Multiscale percolation on k-symmetric mosaic

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Abstract

We study the multiscale (fractal) percolation on d-dimensional k-symmetric mosaic, which can be viewed as a generalization of the classical Mandelbrot percolation model. The obtained result is applied to the multiscale bond percolation model.

Keywords: multiscale percolation, critical probability, k-symmetric mosaic, bond percolation

In this note we consider the multiscale (fractal) percolation model on d-dimensional k-symmetric mosaic, $d \geq 2$. This model is a natural generalization of the Mandelbrot percolation model (cf., for example, Chayes et al. (1988), Falconer and Grimmett (1992), Menshikov et al. (2000), Orzechowski (1996), White (1999)). The latter model is constructed by using a cubic tesselation of $\mathbb{R}^d$, and only the neighboring cells are

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connected. In contrast to the Mandelbrot model, here we allow tesselations of more general types, and the adjacency relation also can be more complicated.

One of the purposes of this paper is to underline that the methods developed by Menshikov et al. (2000) are applicable not only to Mandelbrot and Poisson fractal percolation models, but also to multiscale models where the adjacency relation is of rather general nature.

Let us describe some possible applications of this work. Consider the Poisson fractal percolation model in $\mathbb{R}^d$ (cf. Menshikov et al. (2000)). The model can be formulated as follows. For fixed scaling parameter $R > 1$ the level-i model is the following: first, take a Poisson field of points with rate $\lambda_i = cR^{id}$, and second, to each of those points place a ball with radius R^{-i} (this is called Poisson percolation model). The models of different levles are independent, and the Poisson fractal percolation model is just their union. A natural approach to prove the absence of percolation in this multiscale model would be to use the discretization method (cf. Menshikov and Sidorenko (1987)). Roughly speaking, it means that a (cubic) mosaic is placed on $\mathbb{R}^d$, the cell is selected if it contains a point of Poisson field, and the cells adjacent to the selected cell are those covered by the defect set placed at that point. So, nonnearest-neighbor connections appear in a natural way here. Although the final version of Menshikov et al. (2000) uses another approach, we think that the method described here may prove useful in other problems of continuum multiscale percolation. Another interesting application, which we call *multiscale bond percolation*, is discussed later in this note.

Let us now make the necessary definitions.

Definition 1 *Let $\mathcal{G}$ be the countable collection of compact subsets of $\mathbb{R}^d$ with the following properties:*

- *the union of all elements ("cells") of $\mathcal{G}$ is $\mathbb{R}^d$;*
- *every element of $\mathcal{G}$ is homeomorphic to the unit ball in $\mathbb{R}^d$;*
- *if G_1 and G_2 are two distinct elements of $\mathcal{G}$, then $(G_1 \setminus \partial G_1) \cap (G_2 \setminus \partial G_2) = \emptyset$, i.e. different cells may only "touch";*

- *there exist a partition $\mathcal{G} = \mathcal{A}_1 \cup \ldots \cup \mathcal{A}_k$, such that for every pair $G_1, G_2 \in \mathcal{A}_i$ there exists $a \in \mathbb{R}^d$ and a unitary operator U such that the map $x \mapsto Ux + a$ is an automorphism of $\mathcal{G}$ which maps G_1 to G_2 and preserves the collections $\mathcal{A}_1, \ldots, \mathcal{A}_k$ for every $i = 1, \ldots, k$. In that case we say that $\mathcal{G}$ has k types of cells, and if $G \in \mathcal{A}_i$, then we say that G is of type i. Note that any two cells of the same type are congruent.*

Such $\mathcal{G}$ is called mosaic.

Fix some $R > 1$ and consider the rescalings of $\mathcal{G}$: $\mathcal{G}_m = R^{-m}\mathcal{G}$, $m = 0, 1, \ldots$ In other words, $\mathcal{G}_m$ is obtained from $\mathcal{G}$ by applying the transformation $x \mapsto R^{-m}x$. We call $\mathcal{G}_m$ *level-m mosaic.*

Consider any adjacency relation between the cells of $\mathcal{G}$ which satisfies the following condition.

Condition A:

- The adjacency relation is translation invariant, i.e., it is preserved by all the automorphisms of the mosaic;
- there exists $L < \infty$ such that any $G_1, G_2 \in \mathcal{G}$ with $\operatorname{dist}(G_1, G_2) > L$ are not adjacent, where $\operatorname{dist}(G_1, G_2)$ denotes the usual distance between two subsets of $\mathbb{R}^d$;
- if $\operatorname{dist}(G_1, G_2) = 0$ (i.e. $G_1 \cap G_2 \neq \emptyset$), then G_1 and G_2 are adjacent.

Here it is important to note that not only the nearest neighbors are allowed to be adjacent.

The adjacency relation between the cells of arbitrary levels is defined in a natural way. Two cells $G_1, G_2 \in \mathcal{G}_m$ are adjacent iff $R^m G_1$ and $R^m G_2$ are adjacent in $\mathcal{G}$. Two cells of *different* levels $G_1 \in \mathcal{G}_m$, $G_2 \in \mathcal{G}_l$, $m > l$ are adjacent iff G_1 is adjacent to some $G \subset G_2$, $G \in \mathcal{G}_m$.

Now we are going to define the random set of interest. Suppose that one has a collection of probabilities $\vec{p} = (p_1, \ldots, p_k) \in [0,1]^k$. Each cell of type i of level-m

mosaic is selected with probability p_i independently of the other cells of all levels, $m = 0, 1, 2, \dots$

The main objects of study will be the following random sets:

$$D_n = \{\text{union of all selected cells of levels } 0, \dots, n\} \subset \mathbb{R}^d$$

and

$$D = D(\mathcal{G}, R, \vec{p}) = \cup_{n=0}^{\infty} D_n = \{\text{union of all selected cells of all levels}\}.$$

We say that percolation occurs in D_n iff there exists an infinite sequence of distinct cells

$$\left\{ G_i \in \bigcup_{j=1}^{n} \mathcal{G}_j,\, G_i \text{ is selected}, G_i \text{ and } G_{i+1} \text{ are adjacent} \right\}_{i=0}^{\infty}.$$

Now we define percolation in D. Let ω be a configuration of the selected cells (of all levels). For $x \in \mathbb{R}^d$ define a random variable m_x as

$$m_x = \min\{m : \text{ there exists } G \in \mathcal{G}_m \text{ such that } G \text{ is selected and } x \in G\}$$

and denote by $G(x)$ the selected cell from $\mathcal{G}_{m_x}$ containing x (in the case when two or more selected cells from $\mathcal{G}_{m_x}$ contain x, one may denote by $G(x)$ any of them). Note that $m_x < \infty$ for Lebesgue–almost all x almost surely. Suppose that $m_x < \infty$, and, consequently, $G(x)$ exists. The set of points directly connected with x is defined as

$$A_x(\omega) = G(x) \cup \{y \in \mathbb{R}^d : G(y) \text{ exists and } G(y) \text{ is adjacent to } G(x)\}.$$

One can easily see that $y \in A_x(\omega)$ iff $x \in A_y(\omega)$.

Definition 2 *For given ω we say that percolation occurs in D iff there exists a sequence*

$$\{x_i : G(x_i) \text{ exists and } x_{i+1} \in A_{x_i}(\omega)\}_{i=0}^{\infty}$$

which is not contained in any finite box.

As an example, one can consider a particular case of this model known as Mandelbrot's model. The latter model is constructed in the following way. For fixed $N \in \mathbf{N}$

and $p \in (0,1)$ the space is divided into hypercubes with side 1, each of them is selected with probability p independently of the others; then, every nonselected cube is divided into N^d equal cubes with side N^{-1} and each of those smaller cubes is selected with probability p independently of everything, and so on. The percolation in the set of selected cubes of all levels as well as the percolation in the closure of the complementary set are of interest.

To formulate the main theorem we need to define the graph $\Gamma(\mathcal{G}) = (V(\mathcal{G}), E(\mathcal{G}))$ corresponding to $\mathcal{G}$, where $V(\mathcal{G})$ and $E(\mathcal{G})$ denote the sets of vertices and edges respectively. It will be done in a natural way: the vertices of $\Gamma(\mathcal{G})$ correspond to the cells of $\mathcal{G}$, and two vertices are connected with an edge iff the corresponding cells are adjacent. Note that as $\mathcal{G}$ has k types of cells, $\Gamma(\mathcal{G})$ has k types of vertices, i.e. there exist a partition $V(\mathcal{G}) = V_1 \cup \ldots \cup V_k$ such that for every pair $v, v' \in V_i$ there exists an automorphism which maps v to v' and preserves the sets $V_1, \ldots, V_k$, $i = 1, \ldots, k$. Declare each vertex of type i to be open (respectively closed) with probability p_i (respectively $1 - p_i$) and consider a percolation by open vertices (site percolation) on $\Gamma(\mathcal{G})$. For fixed $\vec{p}$ let $\theta(\vec{p})$ be the probability that some fixed vertex from $V(\mathcal{G})$ belongs to an infinite cluster and $H = \{\vec{p} : \theta(\vec{p}) = 0\}$. Let $\mathring{H} = H \setminus \partial H$ be the interior of H. Denote also $\vec{p}_\varepsilon = (p_1 + \varepsilon, \ldots, p_k + \varepsilon)$. Our main result is the following

Theorem 1 *Suppose that $\vec{p}$ is such that $\vec{p}_\varepsilon \in \mathring{H}$ for some $\varepsilon > 0$. Then there exists $R_0 = R_0(\vec{p})$ such that for all $R \geq R_0$ there is no percolation in the set D.*

Note that although the set $\mathbb{R}^d \setminus D$ has Lebesgue measure 0, there is no percolation in the set D. As Mandelbrot's model is covered by present formalism, Theorem 1 is indeed the generalization of Theorem 2 of Orzechowski (1996).

Example: multiscale bond percolation.

Multiscale bond percolation model is defined in the following way. Fix $R \in \mathbb{N}$ and consider the rescalings of the lattice $\mathbb{Z}^d$: $\mathcal{Z}_m = R^{-m}\mathbb{Z}^d$, $m = 0, 1, \ldots$ We call $\mathcal{Z}_m$ level-m lattice. Each bond of $\mathcal{Z}_m$ is open with probability $p > 0$, independently of all other bonds (of all levels). Denote the bond between nearest neighbors x and y by

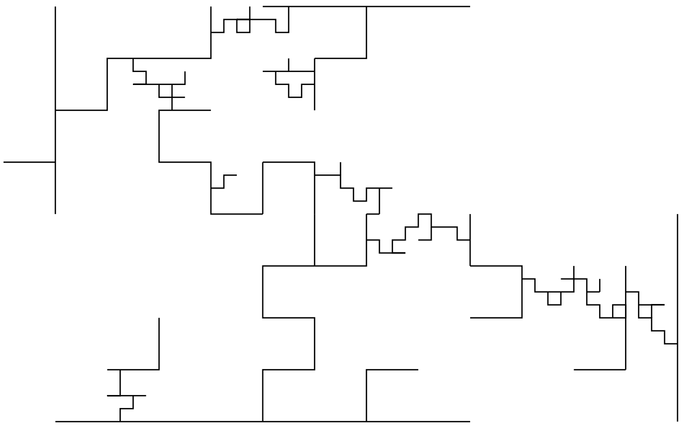

Figure 1: A connected component in a multiscale bond percolation, $R = 4$

$\langle x, y\rangle$ (we identify it with the corresponding closed segment in $\mathbb{R}^d$). Let

$$
\begin{aligned}
D^{(m)} &= \cup\{\langle x, y\rangle \in \mathcal{Z}_m : \langle x, y\rangle \text{ is open}\} \subset \mathbb{R}^d \text{ and} \\
D &= \bigcup_{m=0}^{\infty} D^{(m)} \subset \mathbb{R}^d.
\end{aligned}
$$

We say that in multiscale bond percolation model percolation occurs if D a.s. contains an unbounded connected component (see Fig. 1). Let p_{bond} be the critical probability for the usual bond percolation model on $\mathbb{Z}^d$ (that is, in $\mathcal{Z}_0$).

One can associate the usual bond percolation on $\mathbb{Z}^d$ with the percolation on the following mosaic $\mathcal{G}$ (see Fig. 2). White (hexagonal) cells are selected with probability $p_1 := p$, and the dashed cells are selected with probability $p_2 := 0$. The rescalings of $\mathbb{Z}^d$ correspond to the rescalings of $\mathcal{G}$. One can easily see that the multiscale percolation on rescalings of $\mathcal{G}$ majorizes the multiscale bond percolation. So we have

Corollary 1 *For any $p < p_{bond}$ there exists $R_0 = R_0(p) \in \mathbb{N}$ such that for all $R > R_0$ there is no percolation in multiscale bond percolation model.*

Note that the fact that $(p + \varepsilon, \varepsilon) \in \mathring{H}$ for $p < p_{bond}$ follows from the results of Menshikov (1986, 1987).

Proof of Theorem 1.

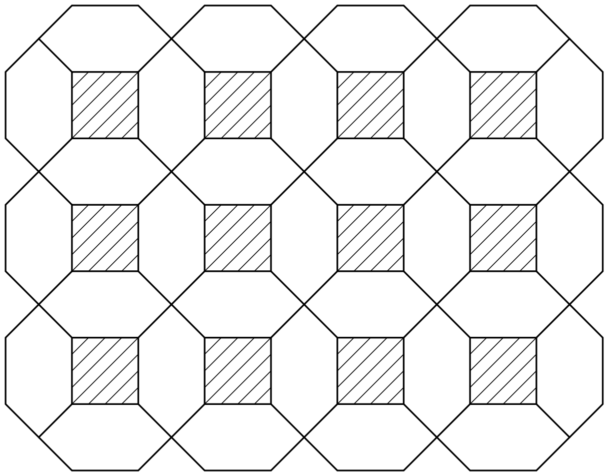

Figure 2: The mosaic associated to the bond percolation in $\mathbb{Z}^2$

To prove the absence of percolation in D it is sufficient to prove that the sets D_n are in the subcritical phase uniformly in n (i.e. that the probability of existence of the path from the origin to the boundary of big box is small uniformly in n). See Menshikov et al. (2000) for more detailed discussion. So, let us fix some n and consider percolation problem in D_n.

First, denote by $N_{\mathcal{G}}$ the minimal number of cells in the path by level-1 cells connecting two non-intersecting level-0 cells.

Remark. Denote $\rho = \inf\{\mathrm{dist}(G_1, G_2) : G_1, G_2 \in \mathcal{G}, G_1 \cap G_2 = \emptyset\}$. Note that $\rho > 0$. Then one can easily see that $N_{\mathcal{G}} \geq \rho R/L$ (where L is from Condition A). That is, by choosing large R we can make $N_{\mathcal{G}}$ as large as we want.

The next Definitions 3 and 4 and Lemma 1 follow Definitions 2.2 and 2.3 and Lemma 2.1 from Menshikov et al. (2000).

Definition 3 *The level-n cell G is good iff it is selected. For $l < n$ the level-l cell G' is passable iff it intersects with a path by good level-$(l+1)$ cells of length (the number of cells in it) greater or equal than $N_{\mathcal{G}}$. Finally, G' is good iff it is passable or selected.*

Lemma 1 *If percolation occurs in D_n, then there exists an infinite sequence of adjacent good level-0 cells.*

The proof of this lemma is a straightforward adaptation of the proof of Lemma 2.1 from Menshikov et al. (2000) and is therefore omitted.

Associate three random fields $\eta_m^s = \{\eta_m^s(v)\}_{v\in V(\mathcal{G}_m)}$, $\eta_m^g = \{\eta_m^g(v)\}_{v\in V(\mathcal{G}_m)}$, and $\xi_{\vec{r},m} = \{\xi_{\vec{r},m}(v)\}_{v\in V(\mathcal{G})}$ to the set of level-m cells, where

$$\begin{aligned}\eta_m^s(v) &= \mathbf{1}\{\text{the cell corresponding to } v \text{ is passable}\},\\ \eta_m^g(v) &= \mathbf{1}\{\text{the cell corresponding to } v \text{ is good}\},\end{aligned}$$

and $\xi_{\vec{r},m} \in \{0,1\}^{V(\mathcal{G}_m)}$ is the Bernoulli random field of level-m cells with parameter $\vec{r} \in [0,1]^k$.

We need the following definition:

Definition 4 *For random field $\{\zeta(x)\}_{x\in V(\mathcal{G})}$, $\zeta(x) \in \{0,1\}$ define the cluster $W_\zeta(x)$ of the vertex $x \in V(\mathcal{G})$ as the set of all $y \in V(\mathcal{G})$ such that there exist m and a sequence $y_0 = x, y_1, \dots, y_m = y$ such that $\zeta(y_i) = 1$ and y_{i-1} is adjacent to y_i, $i = 1, \dots, m$.*

Fix $\varepsilon > 0$ such that $\vec{p}_\varepsilon \in \mathring{H}$.

Lemma 2 *There exists $R_0 > 1$ such that for any $R > R_0$ and for any $n > 1$ the random field of good level-0 cells can be stochastically dominated by Bernoulli random field with parameter $\vec{p}_\varepsilon$.*

Proof. The proof of the lemma is similar to that of Lemma 2.3 from Menshikov et al. (2000).

The field $\eta_n^g = \xi_{\vec{p},n}$ is dominated by $\xi_{\vec{p}_\varepsilon,n}$, so the statement of the lemma holds for level-n.

Suppose that the field η_{k+1}^g is dominated by the Bernoulli field $\xi_{\vec{p}_\varepsilon,k+1}$.

From Menshikov (1986) it follows that if $\vec{p}_\varepsilon \in \mathring{H}$ then there exists $\beta(\vec{p}_\varepsilon) > 0$ such that for all ν and all $x \in V(\mathcal{G})$

$$\mathbf{P}\{|W_{\xi_{\vec{p}_\varepsilon}}(x)| \geq \nu\} \leq \exp(-\beta(\vec{p}_\varepsilon)\nu). \tag{1}$$

Fix $G \in \mathcal{G}_k$. Using (1) and the induction assumption (i.e. that the field η^g_{k+1} is dominated by Bernoulli field $\xi_{\vec{p}_\varepsilon,k+1}$), analogously to the formula (3) from Menshikov et al. (2000), we have

$$\mathbf{P}\{G \text{ is passable}\} \le N^d_{\mathcal{G}} \exp(-\beta(\vec{p}_\varepsilon)N_{\mathcal{G}}) =: \varepsilon_1(N_{\mathcal{G}}). \tag{2}$$

Note that the event $\{G$ is passable$\}$ depends only on what happens in some finite number of cells of the same level. So there exists a bounded set $\Lambda(G) \subset \mathbb{R}^d$, $G \subset \Lambda(G)$, such that the state of G does not depend on what happens outside of $\Lambda(G)$. As $\varepsilon_1(N_{\mathcal{G}})$ is small for $N_{\mathcal{G}}$ large (i.e. for R large), the result of Liggett et al. (1997) implies that the random field of passable level-k cells can be dominated by Bernoulli random field with parameter $\varepsilon_2(R)$ which can be made arbitrarily small by choosing $\varepsilon_1(N_{\mathcal{G}})$ small (i.e. choosing R large). Note that Theorem 2.1 of Menshikov et al. (2000) also could have been applied here.

Take R such that $\varepsilon_2(R) < \varepsilon$. We have that η^s_k is dominated by Bernoulli random field with parameter $\varepsilon_2(R)$, the field of selected cells of level-k is Bernoulli with parameter $\vec{p}$, so the field η^g_k is dominated by $\xi_{\vec{p}_\varepsilon,k}$. As the choice of R depends only on $\vec{p}_\varepsilon$, it can be taken the same for all levels of cells. This proves Lemma 2. ■

Combining Lemmas 1 and 2, we complete the proof of Theorem 1. ■

References

[1] Chayes, J.T., Chayes, L., Durrett, R., 1988. Connectivity properties of Mandelbrot's percolation process. Probab. Theory Relat. Fields 77, 307–324.

[2] Falconer, K.J., Grimmett, G.R., 1992. On the geometry of random Cantor sets and fractal percolation. J. Theoret. Probab. 5 (3), 465–485.

[3] Liggett, T.M., Schonmann, R.H., Stacey, A.M., 1997. Domination by product measures. Ann. Probab. 25 (1), 71–95.

[4] Menshikov, M.V., 1986. Coincidence of critical points in percolation problems. Sov. Math. Doklady 33, 856–859.

[5] Menshikov, M.V., 1987. Qualitative estimates and rigorous inequalities for critical points of a graph and its subgraphs. Probab. Theory Appl. 32, 544–547.

[6] Menshikov, M.V., Popov, S.Yu., Vachkovskaia, M., 2000. On the connectivity properties of the complementary set in fractal percolation models. To appear in: Probab. Theory Relat. Fields.

[7] Menshikov, M.V., Sidorenko, A.F., 1987. Coincidence of critical points for Poisson percolation models. Probab. Theory Appl. 32, 603–606.

[8] Orzechowski, M.E., 1996. On the phase transition to sheet percolation in random Cantor sets. J. Statist. Phys. 82 (3/4), 1081–1098.

[9] White, D.G., 1999. On fractal percolation in $\mathbb{R}^2$. Statist. Probab. Letters 45, 187–190.