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ONE-DIMENSIONAL BRANCHING RANDOM WALK IN A MARKOVIAN RANDOM ENVIRONMENT

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Abstract

We study one-dimensional supercritical branching random walk in non-i.i.d. random environment, which considers both the branching mechanism and the step transition. This random environment is constructed using a recurrent Markov chain on finite or countable state space. Criteria of (strong) recurrence and transience are presented for this model.

BRANCHING RANDOM WALK; RANDOM ENVIRONMENT; STRONG TRANSIENCE; STRONG RECURRENCE

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Introduction

We study a discrete time branching random walk in a random environment. The random environment is "frozen", which means that first we fix a realization of the random sequence of parameters (branching and transition probabilities), and then start the branching random walk. Similar models have been studied in [1, 2, 5, 6, 7, 12]. In the papers [1, 6], systems with infinite number of particles are considered

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and results about global and local particle density are obtained. The limit of finite systems (as the number of sites increases) is studied in [5]. The main object of study there is the total population size. Branching random walks in *many-dimensional* random environment are considered in [7, 12]. The local particle density is studied in [12], while the main question of [7] is recurrence/transience, that is, whether the whole cloud of particles touch a fixed site almost surely.

In the papers [1, 5, 6, 7, 12], the random environment plays a role *only* to the branching mechanism. One-dimensional branching random walk in i.i.d. random environment which includes both the branching mechanism and the step transition is considered in [2]. In that work a classification with respect to recurrence and transience is obtained. Here the goal is to generalize the results of [2] to the case of non-i.i.d. random environment. Namely, in this paper the results are obtained for an environment governed by a recurrent Markov chain on a finite or countable state space $\mathcal{A}$.

The main tool used here is the method of Lyapunov (test) functions (for a general introduction to the subject see [4]). In [3] this method is applied to a non-branching Markov processes in random environment. Adaptations of this method to branching random walks can be found in [2, 8, 9, 11]; see also in [10] a related work for the case of continuous space and time.

The paper is organized as follows. In Section 1 we define a simple version of one-dimensional branching random walk and give an appropriate classification toward recurrence and transience. The random environment is constructed in Section 2, the results on recurrence/transience (Theorems 3.1 and 3.2) are formulated in Section 3. In Section 4 we formulate the necessary criteria for the general branching random walks, and then the proof of Theorem 3.1 is given.

For what follows, $\mathbf{P}$ means probability with respect to the random environment, while P and E stand for the probability and expectation related to the branching random walk (after the environment is fixed).

1. One-dimensional branching random walk: definition and classification

Let $\mathcal{L}$ be a discrete-time homogeneous Markov chain on $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ with transition probabilities $\tilde{p}_{x,y}$. We describe a simple version of the branching random walk as follows. At time 0 there is one single particle at some site $x_0 \neq 0$. Then, if at some moment there is a particle in x, it is substituted by i offsprings with probability $r_i(x)$, $i = 1, 2, 3, \ldots$ Note that $r_0(x) = 0$, so the total number of particles cannot decrease. We suppose also that for all $x \in \mathbb{Z}_+$

$$r(x) := \sum_{i=1}^{\infty} i r_i(x) < \infty.$$

The generated offsprings evolve independently, jumping from site to site of $\mathbb{Z}_+$ according to the Markov chain $\mathcal{L}$, generating their own offsprings and being substituted by them. Note that here there is no "one-particle-per-site" rule. In this paper we will deal only with *nearest-neighbor* case: each particle can either jump to its left, to stay on the site where it is or jump to its right site as well. Particles take that decision according to the probabilities $p(x) := \tilde{p}_{x,x-1}$, $s(x) := \tilde{p}_{x,x}$, $q(x) := \tilde{p}_{x,x+1}$. Some restrictions should apply to those parameters in order to avoid situations like traps or total asymmetry.

As the number of particles on the system is always finite but grow to infinity, we cannot use an ordinary definition of transience and recurrence. In order to have a clean definition of transience and recurrence we declare 0 to be an *absorbing state* in the sense that, once a particle comes to that site, it stays there forever without having any further offsprings. That means that

$$\tilde{p}_{00} = 1, \quad r_1(0) = 1.$$

Let us define the random variables $N^x(t)$ as the number of particles at site x at time t. Since 0 is the absorbing state, the sequence $N^0(t)$, $t = 0, 1, 2, \ldots$ is non-decreasing, so the limit

$$\nu = \lim_{t \to \infty} N^0(t)$$

exists for all realizations of the process. Of course, ν is a random variable, and it may assume the value $+\infty$.

The next definition follows Definition 2.1 of [2].

Definition 1.1 We say that the branching random walk is

- *recurrent, if* $\mathsf{P}\{\nu \geq 1\} = 1$*;*
- *strongly recurrent, if it is recurrent and* $\mathsf{E}\nu < \infty$*;*
- *transient, if* $\mathsf{P}\{\nu \geq 1\} < 1$*;*
- *strongly transient, if it is transient and* $\mathsf{E}\nu < \infty$*.*

Note that the position x_0 of the initial particle (as long as $x_0 \neq 0$) is not essential for this definition.

2. Construction of the random environment

In this paper we work with the following source of randomness: first, the transition and branching parameters are sampled from a set of probability distributions. Then, this environment is fixed, and we start the branching random walk. We are particularly interested in the properties which hold for $\mathbf{P}$-almost all environments.

Define M as the set of all infinite sequences $\xi = (p^\xi, s^\xi, q^\xi, r_1^\xi, r_2^\xi, \dots)$ of non-negative numbers such that

$$p^\xi + s^\xi + q^\xi = 1, \quad \sum_{j=1}^{\infty} r_j^\xi = 1, \quad r^\xi := \sum_{j=1}^{\infty} j r_j^\xi < \infty.$$

Let $\mathcal{A}$ be some finite or countable set. Let $\tilde{Q}_i, i \in \mathcal{A}$, be a collection of probability distributions on M which satisfy the following two conditions:

Condition Q. $\tilde{Q}_j\{\xi \in M : p^\xi = 0 \text{ or } q^\xi = 0\} = 0$ for all $j \in \mathcal{A}$.

Condition R. $\tilde{Q}_{j_0}\{\xi \in M : r^\xi > 1\} > 0$ for at least one $j_0 \in \mathcal{A}$.

Now we describe the construction of the random environment. Let $\boldsymbol{\eta} = (\eta_n, n \geq 1)$ be an irreducible *recurrent* Markov chain on the state space $\mathcal{A}$ with transition probabilities $P_{ij} = \mathbf{P}\{\eta_{n+1} = j \mid \eta_n = i\}$, and η_1 is chosen according to some initial distribution. Fix a realization of the sequence $\boldsymbol{\eta}$. Consider a sequence of random variables $\boldsymbol{\xi} = (\xi_n, n \geq 1)$, $\xi_i \in M$, with the following properties:

- random variables $(\xi_n, n \geq 1)$ are conditionally independent given the realization of the sequence $\boldsymbol{\eta}$;
- given $\eta_n = i$, the distribution of ξ_n is $\tilde{Q}_i$.

The pair of sequences $(\boldsymbol{\eta}, \boldsymbol{\xi}) = ((\eta_n, \xi_n), n \geq 1)$ is called *random environment.* Then, for $x \neq 0$ we define $p(x) = p^{\xi_x}$, $s(x) = s^{\xi_x}$, $q(x) = q^{\xi_x}$, and $r_i(x) = r_i^{\xi_x}$ for $i = 1, 2, 3, \ldots$

In words, Condition Q is to guarantee that the branching random walk both has no traps on any set of sites and does not have a totally asymmetric environment. Condition R assures that the process is able to branch.

3. Results

In the next few lines we present some definitions which are necessary to make the statement of the results.

First let us consider for $\xi \in M$ the function

$$h^{\xi}(\lambda) = \frac{1}{q^{\xi}}\Big(\frac{1}{r^{\xi}} - s^{\xi} - \frac{p^{\xi}}{\lambda}\Big) \tag{3.1}$$

for $0 < \lambda \leq +\infty$, and its infimum on the support of the measures indexed by $\mathcal{A}$

$$H_i(\lambda) = \inf_{\xi \in \operatorname{supp} \tilde{Q}_i} h^{\xi}(\lambda), \quad i \in \mathcal{A}, \tag{3.2}$$

with the following convention: $H_i(\lambda) = -\infty$ for $\lambda \leq 0$. Note that the functions H_i may assume the value $(-\infty)$ for some positive λ as well.

For $i \in \mathcal{A}$ define $C(i) = \{j \in \mathcal{A} : P_{ij} > 0\}$.

Theorem 3.1 For $\mathsf{E}\nu$ *to be finite* $\mathbf{P}$*-a.s., it is necessary and sufficient that there exist a collection of positive numbers* $(\lambda_i > 0, i \in \mathcal{A})$ *such that*

$$\lambda_j \leq H_i(\lambda_i) \quad \textit{for all } j \in C(i) \tag{3.3}$$

for all $i \in \mathcal{A}$*. Moreover, if*

$$\lambda_i < 1 \textit{ for all } i \in \mathcal{A}, \tag{3.4}$$

then the process is strongly transient $\mathbf{P}$*-a.s. If*

$$\lambda_i > 1 \text{ for all } i \in \mathcal{A}, \tag{3.5}$$

then the process is strongly recurrent $\mathbf{P}$*-a.s.*

To complete the classification, we state the following

Theorem 3.2 If there is no positive solution to (3.3), then the branching random walk is recurrent (not strongly) $\mathbf{P}$*-a.s.*

Let us now consider the following two examples.

Periodic random environment. Let $\mathcal{A} = \{1, \dots, k\}$ and $P_{12} = \dots = P_{k-1,k} = P_{k1} = 1$. In other words, it is the case when the random variables ξ_n are independent, and ξ_{ki+j}, $i \geq 0$, $1 \leq j \leq k$, has distribution $\tilde{Q}_j$. In this case (3.3) becomes the following: there exist $\lambda_1, \dots, \lambda_k > 0$ such that $\lambda_{i+1} \leq H_i(\lambda_i)$ for $i = 1, \dots, k-1$, and $\lambda_1 \leq H_k(\lambda_k)$. Moreover, from Section 4 below one can get that it is possible to substitute the independence condition by the following two conditions:

- $((\xi_{nk+1}, \dots, \xi_{nk+k}), n \geq 0)$ is a stationary ergodic sequence;
- if $\mathbf{P}\{\xi_{x_i} \in A_i\} > 0$, $i = 1, \dots, n$, then $\mathbf{P}\{\xi_{x_1} \in A_1, \dots, \xi_{x_n} \in A_n\} > 0$ for any $x_1, \dots, x_n \in \mathbb{Z}_+$ and $A_1, \dots, A_n \subset M$,

keeping the criterion the same.

Mixing random environment. Suppose that $P_{ij} > 0$ for all $i, j \in \mathcal{A}$, i.e. $C(i) = \mathcal{A}$ for all $i \in \mathcal{A}$. Denote $H(\lambda) = \inf_{i \in \mathcal{A}} H_i(\lambda)$. One gets that (3.3) now reduces to the following: there exists $\lambda > 0$ such that $\lambda < H(\lambda)$. It is straightforward to verify that this is equivalent to the condition "$K_{\tilde{Q}} \neq \emptyset$" from [2].

Before going further, let us give some remarks on the restriction to the nearest-neighbor case. In the non-nearest-neighbor case (i.e. when the branching random walk admits not only nearest-neighbor jumps) it is straightforward, after making suitable changes to (3.1), to prove the sufficiency part of Theorem 3.1. However, the necessity part is still beyond our reach. Moreover, it is still very unclear to us whether the criterion (3.3) will be the same in the non-nearest-neighbor case.

4. Proofs

First, we note that the proof of Theorem 3.2 is a straightforward adaptation of the proof of Theorem 4.3 from [2], and is therefore omitted. To prove Theorem 3.1, we need the following two results:

Lemma 4.1 For $\mathsf{E}\nu$ *to be finite, it is necessary that there exist a positive function* $f(n)$ *such that for all* $n \neq 0$

$$p(n)f(n-1) + s(n)f(n) + q(n)f(n+1) = \frac{f(n)}{r(n)} \tag{4.1}$$

for $n \in \mathbb{Z}_+$. *It is sufficient for* $\mathsf{E}\nu$ *to be finite that there exist a positive function* $f(n)$ *such that for all* $n \neq 0$

$$p(n)f(n-1) + s(n)f(n) + q(n)f(n+1) \leq \frac{f(n)}{r(n)}. \tag{4.2}$$

Moreover, if $f(n) \to 0$ *as* $n \to \infty$, *then the branching random walk is strongly transient. If* $f(n) \to \infty$ *as* $n \to \infty$, *then the branching random walk is strongly recurrent.*

Lemma 4.2 Let f_1 *and* f_2 *be two solutions of (4.1) such that* $f_1(0) = f_2(0) > 0$ *and that* $0 < f_1(1) < f_2(1)$. *Let*

$$\bar{x} = \max\{x : f_1(x') > 0 \textit{ for all } x' \leq x\} \tag{4.3}$$

Then $f_1(x) < f_2(x)$ *for all* $0 < x \leq \bar{x}$.

Lemma 4.1 is the composition of Theorems 2.1 and 2.2 of [2] which in turn were inspired by the corresponding results in [8, 9, 11]. Lemma 4.2 is Lemma 6.3 from [2].

Proof of Theorem 3.1. First, let us show that if the positive solution of (3.3) exists, then only the cases (3.4)–(3.5) are possible. Indeed, from (3.1)–(3.2) one gets that $H_i(1) \leq 1$ for all $i \in \mathcal{A}$. Moreover, if $\tilde{Q}_i\{\xi \in M : r^\xi > 1\} > 0$, then $H_i(1) < 1$. Now, if $\lambda_i = 1$ for all $i \in \mathcal{A}$, then, being j_0 from Condition R, we get $H_{j_0}(\lambda_{j_0}) = H_{j_0}(1) < 1 = \lambda_i$ for $i \in C(j_0)$, which contradicts (3.3). If there exist i_1, j_1 such that $\lambda_{i_1} < 1, \lambda_{j_1} \geq 1$, then, due to the irreducibility of the matrix (P_{ij}), there exist $i_2, j_2 \in C(i_2)$ such that $\lambda_{i_2} < 1, \lambda_{j_2} \geq 1$. In this case we get $H_{i_2}(\lambda_{i_2}) < H_{i_2}(1) \leq 1 \leq \lambda_{j_2}$, which again contradicts (3.3). The case when there exists i_3, j_3 such that $\lambda_{i_3} \leq 1, \lambda_{j_3} > 1$ is treated similarly. So, (3.4)–(3.5) cover all possible situations.

Let us prove the sufficient condition for $\mathsf{E}\nu$ to be finite. For this we show that the existence of solution of (3.3) implies the existence of function $f(x)$ satisfying (4.2). Define

$$f(n) = \prod_{i=1}^{n} \lambda_{\eta_i}$$

for $n \geq 1$, and $f(0) = 1$. It is straightforward to get from (3.1)–(3.3) that for this function the condition (4.2) is satisfied, thus proving the sufficiency in Theorem 3.1. Now, since the sequence $\boldsymbol{\eta}$ is recurrent, if $\lambda_i < 1$ for all $i \in \mathcal{A}$, then $f(x) \to 0$, and if $\lambda_i > 1$ for all $i \in \mathcal{A}$, then $f(x) \to \infty$, and we apply Lemma 4.1 again.

Now we prove the necessary condition for $\mathsf{E}\nu$ to be finite. From now on we assume that $\mathsf{E}\nu < \infty$ and that there is no solution of (3.3), to reach a contradiction. By Lemma 4.1 we know about the existence of a positive function satisfying (4.1). That function can be specified uniquely through the following sequence of quotients

$$\rho_x = \frac{f(x)}{f(x-1)}, \quad x = 1, 2, 3, \ldots$$

together with the value of $f(0)$.

Recalling how the quantities $p(x), s(x), q(x), r(x)$ were obtained from the random environment $(\boldsymbol{\eta}, \boldsymbol{\xi})$, we get from (3.1) and (4.1) that

$$\rho_{i+1} = h^{\xi_i}(\rho_i), \quad i = 1, 2, 3 \ldots, \tag{4.4}$$

so, given the environment, the sequence ρ_x (and, consequently, $f(x)$ up to a constant factor) is completely specified by ρ_1. Note that if we prove that no positive solution of (4.1) exists for $\rho_1 = +\infty$, then Lemma 4.2 would imply that no positive solution of (4.1) exists for any ρ_1.

To proceed, we need the following result about the properties of the functions H_i:

Lemma 4.3 (i) *For $i \in \mathcal{A}$ the function $H_i(\cdot)$ is increasing and continuous on $\mathcal{D}_i := \{\lambda : H_i(\lambda) > -\infty\}$. Moreover, there are only three alternatives: $\mathcal{D}_i = (0, \infty)$, $\mathcal{D}_i = \emptyset$, or $\mathcal{D}_i = [\alpha_i, \infty)$ for some $\alpha_i > 0$.*

(ii) *For all $i \in \mathcal{A}$ there exists $b_i > 0$ such that $H_i(\lambda) < 0$ for $\lambda < b_i$.*

The proof of this result uses only elementary calculus, and is therefore omitted.

Let $\mathcal{D} = \mathcal{A} \times (\mathbb{R} \cup \{-\infty, +\infty\})$ and fix any $a_1 \in \mathcal{A}$. Consider a sequence of sets $B_1, B_2, B_3, \ldots \subset \mathcal{D}$ constructed in the following way: $B_1 = \{(a_1, +\infty)\}$ and, inductively, if $B_j = \{(i_1, \beta_1), \ldots, (i_m, \beta_m)\}$, then

$$B_{j+1} = \bigcup_{n=1}^{m} C(i_n) \times \{H_{i_n}(\beta_n)\}.$$

Let $B = \cup_{i=1}^{\infty} B_i \subset \mathcal{D}$. Denote $\lambda_i = \inf\{\beta : (i, \beta) \in B\}$, $i \in \mathcal{A}$. From the irreducibility of (P_{ij}) and Lemma 4.3 (ii) we get that either $\lambda_i = -\infty$ for all $i \in \mathcal{A}$, or $\lambda_i \geq b_i$ for all $i \in \mathcal{A}$. In the latter case, due to Lemma 4.3 (i), $(\lambda_i, i \in \mathcal{A})$ provide the solution of (3.3).

Say that an element of $\mathcal{D}$ is negative if its second coordinate is negative. Denote

$$N = \min\{i : \text{ the set } B_i \text{ contains a negative element}\}.$$

Then, from the construction of the sequence B_i, it follows that if there is no solution to (3.3), then the following holds: There exist a sequence $i_1 = a_1, i_2, \ldots, i_{N-1} \in \mathcal{A}$ such that $i_{n+1} \in C(i_n)$, $n = 1, \ldots, N-2$, which implies that for that sequence

$$\mathbf{P}\{\eta_2 = i_2, \ldots, \eta_{N-1} = i_{N-1} \mid \eta_1 = 1\} > 0, \tag{4.5}$$

in addition

$$H_{i_{N-1}} \circ \cdots \circ H_{i_1}(+\infty) < 0. \tag{4.6}$$

Denote also $\beta_n = H_{i_{n-1}} \circ \cdots \circ H_{i_1}(+\infty)$. Fix some $\varepsilon > 0$. For $1 \leq n \leq N-1$ define

$$A_n^{(\varepsilon)} = \{\xi \in M : h^{\xi}(\beta_n) \leq H_{i_n}(\beta_n) + \varepsilon\}.$$

Due to the definition of H_i and (4.5), we have that

$$\mathbf{P}\{\xi_n \in A_n^{(\varepsilon)}, n = 2, \ldots, N-1 \mid \eta_1 = 1\} > 0. \tag{4.7}$$

Now we go back to the sequence (ρ_i) constructed at (4.4) with $\rho_1 = +\infty$. From (4.6) and Lemma 4.3 one gets that if $\xi_n \in A_n^{(\varepsilon)}$, $n = 1, \ldots, N-1$, then $\rho_N < 0$ for ε small enough. Using Lemma 4.2, we see that to guarantee the non-existence of solution of (4.1) it is enough to find some τ such that $\xi_{\tau+n} \in A_n^{(\varepsilon)}$, $n = 1, \ldots, N-1$. The construction of the random environment together with (4.7) guarantees that τ is finite $\mathbf{P}$-almost surely.

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