ε_6 is such that

$$\frac{\varepsilon_3}{2} + \varepsilon_4 < \varepsilon_6 < \frac{\varepsilon}{d}. \tag{2.10}$$

Consider now the cubes $K_{i_1\dots i_d}$ which have nonempty intersection with the ball U. The number ℓ of them is of order $\|x\|^{\frac{d}{2}-d(\varepsilon_6-\frac{\varepsilon_3}{2}-\varepsilon_4)}$. Denote these cubes by $K_1, \dots, K_\ell$. For $z \in \mathbb{Z}^d$ and $W \subset \mathbb{Z}^d$, denote also

$$\begin{aligned} M_W(z) &= \mathbf{E}(\#t \ge 0 : \xi_t \in W \mid \xi_0 = z) = \sum_{i=0}^{\infty} \mathbf{1}\{\xi_i \in W \mid \xi_0 = z\}, \\ P_W(z) &= \mathbf{P}[\text{there exists } t > 0 : \xi_t \in W \mid \xi_0 = z], \\ P'_W(z) &= \mathbf{P}[\text{there exists } t > 0 : \xi'_t \in W \mid \xi'_0 = z], \end{aligned}$$

(remember that ξ'_t is a simple random walk in $\mathbb{Z}^d$).

We have

$$\sum_{k=1}^{m} \mathbf{P}[B_k \mid \xi_0 = x] = \mathbf{E}\sum_{k=1}^{m} \mathbf{1}_{\{\xi_{\tau_i+k} \in G' \mid \xi_0 = x\}} = M_{G'}(x).$$

To prove that there exists $c_7 > 0$ such that $M_{G'}(x) \leq c_7$ for all large enough x, it is sufficient to prove that there exists $c_{13} < 1$, such that for all $z \in G'$ we have $P_{G'}(z) < c_{13}$ (if x is large enough, remember that G' depends on x). For that it is sufficient to prove that there exists c_{13} such that for all z_0 such that there exists a site $z \in G'$, $\|z - z_0\| < D$, we have

$$P'_{G'}(z_0) < c_{13}. \tag{2.11}$$

Note that Borel-Cantelli lemma implies that there exists $b \in \mathbb{N}$ such that for all large enough x each of the cubes $K_1, \dots, K_\ell$ contains at most b green sites.

Now, let us prove (2.11). We suppose that z_0 belongs to the same cube as x. For other sites the proof is analogous. Without loss of generality, we may consider that the site x is in the cube K_1. Denote by V the ball with center in z_0 and radius $\|x\|^{\varepsilon_2}$.

Lemma 2.5 implies that the number of green sites inside V is not greater than $d - 1 < 2d - 1$, so using Lemma 2.6 we get that with probability at least $1 - c_4$ the random walk will never hit any green site in V. Let us denote $G'_1 = G' \cap (K_1 \setminus V)$, and $G'_i = G' \cap K_i$ for $i = 2, \dots, \ell$. We have

$$P'_{G'}(z_0) < c_4 + P'_{G'_1}(z_0) + \sum_{i=2}^{\ell} P'_{G'_i}(z_0). \tag{2.12}$$

It is clear that

$$P'_{G'_1}(z_0) \leq bc_{14}\|x\|^{-\varepsilon_2(d-2)}, \tag{2.13}$$

and

$$\sum_{i=2}^{\ell} P'_{G'_i}(z_0) \leq \sum_{i=2}^{\ell} \frac{bc_{14}}{\rho(z_0, K_i)^{d-2}}, \tag{2.14}$$

where $\rho(z_0, K_i)$ is the distance from z_0 to K_i. An elementary geometric argument implies that

$$\begin{aligned} \sum_{i=2}^{\ell} \frac{1}{\rho(z_0, K_i)^{d-2}} &\simeq \sum_{j=1}^{\|x\|^{1/2d-\varepsilon_7}} \frac{j^{d-1}}{(j\|x\|^{(1/d+\varepsilon_6)})^{(d-2)}} \\ &= \frac{1}{\|x\|^{(1/d+\varepsilon_6)(d-2)}} \sum_{j=1}^{\|x\|^{1/2d-\varepsilon_7}} j \end{aligned}$$

$$\simeq \frac{\|x\|^{(1/2d-\varepsilon_7)2}}{\|x\|^{(1/d+\varepsilon_6)(d-2)}} \leq \|x\|^{-1+3/d-2\varepsilon_7-\varepsilon_6(d-2)}, \quad (2.15)$$

where $\varepsilon_7 = \varepsilon_6 - \varepsilon_3/2 - \varepsilon_4$. Combining (2.12), (2.13), (2.14), (2.15), we get

$$P'_{G'}(z_0) < c_4 + bc_{14}\|x\|^{-\varepsilon_2} + bc_{14}\|x\|^{-1+3/d-2\varepsilon_7-\varepsilon_6(d-2)} < c_{13}$$

for some $c_{13} < 1$ and all x large enough, so (2.11) is proved. Consequently, we get that

$$\sum_{k=1}^{m} \mathbf{P}[\xi_k \in G' \mid \xi_0 = x] = \sum_{k=1}^{m} \mathbf{P}(B_k \mid \xi_0 = x) < c_7. \quad (2.16)$$

Also, we have

$$\mathbf{P}[\overline{C}_i \mid \overline{C}_1 \dots \overline{C}_{i-1}, D_1 \dots D_{i-1}] \leq \sup_{z \in G'} P'_{G'}(z) \leq c_{12} < 1,$$

and $\mathbf{P}[\overline{D_1 \dots D_{i-1}}]$ is exponentially small. Thus, from (2.5)–(2.8) one gets that there exist $c_5, c_6 > 0$ such that for all x large enough

$$\mathbf{P}[\xi_n \in U \text{ for all } n \leq m \mid \xi_0 = x] = \mathbf{P}[A] \geq 1 - e^{-c_5\|x\|^{c_6}}$$

and we have proved *(i1)*. It can be easily seen that (2.16) and *(i1)* imply *(i2)*.

(ii1) The Borel-Cantelli lemma implies that there exists K such that for all x large enough U contains at most K green points. For every green point in U consider a ball centered in it with radius $\frac{1}{3(K+1)}\|x\|^{\frac{1+\varepsilon_3}{2}+\varepsilon_4}$ and let Γ be the set covered by those balls. Note that, as we have at most K green points in U, the following situation is impossible: in Γ there exists a contour around x connected to the boundary of U. Thus, we can draw a contour γ around x such that the distance from γ to x, to any green point and to the boundary of U is at least $\frac{1}{3(K+1)}\|x\|^{\frac{1+\varepsilon_3}{2}+\varepsilon_4}$. So, two cases are possible: or ξ_k stays inside γ for all $k = 1, \dots, m$ and so $\xi_k \in U$ for all $k = 1, \dots, m$ or ξ_k reaches γ for some k. Note that if ξ_k reaches γ, then

$$\begin{aligned} &\mathbf{P}[\xi_n \notin U \text{ for some } n \leq m] \\ &\leq \mathbf{P}\Big[\|\xi'_n\| \geq \frac{1}{3(K+1)}\|x\|^{\frac{1+\varepsilon_3}{2}+\varepsilon_4} \mid \xi'_0 = 0\Big] \leq e^{-c_8\|x\|^{c_9}} \end{aligned}$$

for some $c_8, c_9 > 0$.

(ii2) It is sufficient to prove that for any $y \in U$

$$\begin{aligned} & \mathbf{P}[\xi_n \in G \text{ for some } n \le m \mid \xi_0 = y] \\ &= \mathbf{P}[\xi'_n \in G \text{ for some } n \le m \mid \xi'_0 = y] \\ &\le 1 - \frac{c_{15}}{\log^2 m} \end{aligned} \tag{2.17}$$

for some $c_{15} > 0$. We prove (2.17) by induction on the number ℓ of green sites in U.

If $\ell = 1$, then, by Lemma 2.3, for any y with probability at least $\frac{c_{16}}{\log^2 m}$ SRW starting from y reaches the boundary of U before reaching green site.

Suppose that (2.17) was proved for ℓ and let us prove it for $\ell + 1$. Let $z_1, \dots, z_{\ell+1}$ be the green sites in U. Let D' be the diameter of the set $\{z_1, \dots, z_{\ell+1}\}$. Note that, by Lemma 2.5, $D' \ge \|x\|^{\varepsilon_2}$, so $\log^2 D' \sim \log^2 m$. Consider balls centered in green sites with radius $D'/3(\ell+1)$. Clearly, there exists a contour $\gamma \subset U$ around y such that at least one of those balls is outside γ, say, that with the center in $z_{\ell+1}$, and diameter of γ is at most D'. By the induction assumption, with probability at least $\frac{c_\ell}{\log^2 m}$ random walk ξ_n starting from y reaches γ without hitting any green site in it. Let $W^{(D')}$ be the ball of diameter D' which contains γ and denote $W^{(\frac{3D'}{2})} = \frac{3}{2} W^{(D')}$. Consider also the ball V centered in $z_{\ell+1}$ with radius $2D'$. Note that $W^{(\frac{3D'}{2})} \subset V$. Then, by Lemma 2.3, the random walk starting from any site in γ, with probability at least $1 - \frac{c_{17}}{\log^2 m}$ will reach the boundary of V before hitting $z_{\ell+1}$, as the distance between γ and $z_{\ell+1}$ is at least $D'/3(\ell+1)$. Thus, with probability at least $1 - \frac{c_\ell c_{17}(\ell+1)}{\log^2 m}$, the random walk will reach $W^{(\frac{3D'}{2})}$ without hitting $\{z_1, \dots, z_{\ell+1}\}$. Then, by Lemma 2.3, from the boundary of $W^{(\frac{3D'}{2})}$, with probability at least c_{18}, for some constant $c_{18} > 0$, random walk will reach the boundary of U before reaching $W^{(D')}$ (and thus, without hitting green sites). As the boundary of U lies farther than m from the boundary of $W^{(D')}$, this shows that (2.17) holds. ■

2.3 Proof of Theorem 1.1

Throughout the proof of this theorem, we assume that the environment G is "typical", i.e., what stated in Lemmas 2.4, 2.5, and 2.7 is valid for that G.

2.3.1 Proof of part (a)

We are going to prove the transience of the random walk in random environment ξ_n using Theorem 2.1 with function $f(x) = (\|x\| + 1)^{-1}$.

Define the sequence of the stopping times τ_n by

$$\tau_{n+1} = \tau_n + \left[\|\xi_{\tau_n}\|^{1-\varepsilon_8}\right] + 1,$$

where $[\cdot]$ denotes the integer part, and $\tau_0 = 1$.

To apply Theorem 2.1, first it is necessary to bound

$$\mathbf{E}(\|\xi_{\tau_{n+1}}\| - \|\xi_{\tau_n}\| \mid \xi_{\tau_n} = x)$$

from below and

$$\mathbf{E}((\|\xi_{\tau_{n+1}}\| - \|\xi_{\tau_n}\|)^2 \mid \xi_{\tau_n} = x) \tag{2.18}$$

from above (the environment G is supposed to be fixed, typically we do not indicate the dependence on G in $\mathbf{E}$-s and $\mathbf{P}$-s).

Abbreviate $m := \tau_{n+1} - \tau_n$. By (2.3), it holds that for $k = 1, \dots, m$

$$\mathbf{E}(\|\xi_{\tau_n+k}\| - \|\xi_{\tau_n+k-1}\| \mid \xi_{\tau_n+k-1} = x, x \notin G) = \frac{1}{4\|x\|} + O(\|x\|^{-3}), \tag{2.19}$$

and, from **(A)** and the fact that $\|\mathrm{Proj} M(x)\| \geq a$ it is not difficult to see that

$$\mathbf{E}(\|\xi_{\tau_n+k}\| - \|\xi_{\tau_n+k-1}\| \mid \xi_{\tau_n+k-1} = x, x \in G) \geq a + O(\|x\|^{-1}). \tag{2.20}$$

As $m \simeq \|x\|^{1-\varepsilon_8}$, we get from (2.19) and (2.20) that

$$\begin{aligned}
&\mathbf{E}(\|\xi_{\tau_{n+1}}\| - \|\xi_{\tau_n}\| \mid \xi_{\tau_n} = x) \\
&= \sum_{k=1}^{m} \mathbf{E}(\|\xi_{\tau_n+k}\| - \|\xi_{\tau_n+k-1}\| \mid \xi_{\tau_n} = x) \\
&= \sum_{k=1}^{m} \mathbf{E}(\|\xi_{\tau_n+k}\| - \|\xi_{\tau_n+k-1}\| \mid \xi_{\tau_n} = x, \xi_{\tau_n+k-1} \in G)\mathbf{P}[B_k] \\
&\quad + \sum_{k=1}^{m} \mathbf{E}(\|\xi_{\tau_n+k}\| - \|\xi_{\tau_n+k-1}\| \mid \xi_{\tau_n} = x, \xi_{\tau_n+k-1} \notin G)\mathbf{P}[\overline{B_k}] \\
&\geq \sum_{k=1}^{m} (a + O(\|x\|^{-1}))\mathbf{P}[B_k] + \sum_{k=1}^{m} \Big(\frac{1}{4\|x\|} + O(\|x\|^{-3})\Big)\mathbf{P}[\overline{B_k}]
\end{aligned}$$

$$= \sum_{k=1}^{m} a\mathbf{P}[B_k] + O(\|x\|^{-\varepsilon_8}), \tag{2.21}$$

where $B_k = \{\xi_{\tau_n+k-1} \in G\}$, $k = 1, \ldots, m$. We need then to estimate $\sum_{k=1}^{m} \mathbf{P}[B_k]$ to show that $\sum_{k=1}^{m} a\mathbf{P}[B_k]$ is the main term in (2.21).

Fix some $\delta > 0$ and denote

$$\ell_i(\delta) = \frac{1}{2}\|x\|^{1-\varepsilon_8} + i\|x\|^{(1-\delta)(1-\varepsilon_8)}, \quad i = 0, \ldots, m.$$

We have

$$\begin{aligned} \sum_{k=1}^{\|x\|^{1-\varepsilon_8}} \mathbf{P}[B_k] &\geq \sum_{k=\frac{1}{2}\|x\|^{1-\varepsilon_8}}^{\|x\|^{1-\varepsilon_8}} \mathbf{P}[B_k] \\ &= \sum_{i=1}^{\frac{1}{2}\|x\|^{1-\varepsilon_8}} \sum_{k=\ell_{i-1}(\delta)}^{\ell_i(\delta)} \mathbf{P}[B_k] \\ &\geq \sum_{i=1}^{\frac{1}{2}\|x\|^{1-\varepsilon_8}} \mathbf{P}\Big[\bigcup_{k=\ell_{i-1}(\delta)}^{\ell_i(\delta)} B_k\Big]. \end{aligned} \tag{2.22}$$

By Lemma 2.4, if x is large enough, then for any y such that $\|y-x\| \leq \|x\|^{1-\varepsilon_8}$ there exists $z \in G$ such that $\|y-z\| < \|x\|^{1/2-\varepsilon_1}$. Let us take δ such that $\|y-z\| < \|x\|^{(1-\delta)(1-\varepsilon_8)}$. Then, by Lemma 2.1, as $m = \|x\|^{1-\varepsilon_8}$, we get that (with fixed G) for all x large enough there exist $c_{19}, c_{20} > 0$ such that

$$\begin{aligned} &\mathbf{P}\Big[\bigcup_{k=\ell_{i-1}(\delta)}^{\ell_i(\delta)} B_k\Big] \\ &\geq \mathbf{P}[\xi'_k = z - \xi_{\tau_n+\ell_{i-1}(\delta)} \text{ for at least one } k = 1, \ldots, \|x\|^{(1-\delta)(1-\varepsilon_8)}] \\ &\geq \frac{c_{19}}{\log \|z - \xi_{\tau_n+\ell_{i-1}(\delta)}\|} \\ &\geq \frac{c_{20}}{\log \|x\|}. \end{aligned} \tag{2.23}$$

Thus, by (2.22) and (2.23),

$$\sum_{k=1}^{m} \mathbf{P}[B_k] \geq \frac{c_{20}\|x\|^{\delta(1-\varepsilon_8)}}{\log \|x\|}. \tag{2.24}$$

Now, we turn our attention to the quantity (2.18). Note that for any two random variables X_0 and X_1 it holds that

$$\begin{aligned}
&\mathbf{E}(X_1^2 - X_0^2 \mid X_0 = y_0) \qquad (2.25)\\
&\quad = \mathbf{E}((X_1 - X_0)^2 \mid X_0 = y_0) + 2y_0\mathbf{E}(X_1 - X_0 \mid X_0 = y_0).
\end{aligned}$$

Using (2.25), we get that

$$\begin{aligned}
&\mathbf{E}((\|\xi_{\tau_{n+1}}\| - \|\xi_{\tau_n}\|)^2 \mid \xi_{\tau_n} = x)\\
&= \mathbf{E}(\|\xi_{\tau_{n+1}}\|^2 - \|\xi_{\tau_n}\|^2 \mid \xi_{\tau_n} = x) - 2\|x\|\mathbf{E}(\|\xi_{\tau_{n+1}}\| - \|\xi_{\tau_n}\| \mid \xi_{\tau_n} = x)\\
&= \sum_{k=1}^{m} \mathbf{E}(\|\xi_{\tau_{n+k}}\|^2 - \|\xi_{\tau_{n+k-1}}\|^2 \mid \xi_{\tau_n} = x)\\
&\quad -2\|x\| \sum_{k=1}^{m} \mathbf{E}(\|\xi_{\tau_n+k}\| - \|\xi_{\tau_n+k-1}\| \mid \xi_{\tau_n} = x)\\
&= \sum_{k=1}^{m} \mathbf{E}((\|\xi_{\tau_{n+k}}\| - \|\xi_{\tau_{n+k-1}}\|)(\|\xi_{\tau_{n+k}}\| + \|\xi_{\tau_{n+k-1}}\|) \mid \xi_{\tau_n} = x)\\
&\quad -2\|x\| \sum_{k=1}^{m} \mathbf{E}(\|\xi_{\tau_n+k}\| - \|\xi_{\tau_n+k-1}\| \mid \xi_{\tau_n} = x) \qquad (2.26)\\
&\leq 2(\|x\| + O(\|x\|^{1-\varepsilon_8})) \sum_{k=1}^{m} \mathbf{E}(\|\xi_{\tau_n+k}\| - \|\xi_{\tau_n+k-1}\| \mid \xi_{\tau_n} = x) \qquad (2.27)\\
&\quad -2\|x\| \sum_{k=1}^{m} \mathbf{E}(\|\xi_{\tau_n+k}\| - \|\xi_{\tau_n+k-1}\| \mid \xi_{\tau_n} = x) \qquad (2.28)\\
&\leq c_{21}\|x\|^{1-\varepsilon_8} \sum_{k=1}^{m} \mathbf{E}(\|\xi_{\tau_n+k}\| - \|\xi_{\tau_n+k-1}\| \mid \xi_{\tau_n} = x)\\
&\leq c_{22}\|x\|^{1-\varepsilon_8} \sum_{k=1}^{m} \mathbf{P}[B_k]
\end{aligned}$$

for some $c_{22} > 0$ (here we used the fact that

$$-2\|x\|^{1-\varepsilon_8} \leq \|\xi_{\tau_n+k}\| + \|\xi_{\tau_n+k-1}\| - 2\|x\| \leq 2\|x\|^{1-\varepsilon_8}$$

and the boundedness of jumps).

Now, using the fact that for any $c_{23} > -1$ there exists c_{24} such that

$$(1+\Delta)^{-1} - 1 \leq -\Delta + c_{24}\Delta^2$$

on $\{\Delta > c_{23}\}$, we get

$$
\begin{aligned}
&\mathbf{E}((\|\xi_{\tau_{n+1}}\|+1)^{-1} - (\|\xi_{\tau_n}\|+1)^{-1} \mid \xi_{\tau_n} = x) \\
&\le \|x\|^{-1}\Big(-\frac{1}{\|x\|}\mathbf{E}(\|\xi_{\tau_{n+1}}\| - \|\xi_{\tau_n}\| \mid \xi_{\tau_n} = x) \\
&\quad + \frac{c_{24}}{2\|x\|^2}\mathbf{E}((\|\xi_{\tau_{n+1}}\| - \|\xi_{\tau_n}\|)^2 \mid \xi_{\tau_n} = x)\Big) \\
&\le \|x\|^{-1}\Big(-\frac{a}{\|x\|}\Big(\sum_{k=1}^{m}\mathbf{P}[B_k] + O(\|x\|^{-\varepsilon_8})\Big) \qquad (2.29) \\
&\quad + \frac{c_{24}}{2\|x\|^2}c_{21}\|x\|^{1-\varepsilon_8}\sum_{k=1}^{m}\mathbf{P}[B_k]\Big) < 0
\end{aligned}
$$

for all x large enough. By Theorem 2.1 with $f(x) = (\|x\|+1)^{-1}$ we get that the Markov chain $\hat{\xi}_n = \xi_{\tau_n}$ is transient in dimension 2. As $\tau_{n+1} - \tau_n = [\|\xi_{\tau_n}\|^{1-\varepsilon_8}] + 1$, we have that if $\xi_{\tau_n} \neq 0$, then $\xi_k \neq 0$ for all $\tau_n < k < \tau_{n+1}$, and so the Markov chain ξ_n is transient.

2.3.2 Proof of part (b)

Now we will use Theorem 2.2 with function $f(x) = \|x\|^2$.

As in the proof of (a), define

$$\tau_{n+1} = \tau_n + \left[\|\xi_{\tau_n}\|^{2/d}\right] + 1,$$

$\tau_0 = 1$ and $m := \tau_{n+1} - \tau_n$. Analogously to (2.20), it is not difficult to get that in dimension $d \ge 3$, for all $k = 1, \dots, m$

$$\mathbf{E}(\|\xi_{\tau_n+k}\| - \|\xi_{\tau_n+k-1}\| \mid \xi_{\tau_n+k-1} = x, x \in G) \le -a + O(\|x\|^{-1}). \qquad (2.30)$$

By the condition **(B)** of the definition of the model, there exists $M_0 > 0$ such that for all $k = 1, \dots, m$

$$\mathbf{E}((\|\xi_{\tau_n+k}\| - \|\xi_{\tau_n+k-1}\|)^2 \mid \xi_{\tau_n+k-1} = x, x \in G) \le M_0. \qquad (2.31)$$

From (2.25), (2.30), and (2.31), we have that

$$\mathbf{E}(\|\xi_{\tau_n+k}\|^2 - \|\xi_{\tau_n+k-1}\|^2 \mid \xi_{\tau_n+k-1} = x, x \in G) \le M_0 - 2a\|x\| + O(1). \quad (2.32)$$

So, from (2.32)

$$\begin{aligned}\mathbf{E}(\|\xi_{\tau_n+k}\|^2 - \|\xi_{\tau_n+k-1}\|^2 \mid \xi_{\tau_n+k-1}) \\ \leq \mathbf{P}[\overline{B_k}] + (M_0 - 2a\|\xi_{\tau_n+k-1}\| + O(1))\mathbf{P}[B_k],\end{aligned} \tag{2.33}$$

where, as before, $B_k = \{\xi_{\tau_n+k-1} \in G\}$. By the condition **(B)**, if $\xi_{\tau_n} = x$, then for all $k = 1, \ldots, m$ we have $\|\xi_{\tau_n+k-1}\| \geq \|x\| - \|x\|^{2/d}D$, so there exists a constant $c_{25} > 0$ such that for all x large enough

$$\mathbf{E}(\|\xi_{\tau_n+k}\|^2 - \|\xi_{\tau_n+k-1}\|^2 \mid \xi_{\tau_n+k-1} = x) \leq (M_0 + 1) - c_{25}\|x\|\mathbf{P}[B_k].$$

Using (2.2), analogously to (2.24) we obtain that for some $c_{26} > 0$

$$\sum_{k=1}^{m} \mathbf{P}[B_k] \geq \frac{c_{26}}{\|x\|^{(d-2)(1/d-\varepsilon_1)}}. \tag{2.34}$$

As $m \simeq \|x\|^{2/d}$, we obtain for some $c_{27} > 0$ that

$$\begin{aligned}
&\mathbf{E}(\|\xi_{\tau_{n+1}}\|^2 - \|\xi_{\tau_n}\|^2 \mid \xi_{\tau_n} = x) \\
&= \sum_{k=1}^{m} \mathbf{E}(\|\xi_{\tau_n+k}\|^2 - \|\xi_{\tau_n+k-1}\|^2 \mid \xi_{\tau_n} = x) \\
&\leq (M_0 + 1)\|x\|^{2/d} - c_{25}\|x\| \sum_{k=1}^{m} \mathbf{P}[B_k] \\
&\leq (M_0 + 1)\|x\|^{2/d} - \|x\| \frac{c_{27}}{\|x\|^{(d-2)(1/d-\varepsilon_1)}} \qquad (2.35) \\
&= (M_0 + 1)\|x\|^{2/d} - c_{27}\|x\|^{2/d+(d-2)\varepsilon_1} < 0
\end{aligned}$$

for all x large enough. By Theorem 2.2 with $f(x) = \|x\|^2$, we get that the process ξ_{τ_n} is recurrent. This implies the recurrence of ξ_n, so (b) is proved.

2.3.3 Proof of part (d)

To prove (d), we are going to use Theorem 2.1 with function $f(x) = (\|x\| + 1)^{-\frac{1}{2}}$.

Note that, as the jumps are bounded, there exists a positive number a such that $\|\mathrm{Proj} M(x)\| \leq a$ for all $x \in G$. Take some $\varepsilon_3 < \frac{2\varepsilon}{d}$ and define the sequence of stopping times

$$\tau_{n+1} = \tau_n + [\|\xi_{\tau_n}\|^{1+\varepsilon_3}] + 1,$$

$\tau_0 = 0$. As before, let $m = \tau_{n+1} - \tau_n$ and $B_k = \{\xi_{\tau_n+k-1} \in G\}$.

Fix $0 < \varepsilon_9 < 1/2$ and let us define (for fixed n) the event H by

$$H = \{\|\xi_{\tau_n+i} - \xi_{\tau_n}\| \le m^{\frac{1}{2}+\varepsilon_9} \text{ for all } i = 1, \dots, m\}.$$

Note that by Lemma 2.7 on $\{\xi_{\tau_n} = x\}$ we have

$$\mathbf{P}[H] \ge 1 - e^{-c_5\|x\|^{c_6}}. \tag{2.36}$$

Analogously to (2.30) (note that the inequality now is in the opposite direction), using (2.3) and (2.36), we obtain

$$\begin{aligned}
&\mathbf{E}((\|\xi_{\tau_n+k}\| - \|\xi_{\tau_n+k-1}\|)\mathbf{1}_H \mid \xi_{\tau_n+k-1}) \\
&\quad \ge \Big(\frac{d-1}{2d}\frac{1}{\|\xi_{\tau_n+k-1}\|} + O(\|\xi_{\tau_n+k-1}\|^{-3})\Big)\mathbf{P}(\overline{B_k}) \\
&\qquad - (a + O(\|\xi_{\tau_n+k-1}\|^{-1}))\mathbf{P}(B_k) - \mathbf{P}[\overline{H}] \\
&\quad = \Big(\frac{d-1}{2d}\frac{1}{\|\xi_{\tau_n+k-1}\|} - \Big(a + \frac{d-1}{2d}\frac{1}{\|\xi_{\tau_n+k-1}\|}\Big)\mathbf{P}(B_k) \\
&\qquad + o(\|\xi_{\tau_n+k-1}\|^{-1})\Big) - e^{-c_5\|x\|^{c_6}}.
\end{aligned}$$

By Lemma 2.7, if $\xi_{\tau_n} = x$, we have

$$\begin{aligned}
&\mathbf{E}(\|\xi_{\tau_n+k}\| - \|\xi_{\tau_n+k-1}\| \mid \xi_{\tau_n+k-1}) \\
&\quad = \mathbf{E}((\|\xi_{\tau_n+k}\| - \|\xi_{\tau_n+k-1}\|)\mathbf{1}_H \mid \xi_{\tau_n+k-1}) \\
&\qquad + \mathbf{E}((\|\xi_{\tau_n+k}\| - \|\xi_{\tau_n+k-1}\|)\mathbf{1}_{\overline{H}} \mid \xi_{\tau_n+k-1}) \\
&\quad \ge \Big(\Big(\frac{d-1}{2d} + O\Big(\frac{1}{m^{1/2+\varepsilon_9}}\Big)\Big)\frac{1}{\|x\|} - \Big(a + O\Big(\frac{1}{m^{1/2+\varepsilon_9}}\Big)\Big)\mathbf{P}(B_k)\Big) \\
&\qquad + o(\|x\|^{-1}) - e^{-c_5\|x\|^{c_6}}.
\end{aligned} \tag{2.37}$$

Taking summation from 1 to m in (2.37), for all $\delta > 0$ we obtain

$$\mathbf{E}(\|\xi_{\tau_{n+1}}\| - \|\xi_{\tau_n}\| \mid \xi_{\tau_n} = x) \ge \Big(\frac{d-1}{2d} - \delta\Big)\|x\|^{\varepsilon_3} - (a+\delta)\sum_{k=1}^{m}\mathbf{P}(B_k) \tag{2.38}$$

for all x large enough.

So, Lemma 2.7 and (2.38) imply that

$$\mathbf{E}(\|\xi_{\tau_{n+1}}\| - \|\xi_{\tau_n}\| \mid \xi_{\tau_n} = x) \ge \Big(\frac{d-1}{2d} - \varepsilon_{10}\Big)\|x\|^{\varepsilon_3} \tag{2.39}$$

with arbitrarily small ε_{10} for all x large enough.

Now, we are going to obtain an upper bound for

$$\mathbf{E}((\|\xi_{\tau_{n+1}}\| - \|\xi_{\tau_n}\|)^2 \mid \xi_{\tau_n} = x).$$

Note that, by Lemma 2.7,

$$\begin{aligned}
&\mathbf{E}(\|\xi_{\tau_{n+1}}\|^2 - \|\xi_{\tau_n}\|^2 \mid \xi_{\tau_n} = x) \\
&\quad \leq \sum_{k=1}^{m} D\mathbf{P}[B_k] + \sum_{k=1}^{m} \mathbf{P}[\overline{B_k}] \\
&\quad \leq \|x\|^{1+\varepsilon_3} + O(1)
\end{aligned}$$

and thus, by (2.25),

$$\begin{aligned}
&\mathbf{E}((\|\xi_{\tau_{n+1}}\| - \|\xi_{\tau_n}\|)^2 \mid \xi_{\tau_n} = x) && (2.40)\\
&\quad \leq \|x\|^{1+\varepsilon_3} - 2\|x\|\Big(\frac{d-1}{2d} - \varepsilon_{10}\Big)\|x\|^{\varepsilon_3} + O(1) \\
&\quad = \Big(\frac{1}{d} + 2\varepsilon_{10}\Big)\|x\|^{1+\varepsilon_3} + O(1). && (2.41)
\end{aligned}$$

We are now ready to complete the proof of (d). Note that for any $\varepsilon_{11} > 0$ there exists $\varepsilon_{12} < 1$ such that on $\{\Delta \geq 1 - \varepsilon_{12}\}$ it holds that

$$(1+\Delta)^{-\frac{1}{2}} - 1 \leq -\frac{1}{2}\Delta + \Big(\frac{3}{8} + \varepsilon_{11}\Big)\Delta^2.$$

Define also

$$\tilde{\xi}_n = \begin{cases} \xi_{\tau_n}, & \text{if } \xi_k \neq 0 \text{ for all } k \in [\tau_{n-1}, \tau_n], \\ 0, & \text{otherwise.} \end{cases} \qquad (2.42)$$

Note that

$$\{\xi_k = 0 \text{ for some } k \in [\tau_n, \tau_{n+1}]\} \subset \overline{H}.$$

So, using (2.39) and (2.41), analogously to (2.29), we have (if ε_{10}, ε_{11} are small enough)

$$\begin{aligned}
&\mathbf{E}((\|\tilde{\xi}_{n+1}\| + 1)^{-\frac{1}{2}} - (\|\tilde{\xi}_n\| + 1)^{-\frac{1}{2}} \mid \tilde{\xi}_n = x) \\
&\quad \leq \mathbf{E}((\|\xi_{\tau_{n+1}}\| + 1)^{-\frac{1}{2}} - (\|\xi_{\tau_n}\| + 1)^{-\frac{1}{2}} \mid \xi_{\tau_n} = x) + \mathbf{P}[\overline{H}] \\
&\quad = \mathbf{E}((((\|\xi_{\tau_{n+1}}\| + 1)^{-\frac{1}{2}} - (\|\xi_{\tau_n}\| + 1)^{-\frac{1}{2}})
\end{aligned}$$

$$
\begin{aligned}
&\qquad \times \mathbf{1}\{|\|\xi_{\tau_{n+1}}\| - \|\xi_{\tau_n}\|| < \|x\|^{\frac{1}{2}+\varepsilon_9}\} \mid \xi_{\tau_n} = x) \\
&\quad + \mathbf{E}(((\|\xi_{\tau_{n+1}}\| + 1)^{-\frac{1}{2}} - (\|\xi_{\tau_n}\| + 1)^{-\frac{1}{2}}) \\
&\qquad \times \mathbf{1}\{|\|\xi_{\tau_{n+1}}\| - \|\xi_{\tau_n}\|| \geq \|x\|^{\frac{1}{2}+\varepsilon_9}\} \mid \xi_{\tau_n} = x) + \mathbf{P}[\overline{H}] \\
\leq\ & \|x\|^{-\frac{1}{2}}\Big(- \frac{1}{2\|x\|}\mathbf{E}((\|\xi_{\tau_{n+1}}\| - \|\xi_{\tau_n}\|) \\
&\qquad \times \mathbf{1}\{|\|\xi_{\tau_{n+1}}\| - \|\xi_{\tau_n}\|| < \|x\|^{\frac{1}{2}+\varepsilon_9}\} \mid \xi_{\tau_n} = x) \\
&\quad + \Big(\frac{3}{8} + \varepsilon_{11}\Big)\frac{1}{\|x\|^2}\mathbf{E}((\|\xi_{\tau_{n+1}}\| - \|\xi_{\tau_n}\|)^2 \\
&\qquad \times \mathbf{1}\{|\|\xi_{\tau_{n+1}}\| - \|\xi_{\tau_n}\|| < \|x\|^{\frac{1}{2}+\varepsilon_9}\} \mid \xi_{\tau_n} = x)\Big) \\
&\quad + \mathbf{P}[|\|\xi_{\tau_{n+1}}\| - \|\xi_{\tau_n}\|| \geq \|x\|^{\frac{1}{2}+\varepsilon_9} \mid \xi_{\tau_n} = x] + \mathbf{P}[\overline{H}] \\
\leq\ & \|x\|^{-\frac{1}{2}}\Big(- \Big(\frac{d-1}{2d} - \varepsilon_{10}\Big)\frac{1}{2}\|x\|^{\varepsilon_3 - 1} + \Big(\frac{1}{d} + 2\varepsilon_{10}\Big)\Big(\frac{3}{8} + \varepsilon_{11}\Big)\|x\|^{\varepsilon_3 - 1}\Big) \\
&\quad + 2e^{-c_5\|x\|^{c_6}} \\
=\ & \frac{1}{2}\|x\|^{\frac{1}{2}+\varepsilon_3 - 1}\Big(- \frac{d-1}{2d} + \varepsilon_{10} + \frac{3}{4d} + \frac{3}{2}\varepsilon_{10} + \frac{\varepsilon_{11}}{2d} + \varepsilon_{11}\varepsilon_{10}\Big) \\
&\quad + 2e^{-c_5\|x\|^{c_6}} < 0
\end{aligned}
$$

for all x large enough, as $d \geq 3$. Applying Theorem 2.1 with $f(x) = (\|x\| + 1)^{-\frac{1}{2}}$, we get that $\tilde{\xi}_n$ is transient, and so ξ_n is transient as well, thus completing the proof of (d).

2.3.4 Proof of part (c)

To prove the recurrence, we will use Theorem 2.2 with function $f(x) = \log\log\|x\|$.

As before, we suppose that $\|\mathrm{Proj} M(x)\| \leq a$ for all $x \in G$. Analogously to the proof of (d), we define stopping times in the following way

$$\tau_{n+1} = \tau_n + [\|\xi_{\tau_n}\|^{1+\varepsilon_3}] + 1,$$

$\tau_0 = 0$, $\varepsilon_3 < \varepsilon$. As before, put $B_k = \{\xi_{\tau_n + k - 1} \in G\}$ and abbreviate $m := \tau_{n+1} - \tau_n$. To prove the recurrence of ξ_t, for fixed G we need to bound

$$\mathbf{E}(\|\xi_{\tau_{n+1}}\| - \|\xi_{\tau_n}\| \mid \xi_{\tau_n} = x)$$

from above and

$$\mathbf{E}((\|\xi_{\tau_{n+1}}\| - \|\xi_{\tau_n}\|)^2 \mid \xi_{\tau_n} = x)$$

from below. Analogously to (2.37), using Lemma 2.7, we obtain

$$\begin{aligned}
&\mathbf{E}((\|\xi_{\tau_n+k}\| - \|\xi_{\tau_n+k-1}\|)\mathbf{1}_H \mid \xi_{\tau_n+k-1}) \\
&\quad\le \Big(\frac{1}{4} + O\Big(\frac{1}{m^{1/2+\varepsilon_9}}\Big)\Big)\frac{1}{\|x\|} + \Big(a + O\Big(\frac{1}{m^{1/2+\varepsilon_9}}\Big)\Big)\mathbf{P}(B_k) \\
&\qquad + o(\|x\|^{-1}) - e^{-c_8\|x\|^{c_9}},
\end{aligned} \tag{2.43}$$

where $H = \{\|\xi_{\tau_n+i} - \xi_{\tau_n}\| \le m^{\frac{1}{2}+\varepsilon_9}$ for all $i = 1,\dots,m\}$ and $\varepsilon_9 > \varepsilon_3/2$. Taking summation from 1 to m in the above display, as $m \simeq \|x\|^{1+\varepsilon_3}$, we get

$$\begin{aligned}
&\mathbf{E}(\|\xi_{\tau_{n+1}}\| - \|\xi_{\tau_n}\| \mid \xi_{\tau_n} = x) \\
&\quad= \mathbf{E}((\|\xi_{\tau_{n+1}}\| - \|\xi_{\tau_n}\|)\mathbf{1}_H \mid \xi_{\tau_n} = x) \\
&\qquad + \mathbf{E}((\|\xi_{\tau_{n+1}}\| - \|\xi_{\tau_n}\|)\mathbf{1}_{\overline{H}} \mid \xi_{\tau_n} = x) \\
&\quad\le \Big(\Big(\frac{1}{4} + O\Big(\frac{1}{\|x\|^{(1/2+\varepsilon_9)(1+\varepsilon_3)}}\Big)\Big)\|x\|^{\varepsilon_3} \\
&\qquad + \Big(a + O\Big(\frac{1}{\|x\|^{(1/2+\varepsilon_9)(1+\varepsilon_3)}}\Big)\Big)\sum_{k=1}^{m}\mathbf{P}(B_k)\Big) + o(\|x\|^{\varepsilon_3}) - e^{c_8 m^{c_9}}.
\end{aligned} \tag{2.44}$$

By Lemma 2.7,

$$\sum_{k=1}^{m}\mathbf{P}(B_k) \le c_{10}\log^2\|x\|,$$

so from (2.44) we obtain that

$$\mathbf{E}(\|\xi_{\tau_{n+1}}\| - \|\xi_{\tau_n}\| \mid \xi_{\tau_n} = x) \le \frac{1}{4}\|x\|^{\varepsilon_3} + c_{28}\log^2\|x\| \tag{2.45}$$

for some $c_{28} > 0$ and all x large enough.

By (2.25) and (2.45), we have

$$\begin{aligned}
&\mathbf{E}((\|\xi_{\tau_{n+1}}\| - \|\xi_{\tau_n}\|)^2 \mid \xi_{\tau_n} = x) \\
&\quad= \mathbf{E}(\|\xi_{\tau_{n+1}}\|^2 - \|\xi_{\tau_n}\|^2 \mid \xi_{\tau_n} = x) - 2\|x\|\mathbf{E}(\|\xi_{\tau_{n+1}}\| - \|\xi_{\tau_n}\| \mid \xi_{\tau_n} = x) \\
&\quad= \sum_{k=1}^{m}\mathbf{E}(\|\xi_{\tau_n+k}\|^2 - \|\xi_{\tau_n+k-1}\|^2 \mid \xi_{\tau_n} = x) \\
&\qquad -2\|x\|\mathbf{E}(\|\xi_{\tau_{n+1}}\| - \|\xi_{\tau_n}\| \mid \xi_{\tau_n} = x) \\
&\quad= \sum_{k=1}^{m}\Big(\mathbf{E}((\|\xi_{\tau_n+k}\| - \|\xi_{\tau_n+k-1}\|)^2 \mid \xi_{\tau_n} = x)
\end{aligned}$$

$$
\begin{aligned}
& \quad + 2\|\xi_{\tau_n+k-1}\|\mathbf{E}(\|\xi_{\tau_n+k}\| - \|\xi_{\tau_n+k-1}\| \mid \xi_{\tau_n} = x)\Big) \\
& -2\|x\|\mathbf{E}(\|\xi_{\tau_n+1}\| - \|\xi_{\tau_n}\| \mid \xi_{\tau_n} = x) \\
= \ & \|x\|^{1+\varepsilon_3} - 2\|x\|\frac{\|x\|^{\varepsilon_3}}{4} + O(\|x\|\log^2\|x\|) \\
= \ & \frac{\|x\|^{1+\varepsilon_3}}{2} + O(\|x\|\log^2\|x\|). \qquad (2.46)
\end{aligned}
$$

Here we have used the following two facts:

$$
\begin{aligned}
& \sum_{k=1}^{m}\mathbf{E}((\|\xi_{\tau_n+k}\| - \|\xi_{\tau_n+k-1}\|)^2 \mid \xi_{\tau_n} = x) \\
& \quad = \sum_{k=1}^{m}(\mathbf{E}((\|\xi_{\tau_n+k}\| - \|\xi_{\tau_n+k-1}\|)^2 \mid \xi_{\tau_n} = x, \xi_{\tau_n+k-1} \notin G)\mathbf{P}[\overline{B}_k] \\
& \qquad + \mathbf{E}((\|\xi_{\tau_n+k}\| - \|\xi_{\tau_n+k-1}\|)^2 \mid \xi_{\tau_n} = x, \xi_{\tau_n+k-1} \in G))\mathbf{P}[B_k] \\
& \quad = \sum_{k=1}^{m}\Big(\Big(\frac{1}{2} + O(\|x\|^{-2})\Big)\mathbf{P}[\overline{B_k}] + c_{29}\mathbf{P}[B_k]\Big) \\
& \quad = \frac{1}{2}\|x\|^{1+\varepsilon_3} + O(\log^2\|x\|),
\end{aligned}
$$

and, analogously,

$$
\begin{aligned}
& \sum_{k=1}^{m}\mathbf{E}(\|\xi_{\tau_n+k}\| - \|\xi_{\tau_n+k-1}\| \mid \xi_{\tau_n} = x) \\
& \quad = \sum_{k=1}^{m}(\mathbf{E}(\|\xi_{\tau_n+k}\| - \|\xi_{\tau_n+k-1}\| \mid \xi_{\tau_n} = x, \xi_{\tau_n+k-1} \notin G)\mathbf{P}[\overline{B}_k] \\
& \qquad + \mathbf{E}(\|\xi_{\tau_n+k}\| - \|\xi_{\tau_n+k-1}\| \mid \xi_{\tau_n} = x, \xi_{\tau_n+k-1} \in G))\mathbf{P}[B_k] \\
& \quad = \sum_{k=1}^{m}\Big(\Big(\frac{1}{4\|\xi_{\tau_n+k-1}\|} + O(\|x\|^{-3})\Big)\mathbf{P}[\overline{B_k}] + c_{30}\mathbf{P}[B_k]\Big) \\
& \quad = \frac{1}{4}\|x\|^{\varepsilon_3} + O(\log^2\|x\|).
\end{aligned}
$$

Note that, as for all $x > -1$

$$
\log(1+x) \leq x - \frac{x^2}{2} + \frac{x^3}{3},
$$

we have

$$\log\log(x+\Delta)-\log\log x = \log\frac{\log(x+\Delta)}{\log x}$$

$$\begin{aligned}
&= \log\Big(1+\frac{\log(1+\frac{\Delta}{x})}{\log x}\Big)\\
&\le \log\Big(1+\Big(\frac{\Delta}{x}-\frac{\Delta^2}{2x^2}+\frac{\Delta^3}{3x^3}\Big)\frac{1}{\log x}\Big)\\
&\le \frac{1}{\log x}\Big(\frac{\Delta}{x}-\frac{\Delta^2}{2x^2}+\frac{\Delta^3}{3x^3}\Big)-\frac{1}{2\log^2 x}\Big(\frac{\Delta}{x}-\frac{\Delta^2}{2x^2}+\frac{\Delta^3}{3x^3}\Big)^2\\
&\quad +\frac{1}{3\log^3 x}\Big(\frac{\Delta}{x}-\frac{\Delta^2}{2x^2}+\frac{\Delta^3}{3x^3}\Big)^3 \qquad (2.47)\\
&= \frac{\Delta}{x\log x}-\frac{\Delta^2}{2x^2\log x}-\frac{\Delta^2}{2x^2\log^2 x}+\sum_{\substack{k=1,2,3\\ m=3,\dots,9}}\gamma_{k,m}\frac{\Delta^m}{x^m\log^k x}
\end{aligned}$$

for some constants $\gamma_{k,m}$. Also, from Lemma 2.7 we get for $m\ge 3$

$$\begin{aligned}
&\mathbf{E}((\|\xi_{\tau_{n+1}}\|-\|\xi_{\tau_n}\|)^m \mid \xi_{\tau_n}=x) \qquad (2.48)\\
&= \mathbf{E}((\|\xi_{\tau_{n+1}}\|-\|\xi_{\tau_n}\|)^m\mathbf{1}\{\|\xi_{\tau_{n+1}}-\xi_{\tau_n}\|\le\|x\|^{\frac12+\varepsilon_9}\}\mid\xi_{\tau_n}=x)\\
&\quad +\mathbf{E}((\|\xi_{\tau_{n+1}}\|-\|\xi_{\tau_n}\|)^m\mathbf{1}\{\|\xi_{\tau_{n+1}}-\xi_{\tau_n}\|>\|x\|^{\frac12+\varepsilon_9}\}\mid\xi_{\tau_n}=x)\\
&\le c_{31}\|x\|^{\frac{m}{2}+m\varepsilon_9}
\end{aligned}$$

Now, with the help of (2.45), (2.46), (2.47), and (2.48) we can use Theorem 2.2 with the function $f(x)=\log\log\|x\|$:

$$\begin{aligned}
&\mathbf{E}(\log\log\|\xi_{\tau_{n+1}}\|-\log\log\|\xi_{\tau_n}\|\mid\xi_{\tau_n}=x)\\
&\le \frac{1}{\|x\|\log\|x\|}\Big(\mathbf{E}(\|\xi_{\tau_{n+1}}\|-\|\xi_{\tau_n}\|\mid\xi_{\tau_n}=x)\\
&\qquad -\frac{\mathbf{E}((\|\xi_{\tau_{n+1}}\|-\|\xi_{\tau_n}\|)^2\mid\xi_{\tau_n}=x)}{2\|x\|}\\
&\qquad -\frac{\mathbf{E}((\|\xi_{\tau_{n+1}}\|-\|\xi_{\tau_n}\|)^2\mid\xi_{\tau_n}=x)}{2\|x\|\log\|x\|}+o(1)\Big)\\
&\le \frac{1}{\|x\|\log\|x\|}\Big(\frac14\|x\|^{\varepsilon_3}-\frac{\|x\|^{1+\varepsilon_3}}{4\|x\|}-\frac{\|x\|^{1+\varepsilon_3}}{4\|x\|\log\|x\|}+o(1)\Big)\\
&= \frac{1}{\|x\|\log\|x\|}\Big(-\frac{\|x\|^{\varepsilon_3}}{4\log\|x\|}+o(1)\Big)<0
\end{aligned}$$

for all x large enough. So, by Theorem 2.2, the random walk ξ_{τ_n} is recurrent, thus implying the recurrence of ξ_n. Theorem 1.1 is completely proved. ■

2.4 Proof of Theorem 1.2

Let $\theta = \{\theta_x\}_{x\in\mathbb{Z}}$ be a family of i.i.d. random variables, such that θ_x is uniformly distributed in $[0,1]$ for each $x \in \mathbb{Z}$. Let

$$\zeta_x = \mathbf{1}\{x \in G\} = \mathbf{1}\{\theta_x \le p(x)\}.$$

Take $\beta > 0$ and let $\tilde{p}(x) = \min\{\frac{1}{\beta x}, 1\}$ if $x \neq 0$ and $\tilde{p}(0) = 0$. Define also $\tilde{\zeta}_x = \mathbf{1}\{\theta_x \le \tilde{p}(x)\}$. Consider the function

$$f(\beta, x) = \begin{cases} 1 + \sum_{i=2}^{x} \exp\Big(-\beta \sum_{j=1}^{i-1} \zeta_j\Big), & \text{if } x \ge 2 \\ 1 + \sum_{i=2}^{|x|} \exp\Big(-\beta \sum_{j=1}^{i-1} \zeta_{-j}\Big), & \text{if } x \le -2, \end{cases}$$

$f(\beta, 0) = 0$, $f(\beta, 1) = f(\beta, -1) = 1$. Analogously we define the function $\tilde{f}(\beta, x)$ substituting ζ_j and ζ_{-j} by $\tilde{\zeta}_j$ and $\tilde{\zeta}_{-j}$ correspondingly.

We need the following two lemmas.

Lemma 2.8 *If $f(\beta_0, x) \to +\infty$ as $|x| \to \infty$ a.s., then the random walk ξ_t is recurrent a.s. If there exists a function $M : [0,1]^{\mathbb{Z}} \to \mathbb{R}$ such that for any $x \in \mathbb{Z}$ we have $f(\beta_0, x) < M(\theta)$ a.s., then ξ_t is transient a.s.*

Proof. Let $x \ge 1$. Suppose that $x \notin G$. Then $\zeta_x = 0$ and

$$\begin{aligned} &\mathbf{E}(f(\beta_0, \xi_{t+1}) - f(\beta_0, \xi_t) \mid \xi_t = x) \qquad (2.49)\\ &= \frac{1}{2}(f(\beta_0, x+1) - f(\beta_0, x)) + \frac{1}{2}(f(\beta_0, x-1) - f(\beta_0, x)) \\ &= \frac{1}{2}\Big(\exp\Big(-\beta_0 \sum_{i=1}^{x} \zeta_i\Big) - \exp\Big(-\beta_0 \sum_{i=1}^{x-1} \zeta_i\Big)\Big) = 0. \end{aligned}$$

Note that in dimension 1 we have a monotonicity property: if $p(x) \ge p'(x)$ for all x and the problem with $p'(x)$ is transient, then the problem with $p(x)$ also is transient. So, without loss of generality we can suppose that

$$\limsup_{|x|\to\infty} |x|^{1/2+\varepsilon_{13}} p(x) < \infty \qquad (2.50)$$

for some $\varepsilon_{13} > 0$. Borel-Cantelli lemma and (2.50) imply that there exists a positive function $N = N(\theta)$ such that for any two green points x and y with $|x| > N$ and $|y| > N$ we have $|x - y| > 2D$ a.s. Let $x > N + D$. Suppose that $x \in G$. Then $\zeta_x = 1$ and for any y such that $0 < |x - y| \leq D$ we have $\zeta_y = 0$. Thus,

$$\begin{aligned}
&\mathbf{E}(f(\beta_0, \xi_{t+1}) - f(\beta_0, \xi_t) \mid \xi_t = x) \\
&= \sum_{y \in \mathbb{Z}} P_{xy}(f(\beta_0, y) - f(\beta_0, x)) \\
&= a_+ \exp\Big(- \beta_0 \sum_{i=1}^{x} \zeta_i \Big) + a_- \exp\Big(- \beta_0 \sum_{i=1}^{x-1} \zeta_i \Big) = 0.
\end{aligned} \tag{2.51}$$

In the same way one can prove that (2.49) and (2.51) hold for $x \leq -1$ and $x < -N - D$ correspondingly. Thus, by Theorem 2.2, ξ_t is recurrent.

To prove the transience of ξ_t it is sufficient to prove the transience of the Markov chain with state space $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$ and the following transition probabilities $\hat{P}_{xy} = \hat{P}_{xy}(\theta)$: $\hat{P}_{00} = \hat{P}_{01} = 1/2$, $\hat{P}_{x,x-1} = \hat{P}_{x,x+1} = 1/2$ for all $x = 1, \dots, D$ and $\hat{P}_{xy} = P_{xy}$ for all $x > D$. In order to prove that this new Markov chain is transient we use Theorem 2.1 with the function $\hat{f}$ and the finite set $\hat{A}$ constructed in the following way. Let z be the first green point such that $z > N + 2D$. As $z - D - 1 > N$, we have that $z-D-1, \dots, z-1$ are ordinary points. Let $\hat{f}(x) = M(\theta) - f(\beta_0, x)$ if $x \geq z-D$ and $\hat{f}(x) = M(\theta) - f(\beta_0, z - D - 1)$ if $x < z - D$, $\hat{A} = \{x \in \mathbb{Z}_+ : x < z - D\}$. The proof of Lemma 2.8 is completed. ∎

Lemma 2.9 *If $\beta > \beta_0$, then $\tilde{f}(\beta, x) \to +\infty$ as $|x| \to \infty$. If $\beta < \beta_0$, then there exists a function $\tilde{M} : [0,1]^{\mathbb{Z}} \to \mathbb{R}$ such that for any $x \in \mathbb{Z}$ we have $\tilde{f}(\beta, x) < \tilde{M}(\theta)$ a.s.*

Proof. In order to prove this lemma we use Theorem 2.3.

Let $x > \max\{\beta^{-1}, 1\}$ and $b_x = \log x$. We have

$$\mathbf{E}\tilde{\zeta}_x = \frac{1}{\beta x} \text{ and } \operatorname{Var} \tilde{\zeta}_x = \frac{\beta x - 1}{\beta^2 x^2}.$$

Also,

$$\sum_{x=[\beta^{-1}]+1}^{\infty} \frac{\operatorname{Var} \tilde{\zeta}_x}{b_x^2} = \sum_{x=[\beta^{-1}]+1}^{\infty} \frac{\beta x - 1}{\beta^2 x^2 \log^2 x} < \infty.$$

Theorem 2.3 implies that

$$\frac{\sum_{i=1}^{x}\tilde{\zeta}_i - \sum_{i=1}^{x}(\beta i)^{-1}}{\log x} \to 0 \text{ a.s.}$$

as $x \to +\infty$. Since

$$\lim_{x\to+\infty}\frac{\sum_{i=1}^{x}(\beta i)^{-1}}{\log x} = \beta^{-1},$$

we get that

$$\lim_{x\to+\infty}\frac{\sum_{i=1}^{x}\tilde{\zeta}_i}{\log x} = \beta^{-1} \text{ a.s.}$$

Let $r_k > 0$, $k = 1, 2, \dots$ and $R = \sum_{k=1}^{+\infty} e^{-r_k}$. It is easy to see that

- if
$$\liminf_{k\to+\infty}\frac{r_k}{\log k} > 1,$$
then $R < \infty$;

- if
$$\limsup_{k\to+\infty}\frac{r_k}{\log k} < 1,$$
then $R = \infty$.

Applying this fact, we conclude the proof of Lemma 2.9 for $x > 0, x \to +\infty$. The case $x < 0, x \to -\infty$ is analogous. ■

We now finish the proof of Theorem 1.2.

Suppose that

$$\limsup_{|x|\to\infty}|x|p(x) < \beta_0^{-1}.$$

Then, there exist $\beta > \beta_0$ and $N \in \mathbb{N}$ such that $p(x) \le \tilde{p}(x)$ if $|x| > N$. Therefore, we can couple ζ_x and $\tilde{\zeta}_x$ in such a way that $\zeta_x \le \tilde{\zeta}_x$ for all $|x| > N$. Thus, for every θ if $\lim_{|x|\to\infty}\tilde{f}(\beta, x) = +\infty$, then $\lim_{|x|\to+\infty} f(\beta_0, x) = +\infty$. From Lemmas 2.8 and 2.9 it follows that ξ_t is recurrent.

In the case

$$\liminf_{|x|\to\infty}|x|p(x) > \beta_0^{-1}$$

one can prove in an analogous way that ξ_t is transient. ■

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