On the connection between oriented percolation and contact process

M.V. Menshikov*,[1] S.Yu. Popov†,[2,3] V.V. Sisko‡,[2]

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[1] University of Durham, Department of Mathematical Sciences, South Road, Durham DH1 3LE, UK

E-mail: Mikhail.Menshikov@durham.ac.uk

[2] Department of Statistics, Institute of Mathematics and Statistics, University of São Paulo, mailbox 66.281, CEP 05315–970, São Paulo, SP, Brazil

E-mails: popov@ime.usp.br, valentin@ime.usp.br

[3] Institute for Problems of Information Transmission of Russian Academy of Sciences, B. Karetny 19, 101447, Moscow, Russia

Abstract

We study the contact process in $\mathbb{Z}^d$ and a family of two-parametric oriented percolation models in $\mathbb{Z}^d \times \mathbb{Z}_+$. It is proved that the derivative at the endpoint of the critical curve for percolation exists and its

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absolute value coincides with the critical rate for the corresponding contact process.

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1 Introduction and results

When a percolation model depends on various parameters $\tilde{p}_1, \ldots, \tilde{p}_k$, it is natural to think about the "critical curve" (or, when $k \geq 3$, "critical surface") in the space of parameters, which is defined to be the boundary of the region

$$H = \{\tilde{p}_1, \ldots, \tilde{p}_k \text{ such that there is no percolation}\} \subset [0,1]^k.$$

The continuity of this curve (at least for the case of "good" graph) was proved by Menshikov in [5]. Here "good" graph means that it satisfies some geometrical condition (cf. [5]), which we prefer not to formulate here. We only note that it is satisfied for the most graphs arising in percolation theory. The properties of the critical curve for the so-called *mixed* percolation model were studied by Chayes and Schonmann in [2]. They proved the Lipschitz-continuity of the curve and used that fact to get some relations between the critical probabilities of the respective site and bond models.

The differentiability as well as other analytic properties of this curve (surface) is a much more difficult question. Our approach to this problem can be illustrated by the following example. Consider the oriented percolation model in $\mathbb{Z}_+^2$, where the horizontal oriented bonds are open with probability p, while the vertical oriented bonds are open with probability q. Then, one can observe that if p is close to 0 and q is close to 1, then, after a suitable rescaling in the vertical direction, one gets something which resembles a lot the Harris graphical construction of the one-sided contact process. In this paper this idea is used to prove that for a class of two-parametric oriented percolation models the derivative at the endpoint of the critical curve for percolation exists and its absolute value coincides with the critical rate for the corresponding contact process.

The paper is organized as follows. First, we define the contact process in Section 1.1 and oriented percolation in Section 1.2. The main result is formulated in Section 1.3. Its proof is divided in two parts: First, if the contact process is supercritical, then, using coupling methods, we prove that a certain family of percolation models is in the supercritical phase (Section 2.1). Next, if the contact process is subcritical, then, using exponential decay for the subcritical contact process and weak convergence (cf. [1]), we prove that some other family of percolation models is in the subcritical phase (Section 2.2).

1.1 Contact process

Let us describe the contact process in $\mathbb{Z}^d$ with bounded jumps. The Harris graphical construction is used. Notations: $\mathbb{N} = \{1, 2, \ldots\}, \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, $\mathbb{R}_+ = [0, \infty)$.

Fix some $M \in \mathbb{N}$; let $a_1, \ldots, a_M$ be some vectors from $\mathbb{Z}^d$ ($a_i \neq 0$, $a_i \neq a_j$ for $i \neq j$) and $p_1, \ldots, p_M > 0$ be such that $\sum_{i=1}^M p_i = 1$. From now on those parameters are fixed and will be used to define the contact process as well as percolation models (cf. Section 1.2).

By $\mathrm{CP}[\lambda, \delta]$ denote the contact process defined in the following way. Consider the graph $\mathbb{Z}^d \times \mathbb{R}_+$, in which $\mathbb{Z}^d$ represents the spatial component and $\mathbb{R}_+$ represents time. Along each time line $\{x\} \times \mathbb{R}_+$, $x \in \mathbb{Z}^d$, there is a Poisson process of points (with intensity δ) called deaths. For $i = 1, \ldots, M$ between each pair $\{x_1\} \times \mathbb{R}_+$ and $\{x_2\} \times \mathbb{R}_+$ of time lines with $x_2 - x_1 = a_i$ there is a Poisson process (with intensity λp_i) of crossings oriented in the direction from x_1 to x_2.

Let $\mathfrak{B}_i$ be the set of beginnings of crossings with vector a_i, $\mathfrak{E}_i$ be the set of ends of crossings with vector a_i, and $\mathfrak{D}$ be the set of deaths, $i = 1, \ldots, M$. Of course, $\mathfrak{B}_i, \mathfrak{E}_i, \mathfrak{D} \subset \mathbb{Z}^d \times \mathbb{R}_+$ are random sets. Note that $\mathfrak{E}_i = \mathfrak{B}_i + (a_i, 0)$.

Definition 1.1. We say that there is a path from (x, s) to (y, t) if there is a sequence of times $s = t_0 \leq t_1 \leq \cdots \leq t_n \leq t_{n+1} = t$ and sites $x_0 = x, x_1, \ldots, x_{n-1}, x_n = y$ such that

- for $i = 1, 2, \ldots, n$ there are crossings from x_{i-1} to x_i at time t_i, i.e., $x_{i-1} \in \mathfrak{B}_i \cap (\mathbb{Z}^d \times \{t_i\})$ and $x_i \in \mathfrak{E}_i \cap (\mathbb{Z}^d \times \{t_i\})$.

- if $t_i < t_{i+1}$, then the vertical segments $\{x_i\} \times (t_i, t_{i+1})$ do not contain any deaths, i.e., $(\{x_i\} \times (t_i, t_{i+1})) \cap \mathfrak{D} = \varnothing$, $i = 0, 1, \ldots, n$.

Remark. It is easy to see that almost surely no two points from $\bigcup_{i=1}^M \mathfrak{B}_i \cup \mathfrak{D}$ have equal second coordinates (second coordinate means time).

Let C be the (random) subset of $\mathbb{Z}^d \times \mathbb{R}_+$ defined by

$$C = \{(x,t) : \text{there is a path from } (0,0) \text{ to } (x,t)\}.$$

Point x is called infected at time t if $(x,t) \in C$.

Let $\theta(\lambda, \delta) = \mathbf{P}_{\lambda,\delta}\{C \text{ is unbounded}\}$.

From a time-rescaling argument one gets that $\theta(\lambda,\delta) = \theta(\lambda/\delta, 1)$, where $\delta > 0$. It is clear that $\theta(0,1) = 0$ and that $\theta(\rho, 1)$ is a nondecreasing function of ρ, so one can define the critical value $\rho_c = \sup\{\rho : \theta(\rho,1) = 0\}$. It is known that $0 < \rho_c < \infty$ (cf. [7], Chapter 6).

1.2 Oriented percolation

Let $\langle (x,n),(y,m)\rangle$ be the oriented bond from (x,n) to (y,m), $x, y \in \mathbb{Z}^d$, $n, m \in \mathbb{Z}_+$,

$$\vec{L}_0 = \{\langle (x,n),(x,n+1)\rangle : x \in \mathbb{Z}^d, n \in \mathbb{Z}_+\},$$
$$\vec{L}_i = \{\langle (x,n),(x+a_i,n)\rangle : x \in \mathbb{Z}^d, n \in \mathbb{Z}_+\},$$

and $\vec{L} = \vec{L}_0 \cup (\bigcup_{i=1}^{M} \vec{L}_i)$. Here M, a_i, and $p_i, i \in \{1,\ldots,M\}$ are the same as in Section 1.1.

By OP$[p,q]$ denote the following oriented percolation model on the graph $(\mathbb{Z}^d \times \mathbb{Z}_+, \vec{L})$. Each bond is closed or open independently from other bonds. Each bond from $\vec{L}_0$ is open with probability q and closed with probability $1-q$, while each bond from $\vec{L}_i$ is open with probability pp_i and closed with probability $1-pp_i$. Here $q \in [0,1], p \in [0,\hat{p}]$, where

$$\hat{p} = (\max_{1\le i\le M} p_i)^{-1}.$$

The cluster $\tilde{C}$ is the random subset of $\mathbb{Z}^d \times \mathbb{Z}_+$ defined by

$$\tilde{C} = \{(x, n) : \text{there is a path from } (0,0) \text{ to } (x,n)\}.$$

Let $\tilde{\theta}(p, q) = \mathbf{P}\{\tilde{C} \text{ is unbounded}\}$.

It is clear that for all $q \neq 1$ we have $\tilde{\theta}(0, q) = 0$ and that $\tilde{\theta}(p, q)$ is a nondecreasing function of p when q is fixed, and is a nondecreasing function of q when p is fixed. Therefore one can define the critical curve $p_c(q) = \sup\{p : \tilde{\theta}(p, q) = 0\}$ ($\sup \varnothing$ is supposed to be equal to 0). Clearly, $p_c(1) = 0$, and $p_c(0) = p^* \leq \hat{p}$.

The critical curve is continuous. It is also Lipschitz-continuous. These statements can be proved by the methods of [5] and [6]. Note that analogous methods are used to study the Lipschitz-continuity of the critical curve for the so-called *mixed* percolation model in [2]. But our main goal is to study the properties of $p_c(q)$ in the neighborhood of the point $q = 1$.

1.3 Main result

Our main result is that the critical rate of the contact process coincides with the slope of the tangent line at the endpoint of the critical curve for the percolation.

Theorem 1.1. *There exists the derivative*

$$\frac{dp_c}{dq}(1) = -\rho_c.$$

An interesting consequence of this theorem is the following

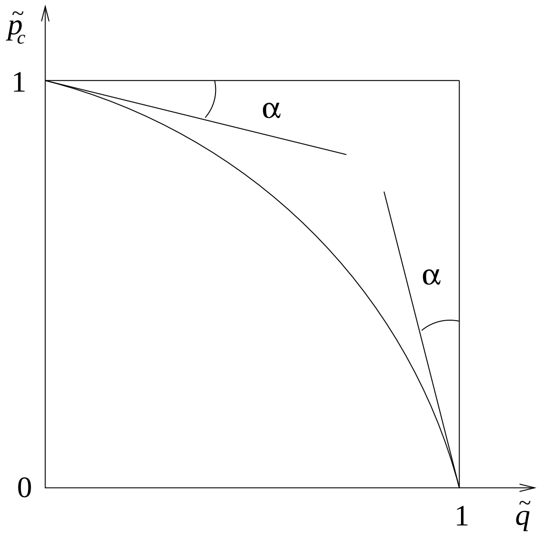

Figure 1: $\cot \alpha = \rho_c$

Corollary.

$$\frac{d\tilde{p}_c}{d\tilde{q}}(1) = -\tilde{\rho}_c, \quad \frac{d\tilde{p}_c}{d\tilde{q}}(0) = -\frac{1}{\tilde{\rho}_c},$$

where $\tilde{\rho}_c$ is the critical rate for the one-sided contact process (cf. Chapter 6 of [3]) and $\tilde{p}_c(\tilde{q})$ is the critical curve for the classic oriented percolation with two parameters in $\mathbb{Z}^2_+$. In this case we use Theorem 1.1 with $d = 1, M = 1, a_1 = 1$, $p_1 = 1$ and the symmetry (see Figure 1).

Note that the simulated value of $\tilde{\rho}_c$ is 3.306 ± 0.002, rigorous estimates are $2.577\ldots \leq \tilde{\rho}_c \leq 3.882\ldots$ (cf. [3], chapter 6, and references therein).

2 Proofs

To prove the theorem we need to prove that

$$\limsup_{h\to 0+} \frac{p_c(1-h)}{h} \le \rho_c, \tag{2.1}$$

$$\liminf_{h\to 0+} \frac{p_c(1-h)}{h} \ge \rho_c. \tag{2.2}$$

In the course of the proof of (2.1) and (2.2) two different approaches are used (cf. Sections 2.1 and 2.2).

2.1 Proof of (2.1)

By $\xi_t, t \in \mathbb{R}_+$, denote the random set of points

$$\{x \in \mathbb{Z}^d: \text{ there is a path from } (0,0) \text{ to } (x,t) \text{ for } \mathrm{CP}[\lambda,\delta]\}.$$

By $\eta_n, n \in \mathbb{Z}_+$, denote the random set of points

$$\{x \in \mathbb{Z}^d: \text{ there is a path from } (0,0) \text{ to } (x,n) \text{ for } \mathrm{OP}[p,q]\}.$$

For the sake of simplicity ξ_t and η_n carry no indices λ, δ, p, q.

Definition 2.1. Let $h > 0$. Say that $\mathrm{CP}[\lambda,\delta]$ is *h-dominated* by $\mathrm{OP}[p,q]$ if there is a coupling such that $\xi_{nh} \subset \eta_n$ for all $n \in \mathbb{Z}_+$. Notation:

$$\mathrm{CP}[\lambda,\delta] \preccurlyeq_h \mathrm{OP}[p,q].$$

The following theorem is vital for the proof of (2.1).

Theorem 2.1. *For any $\lambda, \varepsilon > 0$ there exist $h_0, \sigma_1, \sigma_2 > 0$ such that $\frac{\lambda-\sigma_1}{1+\sigma_2} > \lambda - \varepsilon$ and for all $h \in (0, h_0)$*

$$\mathrm{CP}[\lambda - \sigma_1, 1 + \sigma_2] \preccurlyeq_h \mathrm{OP}[\lambda h, 1 - h].$$

Proof. To prove the theorem, we need some lemmas and definitions.

By $\mathrm{DOP}[p, q]$ denote the locally dependent percolation model defined on the graph $(\mathbb{Z}^d \times \mathbb{Z}_+, \vec{L})$ in the following way:

- each bond from $\bigcup_{i=1}^{M} \vec{L}_i$ is closed or open independently from all other bonds,
- each bond from $\vec{L}_i$ is open with probability pp_i and closed with probability $1 - pp_i$, $i = 1, \ldots, M$,
- let $b \in \vec{L}_0$ and

  $$S_b = \left\{ \tilde{b} \in \bigcup_{i=1}^{M} \vec{L}_i : \text{ the endpoint of } \tilde{b} \text{ coincides with the beginning of } b \right\};$$

  if there exists an open bond $\tilde{b} \in S_b$ then the bond b is open; if all bonds from S_b are closed then b is open with probability q and closed with probability $1 - q$ independently of bonds from $\vec{L} \setminus (S_b \cup \{b\})$.

Note that a definition similar to Definition 2.1 can be given for $\mathrm{DOP}[p, q]$. Only change η_n to η_n^D and $\mathrm{OP}[p, q]$ to $\mathrm{DOP}[p, q]$.

Definition 2.2. Say that $\mathrm{DOP}[p, q]$ is *dominated* by $\mathrm{OP}[\tilde{p}, \tilde{q}]$ if there is a coupling such that $\eta_n^D \subset \eta_n$ for all $n \in \mathbb{N}$. Notation: $\mathrm{DOP}[p, q] \preccurlyeq \mathrm{OP}[\tilde{p}, \tilde{q}]$.

Lemma 2.1. *For all $h, \lambda, \delta > 0$*

$$\mathrm{CP}[\lambda, \delta] \preccurlyeq_h \mathrm{DOP}[1 - \exp(-\lambda h), \exp(-\delta h)].$$

Proof. We construct a coupling of $\mathrm{CP}[\lambda, \delta]$ and $\mathrm{DOP}[1-\exp(-\lambda h), \exp(-\delta h)]$. Let ω be a realization of $\mathrm{CP}[\lambda, \delta]$. Using ω, let us construct the realization $\tilde{\omega}$ of $\mathrm{DOP}[1 - \exp(-\lambda h), \exp(-\delta h)]$ as follows:

- the bond $\langle (x, n), (x + a_i, n) \rangle$ is open if $(x \times [nh, (n+1)h]) \cap \mathfrak{B}_i \neq \varnothing$, and closed otherwise, $i = 1, \ldots, M$;
- the bond $\langle (x, n), (x, n+1) \rangle$ is open if $(x \times (nh, (n+1)h)) \cap \mathfrak{D} = \varnothing$, or $(x \times [nh, (n+1)h]) \cap \mathfrak{E}_i \neq \varnothing$ for some i.

For fixed h let E_h be the event such that

$$\Big(\bigcup_{j=1}^{M} \mathfrak{B}_j \cup \mathfrak{D} \Big) \cap \Big(\mathbb{Z}^d \times \{nh : n \in \mathbb{Z}_+\} \Big) \neq \varnothing.$$

Using the simple fact that the probability of E_h is equal to zero, it is easy to see that the constructed process is really $\mathrm{DOP}[1 - \exp(-\lambda h), \exp(-\delta h)]$.

Now we prove the domination, i.e., $\xi_{kh} \subset \eta_k^D$ for all k. The proof is by induction over k. For $k = 0$ we have $\xi_0 = \{0\} = \eta_0^D$.

Suppose that $\xi_{kh} \subset \eta_k^D$ for some $k \in \mathbb{Z}_+$, let us prove that $\xi_{(k+1)h} \subset \eta_{k+1}^D$.

Let $x \in \xi_{(k+1)h} \cap \xi_{kh}$; then $x \in \eta_k^D$ and either $\mathfrak{D} \cap (\{x\} \times (kh, (k+1)h)) = \varnothing$ or there exists i such that $\mathfrak{E}_i \cap (\{x\} \times [kh, (k+1)h]) \neq \varnothing$, so the bond $\langle (x, k), (x, k+1) \rangle$ is open and $x \in \eta_{k+1}^D$.

Let $x \in \xi_{(k+1)h}$ and $x \notin \xi_{kh}$; then there exists $y \in \xi_{kh}$ such that there is a path from (y, kh) to $(x, (k+1)h)$ for $\mathrm{CP}[\lambda, \delta]$. Therefore there exists a path from (y,k) to (x,k) in the dependent oriented percolation model and there exists i such that $\mathfrak{E}_i \cap (\{x\} \times [kh,(k+1)h)]) \neq \varnothing$, thus $\langle (x,k),(x,k+1)\rangle$ is open and $x \in \eta^D_{k+1}$. $\square$

Lemma 2.2. *For any $\lambda, \varepsilon > 0$ there exist $h_0, \sigma_2, \sigma_1 > 0$, $\sigma_2 > \sigma_1$ such that $\frac{\lambda - \sigma_1}{1+\sigma_2} > \lambda - \varepsilon$ and for all $h < h_0$*

$$\mathrm{DOP}[1-\exp(-(\lambda-\sigma_1)h), \exp(-(1+\sigma_2)h)] \preccurlyeq \mathrm{OP}[\lambda h, 1-h]. \tag{2.3}$$

Proof. For $b \in \vec{L}$ let us introduce a random variable $Z(b)$ in the following way: $Z(b) = 1$ if b is open and $Z(b) = 0$ if b is closed for $\mathrm{OP}[p,q]$. For $\mathrm{DOP}[p,q]$ introduce $Z^D(b)$ analogously. For the sake of simplicity there are no indices p and q in the notations $Z(b), Z^D(b)$.

Let $b_0 \in \vec{L}_0$. Recall that

$$S_{b_0} = \left\{ \tilde{b} \in \bigcup_{i=1}^{M} \vec{L}_i : \text{ the endpoint of } \tilde{b} \text{ coincides with the beginning of } b_0 \right\}.$$

Note that $S_{b_0} = \{b_1, \ldots, b_M\}$, where $b_i \in \vec{L}_i$. Let us remember that the family of random variables $\{Z^D(b_0)\} \cup \{Z^D(b) : b \in S_{b_0}\}$ is independent from the family $\{Z^D(\tilde{b}_0)\} \cup \{Z^D(b) : b \in S_{\tilde{b}_0}\}$ for $b_0 \neq \tilde{b}_0 \in \vec{L}_0$ (cf. the definition of DOP).

Therefore, to prove the domination (2.3), it is sufficient to prove that

$$\begin{aligned} &\mathbf{P}\{Z(b_0) = \alpha_0, Z(b_1) = \alpha_1, \ldots, Z(b_M) = \alpha_M\} \\ &\qquad \geq \mathbf{P}\{Z^D(b_0) = \alpha_0, Z^D(b_1) = \alpha_1, \ldots, Z^D(b_M) = \alpha_M\} \end{aligned} \tag{2.4}$$

for any $(\alpha_0, \alpha_1, \ldots, \alpha_M) \neq (0,0,\ldots,0)$, $\alpha_i \in \{0,1\}$.

Note that

$$\mathbf{P}\{Z^D(b_0) = 0, Z^D(b_1) = \alpha_1, \dots, Z^D(b_M) = \alpha_M\} = 0$$

for any $(\alpha_1, \dots, \alpha_M) \neq (0, \dots, 0)$.

The following calculations (2.5)–(2.8) show that (2.4) holds. Indeed, for h small enough, $\sigma_1 < \lambda$, and $\sigma_2 > \sigma_1$, the right-hand side of (2.5) is greater than the right-hand side of (2.6) for all $A \subset \{1, 2, \dots, M\}$, and the right-hand side of (2.7) is greater than the right-hand side of (2.8).

Let $A \subset \{1, 2, \dots, M\}, A \neq \varnothing, B = \{1, 2, \dots, M\} \setminus A$.

$$\begin{aligned} &\mathbf{P}\{Z(b_0) = 1, Z(b_i) = 1, i \in A, Z(b_j) = 0, j \in B\} \\ &\quad = (1-h)\prod_{i \in A} \lambda h p_i \prod_{j \in B}(1 - \lambda h p_j) = h^{|A|}\lambda^{|A|}\prod_{i \in A} p_i + O(h^{|A|+1}). \end{aligned} \tag{2.5}$$

$$\begin{aligned} &\mathbf{P}\{Z^D(b_0) = 1, Z^D(b_i) = 1, i \in A, Z^D(b_j) = 0, j \in B\} \\ &\quad = \prod_{i \in A}(1 - \exp(-(\lambda - \sigma_1)h))p_i \prod_{j \in B}(1 - (1 - \exp(-(\lambda - \sigma_1)h))p_j) \\ &\quad = h^{|A|}(\lambda - \sigma_1)^{|A|}\prod_{i \in A} p_i + O(h^{|A|+1}). \end{aligned} \tag{2.6}$$

$$\begin{aligned} &\mathbf{P}\{Z(b_0) = 1, Z(b_i) = 0, i = 1, 2, \dots, M\} \\ &\quad = (1-h)\prod_{i=1}^{M}(1 - \lambda h p_i) \\ &\quad = 1 - (1+\lambda)h + O(h^2), \end{aligned} \tag{2.7}$$

recall that $\sum_{i=1}^{M} p_i = 1$.

$$\begin{aligned}
&\mathbf{P}\{Z^D(b_0) = 1, Z^D(b_i) = 0, i = 1, 2, \ldots, M\} \\
&\quad = \exp(-(1+\sigma_2)h) \prod_{i=1}^{M} (1 - (1 - \exp(-(\lambda - \sigma_1)h))p_i) \\
&\quad = 1 - (1 + \lambda + \sigma_2 - \sigma_1)h + O(h^2). \qquad (2.8)
\end{aligned}$$

□

Combining Lemma 2.1 with Lemma 2.2 we complete the proof. □

Using Theorem 2.1, we obtain (2.1) as follows. Let $\lambda > \rho_c$. Then there exists $\varepsilon > 0$ such that $\rho_c < \lambda - \varepsilon$. ¿From Theorem 2.1 it follows that there exist $h_0, \sigma_1, \sigma_2 > 0$ such that

$$\mathrm{CP}[\lambda - \sigma_1, 1 + \sigma_2] \preccurlyeq_h \mathrm{OP}[\lambda h, 1 - h], \qquad (2.9)$$

where $\frac{\lambda - \sigma_1}{1+\sigma_2} > \lambda - \varepsilon$ and $h \in (0, h_0)$. Therefore, $\frac{\lambda-\sigma_1}{1+\sigma_2} > \rho_c$. Hence, $\theta(\lambda - \sigma_1, 1 + \sigma_2) > 0$. Combining this with (2.9), we get $\tilde{\theta}(\lambda h, 1 - h) > 0$. This implies that $p_c(1-h) \leq \lambda h$, for all $h \in (0, h_0)$. As one can choose any $\lambda > \rho_c$, (2.1) follows.

2.2 Proof of (2.2)

The following theorem is vital for the proof of (2.2).

Theorem 2.2. *For any $\lambda < \rho_c$ there exists $h_0 > 0$ such that for all $0 < h < h_0$ the percolation model $OP[\lambda h, 1 - h]$ does not percolate.*

Remark. For the one-sided one-dimensional contact process the coupling method can also be used in the subcritical region of the contact process, but it does not work for the general contact process.

Proof. To prove the theorem we follow this plan:

1. Construct some discrete approximation of $\mathrm{CP}[\lambda, \delta]$. The approximation is referred to as $\mathrm{CP}^h[\lambda, \delta]$, for some $h > 0$.

2. Use exponential decay of cluster size for subcritical process (cf. [1]) to prove that for $\mathrm{CP}[\lambda, \delta]$ the mean number of infected particles at some time t is less than 1 if we begin with one infected particle at time 0 (Lemma 2.3). Prove that $\mathrm{CP}^h[\lambda, \delta]$ has the same property if h is small enough (Lemma 2.6). To prove Lemma 2.6 we use

   - exponential decay of the distribution of the size of the region which contains all the particles which are infected until some fixed time t_0 (Lemma 2.4);
   - the weak convergence of the family of the processes $\mathrm{CP}^h[\lambda, \delta]$ to the process $\mathrm{CP}^0[\lambda, \delta]$ as $h \to 0$ (cf. [1]).

3. Constructing a coupling of $\mathrm{CP}^h[\lambda, \delta]$ and $\mathrm{OP}[\lambda h, 1-h]$, we get that $\mathbf{E}\eta_n < 1$ for some $n \in \mathbb{N}$ (Lemma 2.7); η_n is defined in the beginning of Section 2.1.

4. Use the method of generations (Lemma 2.8) and the coincidence of the critical points (curves) for percolation models (cf. [4]) to get that $\mathrm{OP}[\lambda h, 1-h]$ does not percolate.

Step 1. Let $\mathbf{P}_{\lambda,\delta}$ be the measure corresponding to $\mathrm{CP}[\lambda,\delta]$. For fixed $h>0$ such that $h\delta<1$ and $h\lambda<1$ let $\{X_h(x,k): x\in\mathbb{Z}^d, k\in\mathbb{Z}_+\}$ and $\{Y_h(x,k,i): x\in\mathbb{Z}^d, k\in\mathbb{Z}_+, 1\le i\le M\}$ be independent 0-1 valued random variables with

$$\mathbf{P}\{X_h(x,k)=0\}=h\delta, \quad \mathbf{P}\{Y_h(x,k,i)=1\}=h\lambda p_i.$$

By $\mathrm{CP}^h[\lambda,\delta]$ denote the following model. There is the death point at (x,hk) if $X_h(x,k)=0$ and there is the crossing from x to $x+a_i$ at time $hk+\frac{1}{2}h$ if $Y_h(x,k,i)=1$, $x\in\mathbb{Z}^d, k\in\mathbb{Z}_+, 1\le i\le M$. Write $\mathbf{P}^h_{\lambda,\delta}$ for the measure induced by this construction.

On the space of configurations of $\mathrm{CP}[\lambda,\delta]$ the Skorohod topology can be defined (cf. [1], Section 2). It is known that $\mathbf{P}^h_{\lambda,\delta}$ weakly converges to $\mathbf{P}_{\lambda,\delta}$ as $h\to 0$ (cf. [1]).

Step 2. Let $|\xi_t|$ be the number of infected points for $\mathrm{CP}[\lambda,\delta]$ at time t provided that the process starts with one infected point 0.

Lemma 2.3. *If $\lambda/\delta<\rho_c$, then there exists $t=t(\lambda,\delta)>0$ such that* $\mathbf{E}_{\lambda,\delta}|\xi_t|<1$

Proof. Since $\mathbf{E}_{\lambda,\delta}\int_0^\infty|\xi_t|dt<\infty$ when $\lambda/\delta<\rho_c$ (cf. [1]), we have

$$\int_0^\infty \mathbf{E}_{\lambda,\delta}|\xi_t|dt=\mathbf{E}_{\lambda,\delta}\int_0^\infty|\xi_t|dt<\infty,$$

which easily implies that $\mathbf{E}_{\lambda,\delta}|\xi_t|<1$ for some t. $\square$

Let $\|\xi_t\|=\sup\{\|x\|: x\in\xi_t\}$, where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^d$.

In the sequel, put $\mathrm{CP}^0[\lambda,\delta]:=\mathrm{CP}[\lambda,\delta]$. We define $\xi^h_t, |\xi^h_t|$, and $\|\xi^h_t\|$ for the process $\mathrm{CP}^h[\lambda,\delta]$ in the same way as $\xi_t, |\xi_t|$, and $\|\xi_t\|$ for the process

$\mathrm{CP}[\lambda, \delta]$. Note that ξ_t^0 is only an alternative notation for ξ_t. For the sake of simplicity from now on subscripts λ and δ of $\mathbf{P}$ and $\mathbf{E}$ are omitted.

Lemma 2.4. *Let $\delta = 1$. Then for any t_0, λ there exist $N_0, h_0, \alpha > 0$ such that for all $N > N_0$ and for all $h \in [0, h_0)$*

$$\mathbf{P}\{\|\xi_t^h\| > N \text{ for some } t \leq t_0\} \leq \exp(-\alpha N). \tag{2.10}$$

Remark. The proof of the lemma for $d = 1$ is rather trivial, but for $d > 1$ we need to work more. There are results for Richardson's growth model (cf. [8]), which majorizes the contact process, but we need the estimations uniform in h.

Proof. Let $s \in \mathbb{R}$, $s > h$. We define the model $\mathrm{GR}^{s,h}[\lambda]$ and construct a coupling of $\mathrm{GR}^{s,h}[\lambda]$ with $\mathrm{CP}^h[\lambda, \delta]$ in the following way.

First, we need to introduce a family of random variables $\{U^h(x, k, i)\}$.

For $x \in \mathbb{Z}^d, k \in \mathbb{Z}_+, i \in \{1, \ldots, M\}$, put $U^0(x, k, i) = 1$ if $(\{x\} \times [sk, (k + 1)s)) \cap \mathfrak{B}_i \neq \varnothing$; otherwise $U^0(x, k, i) = 0$.

For $h > 0$, put $U^h(x, k, i) = 1$ if there exists $m \in \mathbb{Z}$ such that $Y_h(x, m, i) = 1$ and $mh + \frac{1}{2}h \in [ks, k(s + 1))$; otherwise $U^h(x, k, i) = 0$.

By $\mathrm{GR}^{s,h}[\lambda]$ denote the following model. There is the crossing from x to $x + a_i$ at time ks if $U^h(x, k, i) = 1$; otherwise there is no crossing there. The process $\mathrm{GR}^{s,h}[\lambda]$ has no deaths.

It is clear that $\theta_t^{s,h} \supset \xi_t^h$ for all $t \in \mathbb{R}_+$, where $\theta_t^{s,h}$ is the random set of points

$$\{x \in \mathbb{Z}^d: \text{ there is a path from } (0, 0) \text{ to } (x, t) \text{ for } \mathrm{GR}^{s,h}[\lambda]\},$$

here we use the notion of path from Definition 1.1.

We have

$$\mathbf{P}\{U^h(x,k,i)=1\} \le 1-(1-h\lambda p_i)^{\lfloor s/h\rfloor+1} \to 1-\exp(-\lambda s p_i)$$

as $h \to 0$ and $\mathbf{P}\{U^0(x,k,i)=1\} = 1-\exp(-\lambda s p_i)$, $i=1,\ldots,M$. Therefore for any $P_0>0$ there exist s and $h_0=h_0(s)$ such that $\mathbf{P}\{U^h(x,k,i)=1\} \le P_0$ for all $h \in [0,h_0)$.

Let $\|\theta_t^{s,h}\| = \sup\{\|x\| : x \in \theta_t^{s,h}\}$ and $a = \max_{i=1,\ldots,M}\|a_i\|$. Note that M is the degree of vertices of the graph $G=(\mathbb{Z}^d, \bigcup_{i=1}^M \vec{L}_i)$. If $\|\theta_t^{s,h}\|>N$ then there is a path formed by at least $\lfloor N/a\rfloor$ crossings ordered in time. Therefore,

$$\mathbf{P}\{\|\theta_t^{s,h}\|>N \text{ for some } t\le t_0\} \le \sum_{n=\lfloor N/a\rfloor}^{\infty} P_0^n (M+1)^{n+\lfloor t_0/s\rfloor+1},$$

where $(M+1)^{n+\lfloor t_0/s\rfloor+1}$ is an upper bound for the number of paths of length n (i.e., n crossings) reaching time t_0. Each path has n crossings in space and less than $\lfloor t_0/s\rfloor+1$ periods of length s in time (the last period can be smaller than s).

One can take P_0 such that $P_0(M+1)<1$ and then choose N_0 and α. $\square$

For $\mathrm{CP}^h[\lambda,\delta], h\ge 0$ we define $\xi_t^h(x)=1$ if there is a path from $(0,0)$ to (x,t) and $\xi_t^h(x)=0$ if there is no such path.

Since $|\xi_t^h| = \sum_{x\in\mathbb{Z}^d}\xi_t^h(x)$, we see that $\mathbf{E}|\xi_t^h| = \sum_{x\in\mathbb{Z}^d}\mathbf{E}\xi_t^h(x)$.

Lemma 2.5. *For any fixed t and x*

$$\mathbf{E}\xi_t^h(x) \to \mathbf{E}\xi_t^0(x) \text{ as } h\to 0. \tag{2.11}$$

Proof. For fixed t let $\Lambda = \Lambda(N) = \{(x,s) : \|x\| \leq N, s \leq t\}$. Let $\xi_t^{h,\Lambda}(x)$ be a 0-1 valued random variable such that $\xi_t^{h,\Lambda}(x) = 1$ if there is a path from $(0,0)$ to (x,t) lying entirely within Λ, and $\xi_t^{h,\Lambda}(x) = 0$ if there is no such path for $\mathrm{CP}^h[\lambda,\delta], h \geq 0$. Clearly,

$$|\mathbf{E}\xi_t^h(x) - \mathbf{E}\xi_t^{h,\Lambda}(x)| \leq \mathbf{P}\{\|\xi_s^h\| > N \text{ for some } s \leq t\}. \tag{2.12}$$

Fix x, t, Λ, we define the event $A = \{\xi_t^{h,\Lambda}(x) = 1\}$. Let ∂A be the boundary of A in the Skorohod topology. It is not difficult to get that $\mathbf{P}_{\lambda,\delta}\{\partial A\} = 0$. Therefore, using the weak convergence, we see that for any fixed x, t, and Λ

$$\mathbf{E}\xi_t^{h,\Lambda}(x) \to \mathbf{E}\xi_t^{0,\Lambda}(x) \text{ as } h \to 0. \tag{2.13}$$

Combining (2.12), (2.13), and (2.10), we complete the proof. $\square$

Lemma 2.6. *For any fixed t*

$$\mathbf{E}|\xi_t^h| \to \mathbf{E}|\xi_t^0| \textit{ as } h \to 0. \tag{2.14}$$

Proof. Using Lemma 2.5, we have that for any fixed N

$$\sum_{\|x\|<N} \xi_t^h(x) \to \sum_{\|x\|<N} \xi_t(x) \tag{2.15}$$

as $h \to 0$.

Clearly,

$$\mathbf{E}\Big(\sum_{\|x\|\geq N} \xi_t^h(x)\Big) \leq \sum_{m=N}^{\infty} Cm^d \mathbf{P}\{\|\xi_t^h\| \geq m\}, \tag{2.16}$$

where C is such that

$$\#\{x \in \mathbb{Z}^d : m \leq \|x\| < m+1\} \leq Cm^d.$$

From (2.16) and Lemma 2.4 it follows that for any $\varepsilon, t > 0$ there exist $N, h_0 > 0$ such that

$$\sum_{\|x\|>N} \mathbf{E}\xi_t^h(x) < \varepsilon \tag{2.17}$$

for all $h \in [0, h_0)$.

Combining (2.15) and (2.17), we complete the proof of Lemma 2.6. $\square$

Step 3. Let $\eta_m(x) = 1$ if there is a path from $(0,0)$ to (x,m) for $\mathrm{OP}[p,q]$; otherwise $\eta_m(x) = 0$. Put $|\eta_n| = \sum_{x\in\mathbb{Z}^d} \eta_n(x)$.

Lemma 2.7. *If for some $t_0, \lambda,$ and δ we have*

$$\mathbf{E}|\xi_{t_0}| < 1, \tag{2.18}$$

then there exists $h_0 > 0$ such that for any $h \in (0, h_0)$ there exists n such that

$$\mathbf{E}|\eta_n| < 1, \tag{2.19}$$

where ξ_t corresponds to $\mathrm{CP}[\lambda, \delta]$ and η_n to $\mathrm{OP}[\lambda h, 1 - \delta h]$.

Proof. From (2.14) and (2.18) it follows that there exist $h_1 > 0$ and $\varepsilon_1 > 0$ such that for all $h \in (0, h_1)$

$$\mathbf{E}|\xi_{t_0}^h| < 1 - \varepsilon_1, \tag{2.20}$$

where ξ_t^h corresponds to $\mathrm{CP}^h[\lambda, \delta]$. Let also h_1 be small enough such that $\lambda h_1 < 1$ and $1 - \delta h_1 > 0$. Now, fix some $h \in (0, h_1)$. We construct a coupling of $\mathrm{CP}^h[\lambda, \delta]$ and $\mathrm{OP}[\lambda h, 1 - \delta h]$ in the following way:

$$Z(\langle (x,n), (x,n+1) \rangle) = X_h(x, n+1),$$

$$Z(\langle (x,n), (x+a_i,n) \rangle) = Y_h(x, n, i),$$

where $i = 1, \ldots, M, x \in \mathbb{Z}^d, n \in \mathbb{Z}_+$, $Z(\cdot)$ is defined in the proof of Lemma 2.2, and $X_h(\cdot)$ and $Y_h(\cdot)$ are from the definition of $\mathrm{CP}^h[\lambda, \delta]$.

Denote $B_h = \{t \in \mathbb{R}_+ : t - \lfloor t/h \rfloor h \geq h/2\}$. Let us prove that for any $t \in B_h$ we have

$$\xi_t^h \supset \eta_{\lfloor t/h \rfloor}. \tag{2.21}$$

To prove the statement we use induction over $k = \lfloor t/h \rfloor$.

Let $k = 0$. If $\eta_0(x) = 1$ then there is a path from $(0,0)$ to $(x,0)$ for $\mathrm{OP}[\lambda h, 1 - \delta h]$. Therefore, there is a path from $(0, h/2)$ to $(x, h/2)$ for $\mathrm{CP}^h[\lambda, \delta]$. So, $\xi_t^h \supset \eta_{\lfloor t/h \rfloor}$ for t such that $\lfloor t/h \rfloor = 0$ and $t \in B_h$.

Suppose that $\xi_t^h \supset \eta_{\lfloor t/h \rfloor}$ for $\lfloor t/h \rfloor = k$ and $t \in B_h$. Let us prove that $\xi_u^h \supset \eta_{\lfloor u/h \rfloor}$ for $\lfloor u/h \rfloor = k + 1$, $u \in B_h$.

Let $x \in \eta_{k+1}$. Then there exists $y \in \eta_k$ such that either there is a path from $(y, k+1)$ to $(x, k+1)$ for $\mathrm{OP}[\lambda h, 1 - \delta h]$ or $x = y$. Therefore $X_h(y, k+1) = 1$, $y \in \xi_t^h$, and there is a path from $(y, (k+1)h + h/2)$ to $(x, (k+1)h + h/2)$ for $\mathrm{CP}^h[\lambda, \delta]$ if $x \neq y$. So, $x \in \xi_u^h$.

This concludes the proof of (2.21).

Using (2.21), for $t \in B_h$ we get

$$\mathbf{E}|\xi_t^h| \geq \mathbf{E}|\eta_{\lfloor t/h \rfloor}|. \tag{2.22}$$

If $t_0 \in B_h$ then (2.19) with $n = \lfloor t_0/h \rfloor$ follows from (2.22) and (2.20).

Suppose that $t_0 \notin B_h$. Then $t_0 - h/2 \in B_h$. Therefore, from (2.22) it follows that

$$\mathbf{E}|\xi_{t_0 - h/2}^h| \geq \mathbf{E}|\eta_{\lfloor (t_0 - h/2)/h \rfloor}|. \tag{2.23}$$

Let us prove that for any $\varepsilon_2 > 0$ there exists $h_2 > 0$ such that for all $h \in (0, h_2)$

$$\left|\mathbf{E}|\xi^h_{t_0}| - \mathbf{E}|\xi^h_{t_0-h/2}|\right| < \varepsilon_2. \tag{2.24}$$

From Lemma 2.4, in the same way as we get (2.17), we obtain that for any $\varepsilon_3 > 0$ there exist $N, h_3 > 0$ such that for all $h \in (0, h_3)$ we have

$$\sum_{\|x\|>N} \mathbf{E}\xi^h_s(x) < \varepsilon_3, \tag{2.25}$$

where $s \in \{t_0, t_0 - h/2\}$.

It is easy to see that for any h

$$|\mathbf{E}\xi^h_{t_0}(x) - \mathbf{E}\xi^h_{t_0-h/2}(x)| \le \mathbf{P}\{X_h(x, \lfloor t_0/h \rfloor) = 0\} \le \delta h. \tag{2.26}$$

Using (2.25) and (2.26), we get (2.24).

¿From (2.20), (2.24) with $\varepsilon_2 = \varepsilon_1$ and (2.23) follows (2.19) with $n = \lfloor (t_0 - h/2)/h \rfloor$. $\square$

Step 4.

Lemma 2.8. *Suppose that there exists $n \in \mathbb{N}$ such that $\mathbf{E}|\eta_n| < 1$. Then $\mathbf{E}|\tilde{C}| < \infty$.*

Proof. The proof of the lemma simply follows from the idea that is analogous to "method of generations" (cf. [9]).

Let $\zeta_0 = \mathbf{E}|\eta_0| = 1, \zeta_n = \mathbf{E}|\eta_n|$. Then $\mathbf{E}|\tilde{C}| = \sum_{i=0}^{\infty} \zeta_i$.

Let $\zeta_n = a < 1$.

Our percolation process can be majorized by the corresponding branching process. It implies that $\zeta_{kn} \le a^k, k \in \mathbb{N}$. Hence, $\sum_{k=1}^{\infty} \zeta_{kn} < \infty$.

It can easily be checked that in our case $\zeta_1 < K < \infty$, where K is some constant. Therefore, $\zeta_{kn+m} \leq \zeta_{kn} K^m$, where $m < n$. Thus, $\sum_{i=1}^{\infty} \zeta_i < \infty$. $\square$

This completes the proof of Theorem 2.2. $\square$

We obtain (2.2) from Theorem 2.2 in the same way as we obtain (2.1) from Theorem 2.1 in Section 2.1.

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