Recurrence and transience of multitype branching random walks [1]

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Abstract

We study a discrete time Markov process with particles being able to perform discrete time random walks and create new particles, known as Branching Random Walk (BRW). We suppose that there are particles of different types, and the transition probabilities, as well as offspring distribution, depend on the type and the position of the particle. Criteria of (strong) recurrence and transience are presented, and some applications (spatially homogeneous case, Lamperti BRW, many-dimensional BRW) are studied.

Key words: multitype branching random walk, strong transience, strong recurrence
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Introduction

The subject of this paper is the so-called branching random walk (BRW). That is a discrete time model of an interacting particle system where particles perform random walks and generate offsprings as in the branching process.

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Once a new particle is born, it is also able to generate offsprings and perform random walk independently of everything else.

In the theory of branching processes some attention has been paid to the case when the particles are of different types (see Chapters II and III of [4]), i.e. when a particle is characterized by some index $a \in \mathcal{A}$, where $\mathcal{A}$ is some finite, countable, or even uncountable set. Following this trend, the main objective of this paper is to study BRW with many types of particles.

We do that basically in three ways. First, we generalize some of the results of F.I. Karpelevich, M.Ya. Kelbert and Yu.M. Suhov [6,7]. By using the Lyapunov function method we are able to extend their results to the multitype case. Second, we improve the results from M.V. Menchikov and S.E. Volkov [10] also to the multitype case. In that situation among other results, we show a criteria for transience based on a matrix inequality for the spatially homogeneous model. Third, we present explicitly the critical value for the average number of offsprings for a couple of important versions of BRWs to be recurrent or transient, which nicely improves a result of M.V. Menchikov and S.E. Volkov [10].

In Section 1 we treat the situation when the number of possible types is finite, say k. At any given time, there are finitely many particles of types $1, 2, \dots, k$, located at sites of a countable space X. Each one of these particles is able to create a new set of particles. These new particles are also of the types from set $\{1, 2, \dots, k\}$ and can jump to other sites of X. Each of those steps (creations and jumps) happen independently of everything else.

In Section 1.1 the dynamics of the process is described, and the concepts of (strong) recurrence and transience are introduced. In Section 1.2 we formulate the criteria of transience and finiteness of the expectation of total number of absorbed particles. Section 1.3 deals with the spatially homogeneous case. We point out that this one-dimensional homogeneous problem with many types of particles is not trivial in comparison with the corresponding model with one type of particle, and can generate interesting effects, as the example in the end of Section 1.3 shows. In Section 1.4 we study BRW with asymptotically zero drift (Lamperti BRW) together with many-dimensional BRW (one type of particle), providing the exact values of critical constants. It is worth noting that studying one-dimensional Lamperti problem helps to get results about many-dimensional models with either zero or asymptotically zero drift (see [1,8,9,11]). Proofs of the main results are placed in Section 1.5.

Section 2 deals with the case of infinite (and, possibly, uncountable) number of types of particles. The proofs of results of this section are analogous to those of Section 1 and therefore omitted. Thus, Section 2 cannot be read independently of Section 1.

1 Finite number of types

1.1 Notations and definitions

Let us describe the evolution of the process. The particles are placed in some countable space X. First we suppose that the configuration (i.e. the distribution of particles in space) at time zero is somehow defined and is nonrandom. We only suppose that this configuration satisfies the following

Condition I. The total number of particles in the initial configuration is finite.

The dynamics is the following: each particle that is in X at time t, first decides independently, whether and how many offsprings of each type $\{1, 2, \ldots, k\}$ it will produce, according to some probabilistic rules. These rules say that a particle of type i located at site x is substituted by R_{ij}^x (a random number) offsprings of type j independently of what other particles do.

The random variables R_{ij}^x, $j = 1, \ldots, k$ have the following joint distribution

$$G_i^x(m_1, \ldots, m_k) := \mathbf{P}\{R_{i1}^x = m_1, \ldots, R_{ik}^x = m_k\},$$

for $i = 1, \ldots, k$ and $x \in X$. The individual distributions are

$$G_{ij}^x(m) = \mathbf{P}\{R_{ij}^x = m\} = \sum_{\substack{m_1,\ldots,m_k: \\ m_j = m}} G_i^x(m_1, \ldots, m_k)$$

having the property $\sum_{m=0}^{\infty} G_{ij}^x(m) = 1$ for all possible i, j, x.

The expected number of offsprings of each type, generated by a particle of type i, when it is located at site x is

$$r_{ij}^x := \sum_{m=1}^{\infty} m G_{ij}^x(m)$$

from now on being assumed to be finite.

After producing offsprings, each existing particle, independently of everything else, decides to which site it will jump, according to the probabilities that follow. Suppose that the particle of type i is in site x, then with probability p_{xy}^i it jumps to y. Clearly, the matrices $\{p_{xy}^i\}_{x,y \in X}$ are Markov transition matrices for $i = 1, \ldots, k$.

Once a particle is created, it is able to create new particles and to perform an independent random walk, also following the probability rules that we have just defined.

To define the concepts of (strong) transience and recurrence, we need one additional construction. Let 0 be some fixed site of X; we turn it into an *absorbing site* in the sense that, once a particle comes to that site, it stays there forever without having any further offsprings. More precisely, for $i = 1, \ldots, k$ we set

$$p^i_{00} = 1, \qquad G^0_{ij}(0) = 1 - \delta_{ij}, \qquad G^0_{ij}(1) = \delta_{ij},$$

where δ_{ij} is the Kronecker delta. We point out that each particle that is placed off the absorbing site is substituted by its set of offsprings, which means that the only way of not seeing any further creations from a given time t is if all particles before that time t became trapped at the absorbing site.

From now on we suppose that the process satisfies the following

Condition A.

i) For any finite set $E \subset X$, $0 \notin E$, and any finite initial configuration ω inside that E, there exists a positive integer $n_0 = n_0(E, \omega)$ such that with positive probability there will be no particles inside $E \cup \{0\}$ after n_0 steps of the process.

ii) For any finite initial configuration ω, there exists a positive integer $n_1 = n_1(\omega)$ such that with positive probability all particles will be absorbed in 0 after n_1 steps.

We suppose also that with probability 1 any particle generates at least one offspring, i.e. the following condition holds

Condition D. $G^x_i(0, \ldots, 0) = 0$ for all x and i.

Let us define the random variables $N^x_i(t)$ as the number of particles of type i at site x at time t. Since 0 is the absorbing state, the sequence $N^0_i(t)$, $t = 0, 1, 2, \ldots$ is nondecreasing, so the limit $\nu_i = \lim_{t\to\infty} N^0_i(t)$ exists for all realizations of the process. Of course, ν_i is a random variable, and it may assume the value $+\infty$.

We are also interested in the random variable $\nu = \sum_{i=1}^k \nu_i$, which represents the total number of particles of all types absorbed by the site 0.

Denote by $\tilde\tau$ the first moment when *all particles* are in the absorbing state 0, i.e. $\tilde\tau$ is the moment when the process completely stops. The next definition follows Definition 2.1 of [2].

Definition 1 *Suppose that Conditions A, D hold. We say that the BRW is*

- *recurrent, if* $\mathbf{P}\{\nu \geq 1\} = 1$;
- *strongly recurrent, if* $\tilde{\tau} < \infty$ *a.s. and* $\mathbf{E}\nu < \infty$;
- *transient, if* $\mathbf{P}\{\nu \geq 1\} < 1$;
- *strongly transient, if it is transient and* $\mathbf{E}\nu < \infty$,

for any initial configuration which satisfies Condition I and such that no particles of it are in $\{0\}$.

Remark 2 *Note that, unlike the case* $k = 1$ *(see [2]), this classification is not complete. It is possible to give an example where the behaviour of the cloud of particles depends on the initial configuration, so the process will be neither recurrent nor transient.*

1.2 Criteria

To formulate our results, we need to consider the moment-generating function for the random vector that represents the number of particles of each type that a particle produces when it is placed at site x

$$\varphi_j^x(z_1, \ldots, z_k) = \sum_{m_1,\ldots,m_k=0}^{\infty} G_j^x(m_1, \ldots, m_k) \prod_{i=1}^{k} z_i^{m_i}. \tag{1.1}$$

Besides that, for $i = 1, \ldots, k$ let $\mathcal{E}_i^x$ be the operator, such that for any function $f(x)$

$$\mathcal{E}_i^x f = \sum_{y \in X} p_{xy}^i f(y), \tag{1.2}$$

$x \in X$. It is important to stress that this linear operator is a convex combination of the function f, evaluated on sites to which the particles can jump to, from the site where it is placed. As a consequence it is bounded whenever the function f is.

First, we give the necessary and sufficient conditions for the process to be transient.

Theorem 3 *The branching random walk is transient if and only if there exist* k *functions* $0 \leq f_1(x), \ldots, f_k(x) \leq 1$ *such that for all* $i = 1, \ldots, k$

$$\varphi_i^x(\mathcal{E}_1^x f_1, \ldots, \mathcal{E}_k^x f_k) \geq f_i(x) \tag{1.3}$$

for all x *except possibly for a finite set* $M \subset X$, $0 \in M$, *and there exists a finite set* $D \subset X$ *such that for all* $i = 1, \ldots, k$

$$f_i(x) > \max_{\substack{x \in M, \\ j=1,\ldots,k}} f_j(x) \tag{1.4}$$

for all $x \in X \setminus D$.

Next, we deal with the strong transience and recurrence.

Theorem 4 *For $\mathbf{E}\nu$ to be finite, it is necessary and sufficient that there exist k positive functions $f_1(x), \ldots, f_k(x)$ such that for all $x \neq 0$*

$$\sum_{j=1}^{k} r_{ij}^x \mathcal{E}_j^x f_j \leq f_i(x) \tag{1.5}$$

for $i = 1, \ldots, k$. Moreover, if there exists a finite set $D \subset X$ such that for any $x \in X \setminus D$

$$f_i(x) < \min_{j=1,\ldots,k} f_j(0) \tag{1.6}$$

for $i = 1, \ldots, k$, then the process is strongly transient. If for any i $f_i(x) \to \infty$ as $x \to \infty$, then the process is strongly recurrent.

Corollary 5 *If (1.5) holds only for $x \geq x_0$ where $x_0 > 1$, and $f_i(x) \to 0$ (or $f_i(x) \to \infty$) for $i = 1, \ldots, k$, then we can say only that the BRW is transient (recurrent).*

1.3 Spatially homogeneous model

In this section we treat the case when the transition probabilities and offspring distributions do not depend on the spatial coordinate x. The space X is equal now to the set of nonnegative integers $\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$, where 0 is the absorbing state. To simplify the matter, we suppose also that the transitions from site x to site y can only occur if $|x - y| \leq 1$.

Let us introduce now the necessary notations. The transition probabilities p_{xy}^i, and the average numbers of offsprings r_{ij}^x do not depend on x as long as $x \neq 0$, so we denote $p_i := p_{x,x-1}^i$, $q_i := p_{x,x+1}^i$, $s_i := p_{xx}^i$, and $r_{ij} := r_{ij}^x$. This means in words that a particle of type i generates in mean r_{ij} offsprings of type j, and jumps to the left (to the right, holds the position) with probability p_i (q_i, s_i). We introduce also four $k \times k$-matrices $R = (r_{ij})$, $P = (p_{ij})$, $Q = (q_{ij})$, and $S = (s_{ij})$, where, being δ_{ij} the Kronecker delta, $p_{ij} = p_i \delta_{ij}$, $q_{ij} = q_i \delta_{ij}$, and $s_{ij} = s_i \delta_{ij}$.

A matrix is called nonnegative (positive), if all its elements have that property. For two matrices A, B we say that $A \geq B$ ($A > B$) if $A - B$ is nonnegative (positive).

Sometimes we will need to impose the following condition on the matrix R:

Condition P. For some n the matrix R^n is positive.

Let us state now the main result of this section.

Theorem 6 *For* $\mathbf{E}\nu$ *to be finite, it is necessary and sufficient that there exists a nonnegative* $k \times k$*-matrix* $\Lambda = (\lambda_{ij})$ *such that*

$$\Lambda \geq RQ\Lambda^2 + RS\Lambda + RP. \tag{1.7}$$

Moreover, the process is

- *strongly transient, if* $\mu < 1$*;*
- *strongly recurrent, if Condition P holds,* $p_i > 0$ *for* $i = 1, \dots, k$ *and* $\mu > 1$*,*

where μ *is the maximal eigenvalue of* Λ*.*

Note that the inequality (1.7) is easy to wright, but is not easy to solve (since there are k^2 unknown variables). But, in the important case when Condition P holds (i.e. every type of particle can have offsprings of any type in some generation), it is possible to provide more verifiable criterion:

Theorem 7 *Suppose that Condition P holds. For* $\mathbf{E}\nu$ *to be finite, it is necessary and sufficient that there exist* $\mu > 0$ *and* $\alpha_1, \dots, \alpha_k > 0$ *such that*

$$\sum_{j=1}^{k} r_{ij}\alpha_j(p_j\mu^{-1} + s_j + q_j\mu) \leq \alpha_i \tag{1.8}$$

for $i = 1, \dots, k$*. Moreover, suppose that* $p_i > 0$ *for* $i = 1, \dots, k$*. Then the process is*

- *strongly transient, if* $\mu < 1$*;*
- *strongly recurrent, if* $\mu > 1$*.*

Remark 8 *In fact, since the vector* $(\alpha_1, \dots, \alpha_k)$ *can be multiplied by arbitrary positive constant, here there are only* k *unknown parameters. Moreover, from the proof it will be seen that for* $\mathbf{E}\nu$ *to be finite, it is necessary that there exist* $\mu > 0$ *such that the maximal eigenvalue of the matrix* $\mu QR^T + SR^T + \mu^{-1}PR^T$ *is less than* 1*, where* T *means transposition (or, equivalently, (1.8) holds with equality).*

In general, the fact that $\mathbf{E}\nu = \infty$ does not imply the recurrence of the process (see [2] for the counterexample). But for the homogeneous model (as well as for the model of [2]) it does imply the recurrence in the case when Condition P holds, i.e. we have the following result:

Theorem 9 *If in the homogeneous model* $\mathbf{E}\nu = \infty$ *and Condition P holds, then* $\mathbf{P}\{\nu \geq 1\} = 1$*, i.e. the process is not transient.*

If Condition P does not hold, we can only prove a weaker result:

Theorem 10 *If the homogeneous model with the mean-offspring matrix R is transient, then for any $\varepsilon > 0$ the model with the mean-offspring matrix $(1-\varepsilon)R$ and the same transition probabilities is strongly transient.*

We end this section with an example.

Example. There are particles of two types. The probability that a particle of type i jumps to the right (to the left) is denoted by q_i (p_i), $i = 1, 2$. Particles of type 1 can generate particles of types 1 and 2, while particles of type 2 can only generate particles of its own type, so the mean offspring matrix R looks like

$$R = \begin{pmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{pmatrix}.$$

Suppose now that $q_i > 1/2$ for $i = 1, 2$, and the following two quadratic equations

$$\lambda = r_{11} q_1 \lambda^2 + r_{11} p_1, \tag{1.9}$$
$$\lambda = r_{22} q_2 \lambda^2 + r_{22} p_2 \tag{1.10}$$

have positive solutions. Results of [2,7] show that, do not taking into account particles of type 2 generated by particles of type 1 (i.e., setting $r_{12} = 0$), the both types are strongly transient. Then, common sense might suggest that putting the interaction parameter r_{12} small enough one can preserve the transience, while choosing r_{12} big enough, the model could become not transient. However, the situation turns out to be completely different.

The proof of Theorem 6 shows that if $\mathbf{E}\nu < \infty$, then there must exist a matrix Λ such that (1.7) holds with equality, and it is obvious that in this case we have to look for the matrix of the form

$$\Lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ 0 & \lambda_{22} \end{pmatrix}.$$

Elementary calculations show that

(1) λ_{11} and λ_{22} are equal to the smallest roots of (1.9) and (1.10), and these roots are positive numbers less than 1;
(2) λ_{12} is given by the following expression:

$$\lambda_{12} = \frac{r_{12}(p_2 + q_2 \lambda_{22}^2)}{1 - q_1 r_{11}(\lambda_{11} + \lambda_{22})}. \tag{1.11}$$

So, (1.11) implies that, as long as $r_{12} > 0$, the sign of λ_{12} (and, consequently, the fact that (1.7) has a positive solution) do not depend on r_{12}. Using Theorem 10, we conclude that

- if $q_1 r_{11}(\lambda_{11} + \lambda_{22}) < 1$, then the process is still strongly transient for any value of r_{12};
- if $q_1 r_{11}(\lambda_{11} + \lambda_{22}) > 1$, then $\mathbf{E}\nu = \infty$ and the process is not transient for any $r_{12} > 0$ (but note that it is not recurrent either, because if we start from one particle of type 2 then no type 1 will ever appear, and so with positive probability nothing will visit 0).

1.4 Lamperti branching random walk

In this section we study BRW with asymptotically zero drift with one and with many types of particles. We also present a result for a many-dimensional BRW. In particular, we improve here the results of [10] by getting the exact values of critical constants.

Let us describe the model. There are k types of particles; the state space X is equal to $\mathbb{Z}_+$ (although it is possible to use the same technique when X is a countable subset of $\mathbb{R}_+$, or even the whole $\mathbb{R}_+$). Let us make the following assumptions on the first and second moments of the jumps of the particle of type i (these conditions correspond to the Lamperti model [8,9]):

$$\mu_1^i(x) := \sum_y y p_{xy}^i - x = \frac{m_i}{x} + o(x^{-1}), \tag{1.12}$$

$$\mu_2^i(x) := \sum_y (y-x)^2 p_{xy}^i = b_i + o(1). \tag{1.13}$$

The mean offspring matrix $R_x = (r_{ij}^x)$ is equal to $I + R/x^2 + o(x^{-2})$, where R is some $k \times k$ nonnegative matrix. When referring to the problem with only one type of particles (i.e. when $k = 1$), we use the notations m instead of m_1, b instead of b_1, and r instead of R.

We suppose that the following condition holds:

Condition L. There exists $K > 0$ such that if $|x - y| > K$, then $p_{xy}^i = 0$ for $i = 1, \dots, k$. Also

$$\alpha_i := m_i - \frac{b_i}{2} > 0,$$

$i = 1, \dots, k$. As before, in the case $k = 1$ we will use α instead of α_1.

If Condition L holds, then the Lamperti random walk (without branching) is transient (see e.g. [1,3,8]). The question is about how much branching we can add (i.e. how large the matrix R can be made) to keep the model transient. The same question can be addressed to the Simple Random Walk in dimension greater or equal than 3. In that case the mean offspring at the site x is equal to $1 + r/\|x\|^2 + o(\|x\|^{-2})$.

The main result of this section is the following

Theorem 11 i) *Let $k = 1$ in the Lamperti BRW. The BRW is*
 · *transient, if $r < \alpha^2/2b$;*
 · *recurrent, if $r > \alpha^2/2b$.*

ii) *Now, let k be an arbitrary positive integer. For the Lamperti BRW to be transient it is sufficient that there exists a matrix A with positive diagonal elements and nonpositive off-diagonal elements such that*

$$A^2 - 2SA + 2BR < 0, \tag{1.14}$$

where $S = (\alpha_i \delta_{ij})$ and $B = (\delta_{ij}/b_i)$. If there is no such A for which (1.14) holds with "$\leq$", then the BRW is recurrent. Moreover, suppose that Condition P holds for the matrix R. In this case, if there exists a positive vector $\bar{c}$ and a positive number a such that

$$2BR\bar{c} < \Phi(a)\bar{c}, \tag{1.15}$$

where $\Phi(a) = (a(\alpha_i - a)\delta_{ij})$, then the BRW is transient. If there is no a, $\bar{c}$ such that (1.15) holds with "$\leq$", then the BRW is recurrent.

iii) *The d-dimensional Simple Random Walk (one type of particle) with branching is*
 · *transient, if $r < (d-2)^2/8d$;*
 · *recurrent, if $r > (d-2)^2/8d$.*

Remark 12 *Assumptions (1.12) and (1.13) may look too strong. We put it that way just to illustrate how our method works. It is possible, for example, to prove transience when the equality is substituted by "$\geq$" in (1.12) and by "$\leq$" in (1.13). We can also consider the original Lamperti's condition $2x\mu_1^i(x) - \mu_2^i(x) > \varepsilon_i > 0$ together with some regularity assumptions on $\mu_1^i(x)$ and $\mu_2^i(x)$.*

1.5 Proofs

Proof of Theorem 3. We use here the ideas of the proof of Theorem 3.1 of [10]. First, we introduce some notations. Considering that at time t there are $n_i(t)$ particles of type i, we denote their coordinates by $x_1^i(t), \ldots, x_{n_i(t)}^i(t)$. Following that notation, the configuration at time t is denoted by

$$\omega(t) = \{x_1^1(t), \ldots, x_{n_1(t)}^1(t), \ldots, x_1^k(t), \ldots, x_{n_k(t)}^k(t)\} \tag{1.16}$$

and the configuration of particles of type i at time t is denoted by

$$\omega_i(t) = \{x_1^i(t), \dots, x_{n_i(t)}^i(t)\}. \tag{1.17}$$

Moreover, we denote by τ the first hitting time of set M (the set from the hypothesis of Theorem 3)

$$\tau = \min\{t : x_j^i(t) \in M \text{ at least for one pair } i, j\}$$

and say that $\tau = \infty$ if the process does not ever visit the set M.

First, we prove that the existence of functions satisfying (1.3) and (1.4) implies transience for the BRW. To this end we define the following stochastic process, whose domain is the configuration of the system regarding types of particles and their positions:

$$\tilde{Q}(t) = \prod_{i=1}^{k} \prod_{j=1}^{n_i(t)} f_i(x_j^i(t)).$$

Let us also define $Q(t) = \tilde{Q}(t \wedge \tau)$, where $a \wedge b := \min\{a, b\}$.

To proceed, we need to prove that the quantity $Q(t)$ is a submartingale, which means that the following inequality holds almost surely:

$$\mathbf{E}[Q(t+1) \mid \omega(t)] \geq Q(t). \tag{1.18}$$

In [10] the inequality (1.18) was verified for the case $k = 1$ (cf. the argument between (3.6) and (3.8) of [10]); the generalization for arbitrary k is straightforward. As the submartingale $Q(t)$ is bounded, we can use the submartingale theorem to guarantee almost sure and L_1 convergence, which means that there exists

$$Q_\infty = \lim_{t\to\infty} Q(t)$$

and

$$\mathbf{E}Q_\infty = \lim_{t\to\infty} \mathbf{E}Q(t) \geq \mathbf{E}Q(0). \tag{1.19}$$

Now, reasoning by absurd, suppose that the process is not transient. This means that if the process starts from some one-point configuration $\omega(0) = \omega_i(0) = \{x_i\}$ for which (1.4) holds (i.e., $x_i \in X \setminus D$), then one of its descendents hits the set M for sure. By its turn this implies that

$$Q_\infty \leq \max_{\substack{x\in M \\ j=1,\dots,k}} f_j(x) < f_i(x_i)$$

which contradicts (1.4) due to (1.19). Thus, if the initial configuration all lies in $X \setminus D$, then the whole progeny will escape the set M with positive probability. Using Condition A, we note that *any* finite initial configuration will leave the set D after some steps with positive probability, finishing this part of the proof.

Let us prove that the transience implies that there exist k functions satisfying (1.3)–(1.4). Take $M = \{0\}$ and let $f_i(x)$ be the probability that the whole progeny of the particle of type i, starting from site x never hit the absorbing site, or

$$f_i(x) = \mathbf{P}\Big\{\text{for all } t \geq 0, \omega(t) \cap \{0\} = \emptyset \mid \omega(0) = \omega_i(0) = \{x\}\Big\}.$$

Now, we show that these functions satisfy (1.3) and (1.4).

Clearly it is true that $0 \leq f_i(x) \leq 1$ for $i = 1, \ldots, k$ and for all $x \in X$, and

$$f_i(0) = 0 \qquad \text{for } i = 1, \ldots, k. \tag{1.20}$$

For the process to be transient, for any i there must be a site x_i such that $f_i(x_i) > 0$, which means that if the process starts with a particle of type i placed on site x_i it might not ever reach the absorbing site. Using Condition A, we get that for any $x \neq 0$

$$f_i(x) > 0 \qquad \text{for } i = 1, \ldots, k. \tag{1.21}$$

The inequality written at (1.4) holds because of (1.20) and (1.21) (we can take $D = M = \{0\}$). Also, analogously to [10] (cf. the last display on page 230 of [10]), one can check that (1.3) is satisfied with equality. $\square$

Proof of Theorem 4. We proceed here in the spirit of the proofs of Theorem 3.2 of [10] and of Theorems 2.1 and 2.2 of [2].

First, let $\mathbf{E}\nu < \infty$. We define the functions $f_i(x)$ to be equal to the mean number of particles of all types absorbed in 0, provided that initially there was only one particle located at site x, and the type of this particle was i, i.e.

$$f_i(x) = \mathbf{E}[\nu \mid \omega(0) = \omega_i(0) = \{x\}]. \tag{1.22}$$

Now, it is straightforward to check that (1.5) holds with equality.

Suppose now that there exist k positive functions $f_1(x), \ldots, f_k(x)$ satisfying (1.5). Let us show that $\mathbf{E}\nu < \infty$. Remembering the notations (1.16)–(1.17), we define the stochastic process $Q(t)$ as follows

$$Q(t) = \sum_{i=1}^{k} \sum_{j=1}^{n_i(t)} f_i(x_j^i(t)).$$

The process $Q(t)$ is a supermartingale (it is straightforward to verify this fact analogously to the proof of the formula (3.11) of [10]), and since it is positive, it converges almost surely to some random variable Q_∞, and, by Fatou Lemma,

$$\mathbf{E}Q_\infty \leq \lim_{t\to\infty} \mathbf{E}Q(t) \leq \mathbf{E}Q(0). \tag{1.23}$$

On the other hand, $Q_\infty \geq \sum_{i=1}^k \nu_i f_i(0) \geq a\nu$, where $a = \min_i f_i(0)$. So

$$\mathbf{E}\nu \leq \frac{\mathbf{E}Q_\infty}{a} \leq \frac{\mathbf{E}Q_0}{a} < \infty, \tag{1.24}$$

thus proving that (1.5) is sufficient for $\mathbf{E}\nu$ to be finite.

Let us prove the transience in the case when (1.6) holds. Choose the initial configuration $\omega(0)$ in the following way: $\omega(0) = \omega_i(0) = \{x\}$, where $x \in X \setminus D$, so $f_i(x) < a$. In this case $Q(0) < a$, so using (1.24), we get $\mathbf{E}\nu < 1$, and consequently $\mathbf{P}\{\nu \geq 1\} < 1$. To prove the transience for any initial configuration, we note that with positive probability it will leave the set D after some steps due to Condition A.

Now, let all the functions $f_i(x)$ tend to ∞. If we suppose that the process is not strongly recurrent, i.e. that with some probability $\kappa_i(y) > 0$ $\tilde{\tau} = \infty$ provided that initially we had only one particle of type i located in y, then with probability at least $\kappa_i(y)$ the process $Q(t) \to \infty$. Indeed, if for a given trajectory of the process $\liminf_{t\to\infty} Q(t) < \infty$, then for an infinite sequence of distinct time moments the total number of particles is uniformly bounded and they all are in some finite neighbourhood of $\{0\}$, which cannot happen due to the second part of Condition A. But if with positive probability $Q(t) \to \infty$, then $\mathbf{E}Q_\infty = \infty$, which contradicts (1.24). $\square$

Proof of Theorem 6. First, we show that $\mathbf{E}\nu < \infty$ implies the existence of matrix Λ satisfying (1.7). Let us denote by λ_{ij}^n the mean number of particles of type j which reach the site 0 if the system started from one particle of type i located at site n for $n \geq 1$. The matrix Λ is defined in the following way: $\Lambda := (\lambda_{ij}^1)_{i,j=1,\dots,k}$, and we denote by $\lambda_{ij}^{(n)}$ the elements of nth power of Λ.

It is important to note that, due to spatial homogeneity of transition probabilities and offspring distributions, $\lambda_{ij}^n = \lambda_{ij}^{(n)}$ for $n \geq 1$. Now, using a standard probabilistic argument, it is straightforward to check that (1.7) holds with equality.

Let us prove now that the existence of nonnegative matrix Λ satisfying (1.7) implies that $\mathbf{E}\nu < \infty$. To this end we are going to apply Theorem 4. Denote by $\mathbf{1}$ the column vector of order k with all the coordinates being equal to 1, and e_i stands for the row vector $(0, \dots, 0, 1, 0, \dots, 0)$ of order k with 1 at

ith place. For $i = 1, \dots, k$ let us define the functions $f_i(x)$, $x \in \mathbb{Z}_+$, in the following way:

$$f_i(x) = e_i \Lambda^x \mathbf{1}, \tag{1.25}$$

which is the sum of the elements of ith row of the matrix Λ^x.

The formula (1.5) adapted to our situation is nothing more than

$$\sum_{j=1}^{k} r_{ij}(p_j f_j(x-1) + s_j f_j(x) + q_j f_j(x+1)) \le f_i(x). \tag{1.26}$$

Using (1.25), we rewrite it in matrix form as

$$R(P\Lambda^{x-1}\mathbf{1} + S\Lambda^x\mathbf{1} + Q\Lambda^{x+1}\mathbf{1}) \le \Lambda^x\mathbf{1},$$

or, equivalently,

$$(RP + RS\Lambda + RQ\Lambda^2)\Lambda^{x-1}\mathbf{1} \le \Lambda\Lambda^{x-1}\mathbf{1},$$

which holds due to (1.7).

Now, we need to assure that all the functions $f_i(x)$ are positive. It is obviously true for $x = 0$ because $f_i(0) = 1$ for any i. Suppose for a moment that there exist i_0, x_0 such that $f_{i_0}(x_0) = 0$. Then (1.26) implies that $f_j(y) = 0$ if $r_{i_0 j} p^j_{x_0,y} > 0$ (recall that $p^j_{x_0,x_0-1} = p_j$, $p^j_{x_0,x_0} = s_j$, $p^j_{x_0,x_0+1} = q_j$). Iterating this construction and using the fact that $f_i(0) = 1$ for any i, we obtain a contradiction with Condition A ii), thus finishing the proof of the fact that (1.7) implies $\mathbf{E}\nu < \infty$.

Let us deal with the strong transience and recurrence. If $\mu < 1$, then $\Lambda^x \to 0$ as $x \to \infty$. Consequently, $f_i(x) \to 0$ for $i = 1, \dots, k$, which proves the transience. Now, let R^{n_0} be positive (i.e. Condition P holds) and $p_i > 0$, $i = 1, \dots, k$. Note that, due to (1.7), $\Lambda \ge RP$, so the matrix Λ has at least the same positive elements as the matrix R. So, Λ^x has at least the same positive elements as R^x. Then in this case Λ^{n_0} is also positive, and the Perron-Frobenius Theorem (see, for example, Chapter II.5 of [4]) gives that if $\mu > 1$, then $\Lambda^x \to \infty$ as $x \to \infty$ (this means that all the elements of Λ^x go to infinity). Consequently, $f_i(x) \to \infty$ for $i = 1, \dots, k$. Applying now Theorem 4, we finish the proof of Theorem 6. $\square$

Proof of Theorem 7. First, if there are positive numbers $\mu, \alpha_1, \dots, \alpha_k$ satisfying (1.8), then it is straightforward to check that we can apply Theorem 4 to the functions $f_i(x) = \alpha_i \mu^x$, thus finishing the proof of the sufficiency.

To prove the necessity, we note that if $\mathbf{E}\nu < \infty$, then the proof of Theorem 6 implies that there exists a non-negative matrix Λ satisfying (1.7) with equality, and Λ^{n_0} is positive. Let μ be the maximal eigenvalue of Λ; by Perron-Frobenius Theorem

$$\Lambda^n = \mu^n \tilde{\Lambda} + o(\mu^n) \tag{1.27}$$

as $n \to \infty$, where $\tilde{\Lambda}$ is some constant matrix of rank 1. Put α_i to be the sum of ith row of the matrix $\tilde{\Lambda}$. Now, using (1.27), we insert the functions $f_i(x) = e_i \Lambda^x \mathbf{1}$ (recall (1.25)) into (1.5), divide both sides by μ^x and let $x \to \infty$ to get (1.8) with equality. $\square$

Proof of Theorem 9. Let us denote by $\nu_{11}(x)$ the random variable which represents the number of particles of type 1 absorbed in 0, given that the initial configuration consists of only one particle of type 1. Since $\mathbf{E}\nu = \infty$, Conditions A and P imply that $\mathbf{E}\nu_{11}(x) = \infty$ for all $x \geq 1$. The rest of the proof is the same as the proof of Theorem 4.3 of [2] with the following simplifications: (we use now the notations of [2]) each one of the sets U_i contain only one point, the sets V_i have some fixed length, the numbers k_i do not depend on i, and θ_i, $i = 1, 2, \ldots$ are not random variables, but simply $\theta_i = \theta < 1$ for any i. $\square$

Proof of Theorem 10. First, note that the moment-generating functions $\varphi_i(z_1, \ldots, z_k)$ have the following property: $\frac{\partial \varphi_i}{\partial z_j}\Big|_{\mathbf{1}} = r_{ij}$, so, using the Taylor expansion, we get that for any $\varepsilon > 0$

$$\varphi_i(\mathbf{1} - \bar{x}) \leq 1 - (1 - \varepsilon) \sum_{j=1}^{k} r_{ij} x_j \tag{1.28}$$

when $\bar{x} \leq \mathbf{1}$ is close enough to $\mathbf{1}$.

As we have seen from the proof of Theorem 3, if the model is transient, then there must exist k functions $f_i(x)$ such that (1.3) holds with equality for all $x \neq 0$. Denote $g_i(x) = 1 - f_i(x)$. From the probabilistic interpretation of these functions, one gets that for $i = 1, \ldots, k$, $g_i(x) \to 0$ as $x \to \infty$. Using this and (1.28), we get that there exists some N_0 such that (1.5) holds with f substituted by g for all $x > N_0$. But, since the problem is spatially homogeneous, the sequence $g_i'(x) := g_i(x + N_0)$ satisfies (1.5) for all $x \geq 1$, thus proving Theorem 10. $\square$

Due to the huge amount of calculations involved, we do not present the proof of Theorem 11 in full detail.

Proof of Theorem 11. First, we deal with items i) and ii). To prove the transience, we apply Corollary 5 to different Lyapunov functions (i.e. we will need to check that those functions satisfy (1.5) for x large enough). After some standard calculations (using Taylor formula) one gets that when $k = 1$ and $r < \alpha^2/2b$, the function $f(x) = x^{-a}$, where $a = \alpha/b$, fits. In case ii), when (1.14) holds, suitable functions are $f_i(x) = e_i x^{-A}\mathbf{1}$ for $i = 1, \dots, k$. When (1.15) holds, we take $f_i(x) = \alpha_i x^{-a}$.

Proving the recurrence in i) and ii) is a much more difficult task. The point is that we only have a criterion for *strong recurrence*, and do not have any criterion for *recurrence* itself (unless in the homogeneous case). So, we will have to use another approach (which is an improvement of the method of [10]). We concentrate on the case $k = 1$, since the generalization to an arbitrary k is not so difficult.

The method can be described as follows. Fix a small positive number δ; denoting by $\lfloor\cdot\rfloor$ the integer part, we define the *scale* $\Theta = \{\lfloor(1+\delta)^n\rfloor, n = 1, 2, \dots\}$. The idea is to observe the process only on Θ; it turns out that after this rescaling the Lamperti BRW looks like homogeneous BRW. There are some difficulties to overcome, for example, if the maximal jump K is greater than 1, then particles can overjump the point of Θ. This can be controlled by "enlarging" the points of Θ, i.e. considering the set $\{\lfloor(1+\delta)^n\rfloor, \dots, \lfloor(1+\delta)^n\rfloor + K - 1\}$ instead of the one point $\lfloor(1+\delta)^n\rfloor$. See [10] for more careful discussion about such scales.

Denote by $p(n)$ the probability that, starting from $\lfloor(1+\delta)^n\rfloor$, a particle hits the previous point $\lfloor(1+\delta)^{n-1}\rfloor$ (or the "enlarged set", in the case when $K > 1$) of the scale before hitting $\lfloor(1+\delta)^{n+1}\rfloor$, and $q(n) = 1 - p(n)$. Using standard martingale technique (see e.g. Lemma 4.3 of [10], or Lemma 1 of [5]), one can prove that $p(n)$ and $q(n)$ converge to some numbers p and q as $n \to \infty$, where

$$p = 1 - q \simeq \frac{1}{2} - \frac{\alpha}{4}\delta + O(\delta^2). \tag{1.29}$$

The next step is to determine how much time the particle will spend on the interval $\left[\lfloor(1+\delta)^{n-1}\rfloor, \lfloor(1+\delta)^{n+1}\rfloor\right]$. Denote by ξ_t the Lamperti random walk without branching. We have

$$\mathbf{E}(\xi_{t+1}^2 - \xi_t^2 \mid \xi_t = x) = 2m + b + o(1).$$

So, as $n \to \infty$, this mean time will be approximately equal to

$$\frac{p(n)(1+\delta)^{2(n-1)} + q(n)(1+\delta)^{2(n+1)} - (1+\delta)^{2n}}{2m + b} + o((1+\delta)^n). \tag{1.30}$$

Multiplying this by $d/(1+\delta)^{2(n-1)}$, one gets the lower bound for the mean number of additional particles generated on this interval over this time period. Using (1.29), elementary calculations show that when δ is small and $n \to \infty$, this mean number of additional particles equals $\frac{d}{b}\delta^2 + o(\delta^2)$. The "worst case" for recurrence is when all these additional particles go to the right end of the interval. So, we compare the Lamperti BRW to the homogeneous BRW which produces mean offspring p to the left, and mean offspring $q + \frac{r}{b}\delta^2$ to the right. For this homogeneous BRW to be recurrent, it is sufficient that there is no real solutions for the equation

$$\left(q + \frac{r}{b}\delta^2\right)\lambda^2 - \lambda + p = 0. \tag{1.31}$$

But it is not difficult to see that if $r > \alpha^2/2b$, then for δ small enough (1.31) has no real solution, thus proving recurrence in i).

Part iii) of the theorem reduces to part i) by considering the distance from the particles to the origin. See more details in [10]. □

2 Infinite number of types

2.1 Notations and general criteria

In this section we study the model when the type of the particle is represented by an index $a \in \mathcal{A}$, where $\mathcal{A}$ is some subset of $\mathbb{R}^N$. Since the proofs of all the main results here are analogous to those of Section 1, we do not present them.

Let us describe the process. The main principles are the same: the time is discrete, the particles are placed in some countable space X, each particle is substituted by its offsprings independently of others, and then these offsprings jump according to some Markov transition probabilities. Whenever possible, we will keep the notations similar to that of Section 1, so the probability that a particle of type a jumps from x to y is denoted by p^a_{xy}. We impose one additional condition on the transition probabilities: for any given x, y, p^a_{xy} should be measurable as a function of a.

To define how the particles generate their offsprings, we need some preliminaries (borrowed mostly from [4]). A *point-distribution* $\omega = (a_1, n_1; \dots; a_m, n_m)$ on $\mathcal{A}$ is a set of distinct points $a_1, \dots, a_m \in \mathcal{A}$ with positive integer weight n_i attached to a_i (which corresponds to the number of particles placed in a_i), $i = 1, \dots, m$, where m may be any nonnegative integer including 0, which corresponds to the null point-distribution. The order of the pairs a_i, n_i is not important.

Now, denote by $\Omega_{\mathcal{A}}$ the set of all such point-distributions, and let $\mathcal{P}^x_a$, $x \in X$, $a \in \mathcal{A}$ be a family of probability measures on $\Omega_{\mathcal{A}}$ (see Chapter III of [4] for all necessary formalities). Then we say that if a particle of type a is in the state x, then it generates its offsprings according to $\mathcal{P}_a^x$.

As before, the initial configuration of the process is still supposed to satisfy the Condition I, $0 \in X$ is the absorbing state, Conditions A and D are imposed with obvious notational modifications, and ν still denotes the total number of particles absorbed in 0.

To formulate the result, we introduce more notations. For a nonnegative measurable function $z(a)$ on $\mathcal{A}$ and $\omega = (a_1, n_1; \ldots; a_m, n_m) \in \Omega_{\mathcal{A}}$ define

$$\omega(z) = \prod_{i=1}^{m} (z(a_i))^{n_i},$$

and, being $\mathbf{E}_a^x$ the expectation w.r.t. $\mathcal{P}_a^x$,

$$\Phi_{x,a}(z) = \mathbf{E}_a^x \omega(z)$$

is the moment-generating functional, which is well-defined at least when $0 \leq z(a) \leq 1$ for all $a \in \mathcal{A}$. Also, if $f(a, x)$ is a function on $\mathcal{A} \times X$, define

$$(\mathcal{E}^x f)(a) = \sum_{y \in X} p_{xy}^a f(a, y). \tag{2.1}$$

Now, for a point-distribution ω and $B \subset A$ define $\omega(B)$ to be equal to the number of particles in B. Denote by

$$\mathcal{R}_x(a, B) = \mathbf{E}_a^x \omega(B)$$

the mean number of offsprings in B generated by a particle of type a placed in site x (clearly, we suppose that it is measurable in a). We suppose that the following condition holds:

Condition K. For all x there exists a positive constant K_x such that

$$\mathcal{R}_x(a, \mathcal{A}) \leq K_x$$

for any a.

Due to Condition K, $\mathcal{R}_x$ can also be viewed as a bounded linear operator on functions $z(a)$, $a \in \mathcal{A}$:

$$(\mathcal{R}_x z)(a) = \int_{\mathcal{A}} z(b) \mathcal{R}_x(a, db).$$

The generalization of Theorem 3 is the following

Theorem 13 *The branching random walk is transient if and only if there exists a function $0 \le f(a,x) \le 1$ such that for all a*

$$\Phi_{x,a}(\mathcal{E}^x f) \ge f(a,x) \tag{2.2}$$

for all x except possibly for a finite set $M \subset X$, and there exists a finite set $D \subset X$ such that for any $a \in \mathcal{A}$ and $x \in X \setminus D$

$$f(a,x) > \max_{\substack{x \in M,\\ b \in \mathcal{A}}} f(b,x). \tag{2.3}$$

The criterion of the finiteness of the expectation of ν is given by the following

Theorem 14 *For $\mathbf{E}\nu$ to be finite, it is necessary and sufficient that there exists a positive function $f(a,x)$ such that $\inf_a f(a,0) > 0$ and*

$$(\mathcal{R}_x \mathcal{E}^x f)(a) \le f(a,x) \tag{2.4}$$

for all a and all $x \neq 0$. Moreover, if there exists a finite set $D \subset X$ such that for all $x \in X \setminus D$

$$f(a,x) < \inf_b f(b,0) \tag{2.5}$$

for any $a \in \mathcal{A}$, then the BRW is strongly transient.

2.2 Homogeneous case

Here, following Section 1.3, we consider one-dimensional model with transition probabilities and offspring generation not depending on x as long as $x \neq 0$, and 0 is the absorbing state. Also, we suppose that a jump from x to y is only possible when $|x-y| \le 1$. So, the mean-offspring operator $\mathcal{R}(a,B)$ do not depend on x, and $p(a), q(a), s(a)$ stand for the probabilities that a particle of type a jumps to the left, to the right, holds its position correspondingly. Denote by P, Q, S the operators of multiplication by functions $p(a), q(a), s(a)$.

We say that $\Lambda(a,B)$, $a \in \mathcal{A}$, $B \subset \mathcal{A}$, is nonnegative, if it is a nonnegative bounded function in a when B is fixed, and is a finite measure in B when a is fixed. Note that, similarly to $\mathcal{R}(a,B)$, $\Lambda(a,B)$ can be viewed as a bounded linear operator on functions $z(a)$, $a \in \mathcal{A}$:

$$(\Lambda z)(a) = \int_{\mathcal{A}} z(b)\Lambda(a,db).$$

For two such operators Λ_1, Λ_2 we say that $\Lambda_1 \le \Lambda_2$, if $\Lambda_1(a,B) \le \Lambda_2(a,B)$ for all a, B. Define

$$\Lambda^2(a,B) = \int_{\mathcal{A}} \Lambda(b,B)\Lambda(a,db).$$

Now we are ready to state the criterion of finiteness of $\mathbf{E}\nu$ in the spatially homogeneous model:

Theorem 15 *For* $\mathbf{E}\nu$ *to be finite, it is necessary and sufficient that there exists* $\Lambda(a, B) \geq 0$ *such that*

$$\mathcal{R}Q\Lambda^2 + \mathcal{R}S\Lambda + \mathcal{R}P \leq \Lambda. \tag{2.6}$$

Note that it is quite natural that the operator inequality (2.6) coincides with the matrix inequality (1.7).

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