An improved upper bound for the critical probability of the frog model on homogeneous trees

Élcio Lebensztayn* Fábio P. Machado*† Serguei Popov*

August 30, 2004

Abstract

We study the frog model on homogeneous trees, a discrete time system of simple symmetric random walks whose description is as follows. There are active and inactive particles living on the vertices. Each active particle performs a simple symmetric random walk having a geometrically distributed random lifetime with parameter $(1-p)$. When an active particle hits an inactive particle, the latter becomes active. We obtain an improved upper bound for the critical parameter for having indefinite survival of active particles, in the case of one particle per vertex initial configuration. The main tool is to construct a class of branching processes which are dominated by the frog model and analyze their supercritical behavior. This approach allows us also to present an upper bound for the critical probability in the case of random initial configuration.

Keywords: critical probability, frog model, homogeneous tree.

AMS subject classification (2000): 60J85, 60K35, 82B43

1 Introduction

One of the main features of the study of statistical-mechanics type processes on graphs is the very interesting interplay between the geometry of the graph and the behaviour of the process. While for long it is known that some results

*Department of Statistics, Institute of Mathematics and Statistics, University of São Paulo, Brazil.

†Corresponding author. email: fmachado@ime.usp.br

obtained in statistical mechanics models do not depend on the particular structure of the graphs, its study on homogeneous trees is worthwhile since they are typically simpler and in many cases capture the behaviour of the systems in high dimension Euclidean lattices. Here we deal with a growth process whose agents – random walks having a geometrically distributed random lifetime with parameter $(1-p)$ – move between vertices of a homogeneous tree along its edges. Seeing it as a particular percolation model – in such a way that the existence of the giant component is analogous to the survival of the process – we face the question of understanding the behaviour of the phase transition as a function of the dimension of the graph. Here phase transition and criticality is with respect to the positivity of the probability of the event that there are moving agents at all times. While it is now known that for this model in general the critical parameter is not a monotonic function of the graph (Fontes *et al.* [4]), we do believe that there is strict monotonicity when the process lives on homogeneous trees. In the process of understanding this problem, we improve the upper bound for the critical probability by constructing a class of branching processes which are dominated – in an appropriate sense – by the model we study and for which, by a detailed analysis of their behavior, we present a non-trivial upper bound for the critical probability.

In the model that we study, particles move as a discrete time independent simple symmetric random walks (SSRWs) on the vertices of a graph $\mathcal{G}$, dying after a geometrically distributed random lifetime. Initially there is one particle at each vertex of $\mathcal{G}$. All particles are inactive at time zero, except for the one that is placed at $\mathbf{0}$, the root of $\mathcal{G}$. At each instant of time, each active particle may die with probability $(1-p)$. Once an active particle survives, it jumps on one of its nearest neighbors vertices, chosen with uniform probability, performing a SSRW on the vertices of $\mathcal{G}$. Up to the time it dies, it activates all inactive particles it hits along its way. From the moment they are activated on, every such particle starts to walk, performing exactly the same dynamics, independently of everything else.

The model described above was first proposed by K. Ravishankar as a model for information spreading. The idea is that every active particle has some information and it shares that information with all inactive particles it hits on its way. Once an inactive particle is told about the information, it also starts to move along edges up to the time it dies. The first published paper dealing with this model (with $p = 1$ and $\mathcal{G} = \mathbb{Z}^d$) is due to Telcs and Wormald [10] where it was referred to as the "egg model". They proved that, starting from the one-particle-per-vertex initial configuration, almost surely the origin will be visited infinitely often. Popov [8] proved that the same is true in dimension $d \geq 3$ for the initial configuration constructed as follows: A sleeping particle (or "egg") is added into each $x \neq 0$ with probability $\alpha/\|x\|^2$, where α is a large positive constant. In Alves et al. [3] for the frog model with no death it was proved that, starting from the one-particle-per-vertex initial configuration, the set of the original positions of all active particles, rescaled by the elapsed time, converges to a nonempty compact convex set. In Alves *et al.* [1] a similar result was obtained for the case of random initial configuration. Fontes *et al.* [4]

showed that for this model the critical parameter is not a monotonic function of the graph. A survey can be found in Popov [9] and recent developments in related models are presented in Kurkova *et al.* [6].

Throughout this paper, $\mathcal{G} = \mathbb{T}_d = (\mathcal{V}, \mathcal{E})$ is the homogeneous tree of degree $(d+1)$. Here $\mathcal{V} = \mathcal{V}(\mathbb{T}_d)$ is the set of vertices of $\mathbb{T}_d$, and $\mathcal{E} = \mathcal{E}(\mathbb{T}_d)$ is the set of edges of $\mathbb{T}_d$. Vertices x and y are said to be *neighbors* if they belong to a common edge; we denote this by $x \sim y$. A *path* of $\mathbb{T}_d$ connecting the vertices x and y is the shortest sequence $x = x_0, x_1, \ldots, x_n = y$ of vertices such that $x_i \sim x_{i+1}$ for all $i = 0, \ldots, n-1$. The value n in the latest sentence is the *distance* between x and y, denoted by $\operatorname{dist}(x, y)$.

To define the process in a formal way, let $\{(S_n^x)_{n\in\mathbb{N}}; x \in \mathcal{V}\}$ and $\{(\tau_p^x); x \in \mathcal{V}\}$ be independent sets of independent identically distributed random objects defined as follows. For each $x \in \mathcal{V}$, $(S_n^x)_{n\in\mathbb{N}}$ is a discrete time SSRW on $\mathbb{T}_d$ starting from x (it describes the trajectory of the particle placed initially at x), and τ_p^x, which stands for the lifetime of that particle, is a random variable whose law is given by $\mathbf{P}(\tau_p^x = k) = (1-p)p^{k-1}$, $k = 1, 2, \ldots$, where $p \in [0, 1]$ is a fixed parameter.

Thus, the particle at vertex x, in the event it is activated, follows the SSRW $(S_n^x)_{n\in\mathbb{N}}$ and dies (disappears) τ_p^x units of time after being activated. At the moment the particle disappears, it is not able to activate other particles (as first we decide whether the particle survives, and only after that the particle that survived is allowed to jump). Observe that there is no interaction between active particles, which means that each active particle moves independently of everything else. We call this model the frog model on $\mathbb{T}_d$ with survival parameter p and denote it by $\mathrm{FM}(\mathbb{T}_d, p)$.

Definition 1.1 A particular realization of the frog model *survives* if for every instant of time there is at least one active particle. Otherwise, we say that it *dies out*.

As $\mathbf{P}(\mathrm{FM}(\mathbb{T}_d, p)$ survives) is nondecreasing in p, we define the *critical probability* by

$$p_c(\mathbb{T}_d) = \inf\{p : \mathbf{P}(\mathrm{FM}(\mathbb{T}_d, p) \text{ survives}) > 0\}.$$

In Alves *et al.* [2], the authors study the frog model with random initial configuration, presenting a necessary and sufficient condition for phase transition on $\mathbb{T}_d$. By the results obtained there, one conclude that, for all $d \geq 2$, $\mathrm{FM}(\mathbb{T}_d, p)$ exhibits *phase transition*, which means that $0 < p_c(\mathbb{T}_d) < 1$. An explicit upper bound for the critical probability was known only in the case of one-particle-per-vertex initial configuration (Fontes *et al.* [4]), namely, $p_c(\mathbb{T}_d) \leq (d+1)(2d-2)^{-1}$. Here we present an upper bound for the case of random initial configuration, which as a by-product improves the result obtained in Fontes *et al.* [4] when the initial configuration is one-particle-per-vertex. We note that in percolation models, upper estimates for the critical probability are generally harder to obtain, especially when there is no duality arguments, which is what happens here.

For the sake of clearness, we choose to consider first the case of one-particle-per-vertex initial configuration. From this point on, the paper is organized as follows. In section 2, we explain why the frog model can be seen as an oriented percolation model and show a formula for the probability of an oriented edge to be open. Next, in section 3, we construct a sequence of upper bounds for the critical probability using a comparison to a Galton-Watson branching process. In section 4 (Theorem 4.1), we improve the known upper bound for the critical probability, proving that

$$p_c(\mathbb{T}_d) \leq \frac{d+1}{2d}.$$

This result is generalized for the case of random initial configuration in section 5 (Theorem 5.1). The final section is devoted to a brief discussion about the monotonicity of the critical probability as a function of the graph.

2 FM($\mathbb{T}_d, p$) seen as a percolation model

Here we underline that FM($\mathbb{T}_d, p$) can be seen as a percolation model. Indeed, let

$$\mathcal{R}_x = \{S_n^x : 0 \leq n < \tau_p^x\} \subset \mathcal{V}$$

be the "virtual" set of vertices visited by the particle placed originally at a vertex x. The set $\mathcal{R}_x$ becomes "real" in the case when x is actually visited (and thus the sleeping particle placed there is activated).

Notice that the frog model survives if and only if there exists an infinite sequence of distinct vertices $\mathbf{0} = x_0, x_1, x_2, \ldots$ such that $x_{j+1} \in \mathcal{R}_{x_j}$ for all $j \geq 0$. Thus, we consider the graph $\vec{\mathbb{T}}_d$ with vertex set $\mathcal{V}$ and an oriented edge from x to y for each pair $(x, y) \in \mathcal{V} \times \mathcal{V}$ with $x \neq y$ and define the following oriented dependent long range percolation model on $\vec{\mathbb{T}}_d$: declare the oriented edge from x to y *open* if $y \in \mathcal{R}_x$ and *closed* otherwise. Hence we conclude that the survival of $\mathrm{FM}(\mathbb{T}_d, p)$ is equivalent to the infiniteness of the cluster of the root in that oriented percolation model. Then we introduce the following notations: for $x, y \in \mathcal{V}$ distinct, $\{x \to y\} = \{y \in \mathcal{R}_x\}$, $\{x \nrightarrow y\} = \{y \notin \mathcal{R}_x\}$.

The next result provides a formula for the probability that the oriented edge from x to y is open. We provide a short proof for this supposedly well known fact, since we do not have found a reference. We note that, for the case $p = 1$, the event $\{x \to y\}$ is equivalent to the event "bankrupt" in the famous Gambler's Ruin Problem, considering that the gambler is playing against an infinitely rich adversary, starts with $\mathrm{dist}(x, y)$ units and at each stage either wins or loses 1 unit with respective probabilities $d/(d+1)$ and $1/(d+1)$. For the case $p < 1$, one should consider that the number of stages is a geometrically distributed random variable with parameter $(1-p)$.

Lemma 2.1 For $x, y \in \mathcal{V}$ with $\mathrm{dist}(x, y) = n$ and $p > 0$,

$$\mathbf{P}(x \to y) = (B(p, d))^n, \tag{2.1}$$

where

$$B(p, d) = \frac{d + 1 - \left[(d+1)^2 - 4\,d\,p^2\right]^{1/2}}{2\,d\,p}. \tag{2.2}$$

Proof: Consider $x, y \in \mathcal{V}$ with $\mathrm{dist}(x, y) = n$ and let T_{xy} be the first time when the SSRW starting from x hits y. Conditioning on the lifetime of the particle placed at x, we have that, for $p < 1$,

$$\mathbf{P}(x \to y) = \mathbf{E}(p^{T_{xy}}).$$

Besides, observe that T_{xy} is a sum of n independent copies of $T_{xx'}$, for $x' \sim x$. Hence, $\mathbf{P}(x \to y) = \left(\mathbf{E}(p^{T_{xx'}})\right)^n$. Therefore, by conditioning on the first step of the random walk, we have

$$\mathbf{E}(p^{T_{xx'}}) = \frac{p}{d+1}\left[1 + d\left(\mathbf{E}(p^{T_{xx'}})\right)^2\right]. \tag{2.3}$$

Since $\lim_{p\downarrow 0} \mathbf{E}(p^{T_{xx'}}) = 0$, we stick to (2.2) as the only possible solution for the previous quadratic equation, which finishes the proof for $p < 1$. Observe that, for $p = 1$, $\mathbf{P}(x \to y) = \mathbf{P}(T_{xy} < \infty) = \lim_{p\uparrow 1} \mathbf{E}(p^{T_{xy}}) = d^{-n}$. $\square$

3 Constructing a sequence of upper bounds for $p_c(\mathbb{T}_d)$

In this section, we obtain a sequence of upper bounds for the critical probability $p_c(\mathbb{T}_d)$. For this, we need a few definitions. First, observe that there is a partial order on the set of vertices of $\mathbb{T}_d$. For $x, y \in \mathcal{V}$, we say that $x \leq y$ if x is one of the vertices of the path between $\mathbf{0}$ and y; $x < y$ if $x \leq y$ and $x \neq y$. For $x \neq \mathbf{0}$, we denote by $\mathbb{T}_d^+(x)$ the set $\{y \in \mathcal{V} : x \leq y\}$. We fix an arbitrary vertex z, neighbor of the root, and define $\mathbb{T}_d^+(\mathbf{0}) = \mathcal{V} \setminus \mathbb{T}_d^+(z)$. For $x \in \mathcal{V}$ and $n \geq 1$, we denote $L_n(x) = \{y \in \mathbb{T}_d^+(x) : \mathrm{dist}(x, y) = n\}$.

Definition 3.1 For $x \in \mathcal{V}$ and $y \in L_n(x)$, consider $x_0 = x < x_1 < \cdots < x_{n-1} < x_n = y$ the path connecting x and y. We denote by $\{x \xrightarrow{c} y\}$ the event that $\{x \to y\}$ or there exists a sequence $1 \leq i_1 < \cdots < i_k \leq n-1$ such that

$$\{x \to x_{i_1}\} \bigcap \Big(\bigcap_{1 \leq j \leq k-1} \{x_{i_j} \to x_{i_{j+1}}\} \Big) \bigcap \{x_{i_k} \to y\}.$$

Besides, we denote its complement by $\{x \overset{c}{\nrightarrow} y\}$.

To obtain upper bounds for $p_c(\mathbb{T}_d)$, we compare the frog model on $\mathbb{T}_d$ to the process, also on $\mathbb{T}_d$, in which the particle originally at $y \in L_n(x)$ is activated from the particle placed at x only when the event $\{x \stackrel{c}{\to} y\}$ occurs (assuming the particle from x active). This process starting at the root can be regarded as a Galton-Watson branching process, which is dominated by the frog model (in the sense that the survival of this process implies the survival of the frog model).

Next, we present an important equivalence for the analysis of the function $B(p, d)$ for fixed d. Note that this equivalence is a restatement of equation (2.3) near the end of the proof of lemma 2.1.

Lemma 3.1 For each fixed $d \geq 1$, $B(p, d)$ is increasing in p. In addition, for $V \in [0, 1/d]$,

$$B(p, d) = V \Longleftrightarrow p = \frac{(d+1)V}{1 + dV^2}.$$

The following result is useful to study the Galton-Watson branching process defined above.

Lemma 3.2 For all $n \geq 1$, there exists a function F_n (not depending on d) with domain $[0, 1/d]$ such that $\mathbf{P}(x_0 \stackrel{c}{\to} x_n) = F_n(B(p, d))$. Moreover, for each $n \geq 1$,

(i) F_n is an increasing function;

(ii) There exists a unique root $\bar{B}_n = \bar{B}_n(d)$ for the equation $F_n(B) = 1/d^n$.

Proof: For $1 \leq j \leq n-1$, consider the events

$$\{x_0 \dashv x_j\} := \{x_0 \to x_j, x_0 \nrightarrow x_{j+1}\},$$
$$\{[x_1, \dots, x_j] \xrightarrow{c} x_n\} := \bigcup_{k=1}^{j} \{x_k \xrightarrow{c} x_n\}.$$

Notice that

$$\mathbf{P}(x_0 \dashv x_j) = (B(p,d))^j - (B(p,d))^{j+1} \tag{3.1}$$

and that the quantity $\mathbf{P}([x_1, \dots, x_j] \xrightarrow{c} x_n)$ can be obtained by the inclusion-exclusion formula using the fact that, for $1 \leq k \leq j$ and $1 \leq i_1 < \cdots < i_k \leq j$,

$$\begin{aligned}\mathbf{P}(x_{i_1} \xrightarrow{c} x_n, \dots, x_{i_k} \xrightarrow{c} x_n) &= \mathbf{P}(x_{i_k} \xrightarrow{c} x_n) \prod_{m=1}^{k-1} \mathbf{P}(x_{i_m} \xrightarrow{c} x_{i_{m+1}}) \\ &= \mathbf{P}(x_0 \xrightarrow{c} x_{n-i_k}) \prod_{m=1}^{k-1} \mathbf{P}(x_0 \xrightarrow{c} x_{i_{m+1}-i_m}).\end{aligned}$$

Then, observing that

$$\mathbf{P}(x_0 \xrightarrow{c} x_n) = \mathbf{P}(x_0 \to x_n) + \sum_{j=1}^{n-1} \mathbf{P}(x_0 \dashv x_j)\, \mathbf{P}([x_1, \dots, x_j] \xrightarrow{c} x_n), \tag{3.2}$$

by induction on n, we conclude that the first assertive holds, with F_n polynomial. Since F_n is nondecreasing in B, $F_n(0) = 0$ and $F_n(1/d) \geq 1/d^n$, the result follows. $\square$

Thereby, we can get a sequence of upper bounds for the critical probability of the frog model on the homogeneous tree of degree $(d+1)$:

Theorem 3.1 For any fixed $d \geq 1$ and $n \geq 1$,

$$p_c(\mathbb{T}_d) \leq \frac{(d+1)\bar{B}_n(d)}{1 + d\left(\bar{B}_n(d)\right)^2}.$$

Proof: Consider the Galton-Watson branching process coupled to the frog model $\mathrm{FM}(\mathbb{T}_d, p)$ in which the particle originally at $y \in L_n(x)$ is said to be offspring of the particle of a vertex x if the event $\{x \xrightarrow{c} y\}$ occurs. This branching process is dominated by the frog model and it has mean number of offspring per individual equal to $d^n F_n(B(p,d))$. By lemmas 3.1 and 3.2 (i)-(ii),

$$p > \frac{(d+1)\bar{B}_n(d)}{1 + d\left(\bar{B}_n(d)\right)^2} \Rightarrow B(p,d) > \bar{B}_n(d) \Rightarrow F_n(B(p,d)) > \frac{1}{d^n}.$$

In this case the branching process survives with positive probability. The same happens to the frog model. $\square$

4 Improving the known upper bound for $p_c(\mathbb{T}_d)$

In section 3 we solve in p the equations

$$d^n F_n(B(p,d)) = 1, n \geq 1$$

and this provides a sequence of upper bounds for $p_c(\mathbb{T}_d)$. While numerical computations support the conjecture that, for $d \geq 2$, the upper bounds obtained in theorem 3.1 are strictly decreasing in n, a rigorous proof for that is beyond ordinary calculations. In this section, as an alternative approach, we work on functions G_n such that

$$F_n(B) \geq G_n(B) \text{ for all } B. \tag{4.1}$$

This approach, developed next, leads to the following theorem, which presents an upper bound for $p_c(\mathbb{T}_d)$ that improves the known one.

Theorem 4.1 For all fixed $d \geq 1$,

$$p_c(\mathbb{T}_d) \leq \frac{d+1}{2d}.$$

Having in mind the equations (3.1) and (3.2) of lemma 3.2, we write, for $n \geq 2$,

$$F_n(B) = B^n + \sum_{j=1}^{n-1} [B^j - B^{j+1}] \, \mathbf{P}([x_1, \ldots, x_j] \xrightarrow{c} x_n),$$

where the probabilities are seen as functions of B. Pulling out the powers of B, we get

$$\begin{aligned} F_n(B) \; &= B^n \, \mathbf{P}(x_1 \overset{c}{\nrightarrow} x_n, \ldots, x_{n-1} \overset{c}{\nrightarrow} x_n) + \\ &\sum_{j=1}^{n-2} B^{n-j} \, \mathbf{P}(x_1 \overset{c}{\nrightarrow} x_n, \ldots, x_{n-j-1} \overset{c}{\nrightarrow} x_n, x_{n-j} \xrightarrow{c} x_n) + B \, \mathbf{P}(x_1 \xrightarrow{c} x_n). \end{aligned}$$

But

$$\mathbf{P}(x_1 \overset{c}{\nrightarrow} x_n, \ldots, x_{n-1} \overset{c}{\nrightarrow} x_n) = \mathbf{P}(x_1 \nrightarrow x_n, \ldots, x_{n-1} \nrightarrow x_n) = \prod_{k=1}^{n-1} (1 - B^k),$$

and, for $1 \leq j \leq n-2$,

$$\begin{aligned} &\mathbf{P}(x_1 \overset{c}{\nrightarrow} x_n, \ \ldots \ , x_{n-j-1} \overset{c}{\nrightarrow} x_n, x_{n-j} \xrightarrow{c} x_n) \\ &\qquad = \ \mathbf{P}(x_1 \nrightarrow x_{n-j}, \ldots, x_{n-j-1} \nrightarrow x_{n-j}, x_{n-j} \xrightarrow{c} x_n) \\ &\qquad = \ F_j(B) \prod_{k=1}^{n-j-1} (1 - B^k). \end{aligned}$$

Then, we conclude that

$$F_n(B) = B^n \prod_{k=1}^{n-1} (1 - B^k) + \sum_{j=1}^{n-1} \Big[B^{n-j} \, F_j(B) \prod_{k=1}^{n-j-1} (1 - B^k) \Big]. \tag{4.2}$$

Consider the sequence of functions $\{G_n\}_{n\geq 1}$ (also with domain $[0,1/d]$) inductively given by

$$\begin{aligned} G_1(B) &= B, \\ G_n(B) &= B^n\,(1-B)^{n-1} + \sum_{j=1}^{n-1} B^{n-j}\,G_j(B)\,(1-B)^{n-j-1},\ n\geq 2. \end{aligned}$$

By induction on n, it is elementary to prove that (4.1) holds.

Notice that $G_{n+1}(B) = B\,(2-B)\,G_n(B)$ for $n\geq 1$, therefore

$$G_n(B) = B^n\,(2-B)^{n-1},\ n\geq 1.$$

Now, differentiating $G_n(B)$ with respect to B and noting that $G_n(0)=0$ and $G_n(1/d)\geq 1/d^n$, we prove the next result.

Lemma 4.1 For each $n\geq 1$, the function G_n is increasing in B and there exists a unique root $\bar\beta_n = \bar\beta_n(d)$ for the equation $G_n(B) = 1/d^n$.

Then, using lemmas 3.1 and 4.1 and equation (4.1),

$$p > \frac{(d+1)\bar\beta_n(d)}{1+d\,(\bar\beta_n(d))^2} \Rightarrow B(p,d) > \bar\beta_n(d) \Rightarrow F_n(B(p,d)) \geq G_n(B(p,d)) > \frac{1}{d^n}.$$

Hence, comparing the frog model to the branching process the way it was done in theorem 3.1, we get

Lemma 4.2 For any fixed $d\geq 1$ and $n\geq 1$,

$$p_c(\mathbb{T}_d) \leq \frac{(d+1)\bar\beta_n(d)}{1+d\,(\bar\beta_n(d))^2}.$$

To reach our goal, we need the following property of the sequence of roots.

Lemma 4.3 We have that

$$\lim_{n\to\infty} \bar{\beta}_n(d) = 1 - \left[\frac{d-1}{d}\right]^{1/2}.$$

Proof: Notice that $\bar{\beta}_n\,(2-\bar{\beta}_n) \geq \bar{\beta}_n\,(2-\bar{\beta}_n)^{1-1/n} = 1/d$ for all $n \geq 1$, then $G_{n+1}(\bar{\beta}_n) \geq 1/d^{n+1}$. This implies that $\bar{\beta}_{n+1} \leq \bar{\beta}_n$, so we can define $\bar{\beta} = \bar{\beta}(d) = \lim_{n\to\infty} \bar{\beta}_n(d)$. As the functions $H_n(B) = B(2-B)^{1-1/n}$ converge uniformly to $H(B) = B(2-B)$, we have that $\bar{\beta}\,(2-\bar{\beta}) = 1/d$. Since $\bar{\beta} \leq 1/d$, the result follows. $\square$

Finally, theorem 4.1 is obtained as a direct consequence of lemmas 4.2 and 4.3.

Remarks: One can take advantage of equation (4.2) in order to write an algorithm to generate the functions F_n. Next we present a couple of numerical results. For $n = 150$, the bound provided by theorem 3.1 is approximately 0.645062 for $d = 3$ and 0.545806 for $d = 10$. Compare respectively with the values 0.666667 and 0.55 provided by theorem 4.1. As mentioned, the numerical computations point to the fact that the upper bounds in theorem 3.1 are decreasing in n but there are technical difficulties for proving that.

With respect to the approach of section 4, we observe that there are other possible comparisons. For instance, one could replace $(1-B^k)$ by $(1-B^2)$ for $k \geq 2$ when the functions G_n are defined. Not taking in account the technicalities in the analogous steps of the proof, one obtains that

$$p_c(\mathbb{T}_d) \leq \frac{(d+1)\bar{V}}{1+d\,\bar{V}^2},$$

where $\bar{V}$ is the unique root in $[0,1/d]$ of the polynomial $Q(B) = d^2B^4 - d(d+$

$1)B^3 + 2dB - 1$. The actual approach chosen in this section is satisfactory for its simplicity, cleanness and by the fact that each steps are rigorously justified.

5 Random initial configuration

In this section, we generalize theorem 4.1 for the case of random initial configuration. We follow similar steps of sections 3 and 4, giving a sketch of the proof. The case of one-particle-per-vertex initial configuration was considered first for the sake of clearness.

Let η be a random variable assuming values in $\mathbb{N} = \{0, 1, 2, \ldots\}$, such that $p_0 := \mathbf{P}(\eta = 0) < 1$. We consider the frog model with initial configuration given by independent copies of η at each vertex of $\mathbb{T}_d$, $d \geq 2$, and denote by $p_c(\mathbb{T}_d, \eta)$ its critical probability. See Alves *et al.* [2] for more details. Now for each $x \in \mathcal{V}$, $\mathcal{R}_x$ denotes what we call the virtual range of x, that is, the union of the sets of vertices visited by each of the particles placed originally at x, if any, during their (virtual) lives. Thus, for $x, y \in \mathcal{V}$ distinct, $\{x \to y\} = \{y \in \mathcal{R}_x\}$ is the event that some particle placed originally at x visits y (virtually), and $\{x \nrightarrow y\}$ is the complement of that event. Conditioning on the initial number of particles at vertex x and using lemma 2.1, we obtain that

$$\mathbf{P}(x \to y) = 1 - \varphi(1 - (B(p,d))^n),$$

where $n = \text{dist}(x, y)$ and φ is the probability generating function of η.

Considering the definition 3.1, we still have that

$$\mathbf{P}(x_0 \xrightarrow{c} x_n) = F_n(B(p,d)),$$

but now the functions F_n are not necessarily polynomial in B. Arguing as in the deduction of equation (4.2), we get that

$$F_n(B) = [1 - \varphi(1 - B^n)] \prod_{k=1}^{n-1} \varphi(1 - B^k) + \sum_{j=1}^{n-1} [1 - \varphi(1 - B^{n-j})] \, F_j(B) \prod_{k=1}^{n-j-1} \varphi(1 - B^k).$$

Next, to obtain a function that bounds F_n from below, we define

$$\theta = \varphi(1 - 1/d), \; \gamma = \frac{d}{d-1} \, (\theta - p_0) \; \text{ and } \; \alpha = p_0 + d \, (1 - \theta) \tag{5.1}$$

and use that

$$\gamma s + p_0 \le \varphi(s) \le 1 - (\alpha - p_0) \, (1 - s) \; \text{ for } \; 1 - \frac{1}{d} \le s \le 1,$$

to prove that $F_n(B) \ge G_n(B)$ for all B, where

$$G_n(B) = (1 - p_0) \, B^n \, [\gamma(1 - B) + \alpha]^{n-1}, \; n \ge 1.$$

Using the facts that $\alpha \ge 1$, $\gamma > 0$ and that G_n is increasing in B, one can prove that there exists $N := N(d, \eta)$ such that for all $n \ge N$, there exists a unique root $\bar{\beta}_n := \bar{\beta}_n(d, \eta)$ for the equation $G_n(B) = 1/d^n$.

To obtain upper bounds for $p_c(\mathbb{T}_d, \eta)$, we compare the frog model on $\mathbb{T}_d$ to a Galton-Watson branching process defined in the following way: call $y \in L_n(x)$ offspring of a vertex x if the event $\{x \xrightarrow{c} y\}$ occurs. This branching process has mean number of offspring per individual equal to $d^n F_n(B(p, d))$ and the frog model survives if it does. Notice also that the definition of this branching process is slightly different from the one used in sections 3 and 4. Analogously to lemma 4.2,

Lemma 5.1 For any fixed $d \geq 2$ and n large enough,

$$p_c(\mathbb{T}_d, \eta) \leq \frac{(d+1)\bar{\beta}_n}{1 + d\,(\bar{\beta}_n)^2}. \tag{5.2}$$

Similarly as we have done in section 4, the last step is to prove that

$$\lim_{n\to\infty} \bar{\beta}_n = \frac{d(\alpha+\gamma) - \sqrt{\Delta}}{2d\gamma}, \tag{5.3}$$

where

$$\Delta = d\left[-4\,\gamma + d\,(\alpha+\gamma)^2\right]. \tag{5.4}$$

Therefore, having in mind (5.2) and (5.3), we obtain

Theorem 5.1 For all fixed $d \geq 2$,

$$p_c(\mathbb{T}_d, \eta) \leq \frac{\gamma\ (d+1)\left[\,d\ (\alpha+\gamma) - \sqrt{\Delta}\,\right]}{d\left[(\alpha+\gamma)\left\{\,d\ (\alpha+\gamma) - \sqrt{\Delta}\,\right\} - 2\,\gamma\ (1-\gamma)\right]} < 1,$$

where α, γ and Δ are given in (5.1) and (5.4).

Observe that theorem 4.1 is now a direct corollary of theorem 5.1.

6 On the monotonicity of the critical probability

Clearly, by a standard coupling argument, $p_c(\mathcal{G}, \eta)$ is monotone in η in the sense that, for η and η' such that $\mathbf{P}(\eta > k) \geq \mathbf{P}(\eta' > k)$ for all $k \in \mathbb{N}$, it is true that $p_c(\mathcal{G}, \eta) \leq p_c(\mathcal{G}, \eta')$. A known open question is whether, for a well behaved class of graphs (for example, the homogeneous trees), $\mathcal{G}_1 \subset \mathcal{G}_2$ in this class implies that $p_c(\mathcal{G}_2, \eta) \leq p_c(\mathcal{G}_1, \eta)$. In general, this monotonicity on the graph is not true as pointed out in Fontes *et al.* [4]. This lack of monotonicity is an unexpected fact, since monotonicity of critical probability holds for the usual percolation models; even strict monotonicity can be proved in a rather general situation (see Grimmett [5] and Menshikov [7]).

We discuss this issue for the homogeneous trees and $\eta \equiv 1$. Observe that every vertex with at least one active particle at time $n > 0$ has at least one neighbor vertex whose original particle has been activated prior to time n. Using this fact and a comparison to a Galton-Watson branching process, Alves *et al.* [2] present a lower bound for the critical probability in graphs of bounded degree. For completeness, we present next an alternative proof for the same result for homogeneous trees by using a martingale argument.

Theorem 6.1 For all fixed $d \geq 1$,

$$p_c(\mathbb{T}_d) \geq \frac{d+1}{2d+1}.$$

Proof: Let Y_n be the number of active particles at time n in $\mathrm{FM}(\mathbb{T}_d, p)$. By the fact mentioned in the above paragraph, given that $\{Y_n = k\}$ for a fixed $n \geq 1$, we have that Y_{n+1} is not greater than a sum of k random variables each one assuming the values 0, 1 and 2 with respective probabilities $(1-p)$, $p/(d+1)$ and $p\,d/(d+1)$. Then, for $n \geq 1$,

$$\mathbf{E}(Y_{n+1} \,|\, Y_n = k) \leq \left(\frac{2d+1}{d+1}\right) p\,k,$$

which implies that

$$\mathbf{E}(Y_{n+1}) \leq \left(\frac{2d+1}{d+1}\right) \mathbf{E}(Y_n)\; p.$$

Thus, if $p < (d+1)/(2d+1)$, we have that $\lim_{n\to\infty} \mathbf{E}(Y_n) = 0$. $\square$

It is worth noting that, by theorems 4.1 and 6.1, there exists a sequence $d_1 < d_2 < \cdots$ such that $p_c(\mathbb{T}_{d_1}) > p_c(\mathbb{T}_{d_2}) > \cdots$ (For this, take $d_1 = 2, d_2 = 6, d_3 = 14$). This suggests that, when $\eta \equiv 1$, strict monotonicity of critical probability holds in the class of homogeneous trees. We think however that the method of this paper alone can hardly solve this problem; it should probably be combined with some new ideas.

As a final remark, we point out that the frog model on $\mathbb{T}_d$ with $d = \infty$ corresponds to a Galton-Watson branching process in which each individual can have no offspring with probability $1 - p$ or two offspring with probability p, so that its critical parameter is $1/2$. Therefore the bounds given by theorems 4.1 and 6.1 are asymptotically correct as $d \to \infty$.

Acknowledgements: EL is thankful to FAPESP (01/04416–0), FPM to CNPq (300226/97–7) and SP to CNPq (302981/02–0), for financial support. Thanks are due for the two anonymous referees for their careful reading and nice suggestions which helped us to improve the paper.

References

[1] O. S. M. Alves, F. P. Machado, S. Popov and K. Ravishankar, The shape theorem for the frog model with random initial configuration. *Markov Process. Relat. Fields* **7** (4), 525–539 (2001).

[2] O. S. M. Alves, F. P. Machado and S. Popov, Phase transition for the frog model. *Electron. J. Probab.* **7**, no. 16, 21 p (2002).

[3] O. S. M. Alves, F. P. Machado and S. Popov, The shape theorem for the frog model. *Ann. Appl. Probab.* **12** (2), 534–547 (2002).

[4] L. R. Fontes, F. P. Machado and A. Sarkar, The critical probability for the frog model is not a monotonic function of the graph. *J. Appl. Probab.* **41** (1), 292–298 (2004).

[5] G. Grimmett, *Percolation.* Springer-Verlag, Berlin, 2nd. ed., 1999.

[6] I. Kurkova, S. Popov and M. Vachkovskaia, On infection spreading and competition between independent random walks. *Electron. J. Probab.* **9**, no. 11, 22 p (2004).

[7] M. V. Menshikov, Quantitative estimates and strong inequalities for the critical points of a graph and its subgraphs. *Probab. Theory Appl.* **32**, 544–547 (1987).

[8] S. Popov, Frogs in random environment. *J. Statist. Phys.* **102** (1/2), 191–201 (2001).

[9] S. Popov, Frogs and some other interacting random walks models. *Discrete Math. Theor. Comput. Sci.* AC, 277-288 (2003).

[10] A. Telcs and N. C. Wormald, Branching and tree indexed random walks on fractals. *J. Appl. Probab.* **36**, 999–1011 (1999).