Frogs in random environment

S.Yu. Popov*

July 31, 2000

Department of Statistics, Institute of Mathematics and Statistics, University of São Paulo, Rua de Matão 1010, CEP 05508–900, São Paulo SP, Brazil
AND
Institute for Problems of Information Transmission of Russian Academy of Sciences, Moscow, Russia
e-mail: `popov@ime.usp.br`

Abstract

We study the so-called frog model: Initially there are some "sleeping" particles and one "active" particle. A sleeping particle is activated when an active particle hits it, after that the activated particle starts to walk independently of everything and can activate other sleeping particles as well. The initial configuration of sleeping particles is random with density $p(x)$. We identify the critical rate of decay of $p(x)$ separating transience from recurrence, and study some other properties of the model.

Keywords: random environment, frogs, recurrence, transience

1 Introduction and results

In this note we study the problem of recurrence/transience for random motions in random media. Apparently, F. Solomon in [10] was the first to treat

*The author is thankful to FAPESP (97/12826–6) for financial support

the problems of such kind: He studied one-dimensional random walk in random environment. The model of [10] has been studied extensively since then; see [6] and references therein. Some attention was given to random walks in random environment in more complex spaces, e.g. trees (see [9] and references therein), words from a finite alphabet [1], $\mathbb{Z}^d$ (but in random environment not of Solomon's type) [3, 5]. Branching random walks (which sometimes exhibit different mechanisms of transience/recurrence) in random environment are studied in [2] (one-dimensional random environment of Solomon's type) and in [4] (many-dimensional random environment similar to [5]). As opposed to the cited papers, here the random motion in question is, in some sense, something in the frontier between random walk and branching random walk (cf. Lemma 2.2 in Section 2.1 below). Let us describe the model.

At time 0, there is an "active" particle at site $0 \in \mathbb{Z}^d$, $d \geq 3$, and also an infinite number of "sleeping" particles in other sites of $\mathbb{Z}^d$ (note that in some sites there may be more than one particle, and in some sites there may be no particles at all). The active particle starts to perform a discrete-time simple random walk (SRW), i.e. at each unit of time it jumps to one of its $2d$ neighbors chosen with uniform probability. When an active particle enters a site which contains some sleeping particles, all those become active and start to perform SRW independently. There is no interaction between active particles. The folkloric name for this kind of model is "frog model"; currently some attention is given on how to prove a Richardson-like shape theorem in the case when initially any $x \neq 0$ contains a sleeping particle, and what kind of limiting shape there will be.

In this paper, however, we will study properties of another kind. It is well known that SRW is transient in dimension $d \geq 3$. Given a collection of numbers $(0 \leq p(x) \leq 1, x \in \mathbb{Z}^d \setminus \{0\})$, $d \geq 3$, put a sleeping particle into site x with probability $p(x)$. The configuration of sleeping particles is then fixed (the case of *quenched* random environment), and the process is started. Note that in [4] the random environment is of the same type. Now, what happens when the particle on its course from 0 to infinity activates other particles thus increasing the number of independent random walkers? It is intuitively clear that if the sleeping particles are dense enough, then the model can become recurrent, where by recurrence we mean the following:

Definition 1.1 *The frog model is called recurrent, if the site* 0 *is visited infinitely often a.s.*

Remark 1.1 *By Kolmogorov's zero-one law recurrence either holds for almost all initial configurations of sleeping particles or for almost no initial configuration of sleeping particles. Also, it is not difficult to verify that the recurrence is equivalent to any of the following conditions:*

- *every site $x \in \mathbb{Z}^d$ is hit a.s.*
- *any sleeping particle will be activated a.s.*

So, the main question which we study here is: How can one distinguish transience from recurrence by looking at the density $p(x)$? The next theorem shows that the critical (i.e. separating transience from recurrence) rate of decay of the function $p(x)$ is $\|x\|^{-2}$:

Theorem 1.1 *There exists $\alpha_{cr} = \alpha_{cr}(d)$, $0 < \alpha_{cr} < \infty$, such that*

(i) *if $\alpha < \alpha_{cr}$ and $p(x) \leq \alpha\|x\|^{-2}$ for all x large enough, then the process is transient;*

(ii) *if $\alpha > \alpha_{cr}$ and $p(x) \geq \alpha\|x\|^{-2}$ for all x large enough, then the process is recurrent.*

Notational convention: For what follows, $\mathbf{P}$ means probability with respect to the random environment (i.e. the initial configuration of sleeping particles), while P and E stand for the probability and expectation related to the random walk(s) (after the environment is fixed).

Let

$$A = \{\text{the site } 0 \text{ is visited infinitely often}\}, \tag{1.1}$$

so the transience in fact means that $\mathsf{P}(A) < 1$. The natural question to ask is whether this implies that $\mathsf{P}(A) = 0$, as happens for countable Markov chains. Our conjecture is that the above is true, but for now we can only prove a weaker result:

Theorem 1.2 *There exists $0 < \alpha' \leq \alpha_{cr}$ such that if $p(x) \leq \alpha' \|x\|^{-2}$ for all x large enough, then $\mathsf{P}(A) = 0$.*

To illustrate the difficulties arising when one attempts to prove the above conjecture, we consider the *modified frog model*, and show that for it the situation $0 < \mathsf{P}(A) < 1$ becomes possible. Fix some integer-valued function $N(x)$ such that $N(x) \geq 1$ for all x. The construction of the initial configuration of sleeping particles is the following: a site $x \neq 0$ contains $N(x)$ particles with probability $p(x)$ and is empty with probability $1 - p(x)$. Note that the model which we introduced first is a particular case of modified frog model with $N(x) \equiv 1$.

Theorem 1.3 *For the modified frog model the following holds:*

(i) *there exists β_1 such that if $N(x)p(x) \leq \beta_1 \|x\|^{-2}$ for all x large enough, then $\mathsf{P}(A) = 0$;*

(ii) *there exists β_2, such that if $p(x) \geq \beta_2' \|x\|^{-2-\sigma}$, $N(x) \geq \beta_2'' \|x\|^{\sigma}$ for some $\beta_2', \beta_2'' > 0 : \beta_2'\beta_2'' \geq \beta_2$, $0 < \sigma < d - 2$ and for all x large enough, then $0 < \mathsf{P}(A) < 1$.*

2 Proofs

2.1 Branching random walk and proof of Theorems 1.2 and 1.3 (i)

The main idea of the proof is to dominate the (modified) frog model by a branching random walk (BRW). Let us first describe the latter model. Start with one particle at the origin; it performs a SRW in $\mathbb{Z}^d$ and at each site x produces some new particles according to some probability distribution which depends on x. Those newly born particles as well as their ancestor then jump and produce their offsprings independently. Note that the particles never die. Let $\mu(x) < \infty$ be the mean number of newly born offsprings at site x. The fact we need about this d-dimensional BRW is the following:

Lemma 2.1 *There exist $H, h > 0$ such that if $\mu(x) = 0$ on $\{x : \|x\| < H\}$ and*

$$\mu(x) \leq h\|x\|^{-2} \tag{2.1}$$

on $\{x : \|x\| \geq H\}$, then BRW is strongly transient. This means that if we turn the origin into absorbing state, then the mean number of particles which enter the origin is finite and less than one. This in turn imply that if there is no absorption in 0, then the total number of visits to the origin is finite a.s.

Proof. This fact follows from Theorem 5.1 of [8] together with the remark that the condition on variance of offspring distribution is really nonessential for the proof of the transience. □

Now, given the functions $p(x), N(x)$, let us define BRW(p,N) in the following way: At site x a particle creates nothing with probability $1 - p(x)$, and creates $N(x)$ new particles with probability $p(x)$, so $\mu(x) = N(x)p(x)$. To proceed, we need the following

Lemma 2.2 *The (modified) frog model is dominated by BRW(p,N).*

Proof. Instead of constructing a formal coupling of frogs and BRW(p,N), we prefer to give a verbal explanation. The idea is the following: in the frog model one can postpone the decision about whether to have a sleeping particle(s) in x until the first moment when some active particle hits x. After that first hit, we are prohibited to put more sleeping particles into x. The latter restriction is

absent in the BRW context, which makes evident the statement of Lemma 2.2.
□

The proof of Theorems 1.2 and 1.3 (i) now follows from Lemmas 2.1 and 2.2 and a simple observation that in the frog model finite changes of the configuration of sleeping particles do not affect the fact $\mathsf{P}(A) = 0$. □

2.2 Green's function and hitting probabilities

Let ξ_n be a SRW in $\mathbb{Z}^d$, $d \geq 3$, and

$$g(x,y) = \sum_{n=0}^{\infty} \mathsf{P}\{\xi_n = y \mid \xi_0 = x\}$$

denotes its Green's function. It is known (cf. [7, 11]) that

$$g(x,y) = g(y,x) = g(0, x-y) = \frac{\gamma_d}{\|x-y\|^{d-2}} + O(\|x-y\|^{-d}), \tag{2.2}$$

for some $\gamma_d > 0$. For $x, y, z \in \mathbb{Z}^d$ denote

$$\begin{aligned} q_x(y) &= \mathsf{P}\{\text{there exists } n \geq 0 : \xi_n = y \mid \xi_0 = x\}, \\ q_x(y,z) &= \mathsf{P}\{\text{there exists } n \geq 0 : \xi_n \in \{y,z\} \mid \xi_0 = x\}. \end{aligned}$$

It is straightforward to get (cf., for example, Lemma 2 of [4]) that

$$q_x(y) = \frac{g(x,y)}{g(0,0)}, \qquad q_x(y,z) = \frac{g(x,y) + g(x,z)}{g(0,0) + g(y,z)}. \tag{2.3}$$

When $\|y - z\| \to \infty$, one gets from (2.2)–(2.3) that

$$q_x(y,z) = (q_x(y) + q_x(z))\Big(1 - \frac{\gamma'_d}{\|y-z\|^{d-2}} + O(\Phi)\Big), \tag{2.4}$$

where $\gamma'_d = \gamma_d / g(0,0)$, $\Phi = \|y-z\|^{-\min\{d;2d-4\}}$.

2.3 Proof of Theorems 1.1 and 1.3 (ii)

Proof of Theorem 1.1. From a coupling argument it is easy to check the monotone property: If $p \leq p'$ then the frog model with p is less recurrent than the frog model with p'. Having in mind Theorem 1.2, one gets that it is sufficient to prove the following: If α is large enough and $p(x) \geq \alpha \|x\|^{-2}$ for all x large enough, than the process is recurrent (i.e. $\mathsf{P}(A) = 1$ $\mathbf{P}$-a.s.).

Before going into details, let us make an important observation. What matter for recurrence/transience are only the *trajectories* of the particles; being the trajectory fixed, one can put there any local time counting (i.e. one can "delay" or "accelerate" the particle at will) not affecting the recurrence/transience. In particular, if several active particles eventually passed through the place occupied at time 0 by a sleeping particle, then it does not matter which active particle was the first to pass there. So, if initially there was a sleeping particle in site x, and we know that some active particle hit x, we say that x was *activated* by this active particle, even if it was not the first to pass through x.

In the course of the proof of this theorem we will use some technique from [4]. Denote

$$F = \{x \in \mathbb{Z}^d : \text{at time 0 the site } x \text{ contains a sleeping point}\}.$$

Of course, F is a random set, and, since its density decays like $\|x\|^{-2}$, Theorem 3.1 (a) of [5] implies that it is *trapping*, i.e. with probability 1 a SRW hits F infinitely often. Define for $n \geq 0$

$$\begin{aligned} W_n &= \{x \in \mathbb{Z}^d : 3(2^{n-1}) \leq \|x\| < 2^{n+1}\}, \\ V_n &= \{x \in \mathbb{Z}^d : \|x\| < 2^n\}, \end{aligned}$$

and $F_n = F \cap W_n$.

Let us suppose that at some moment there are M active particles located in $x_1, \dots, x_M \in \mathbb{Z}^d$, and that none of those particles ever went out of the set V_{n_0} for some n_0. Suppose also that $M \leq 2^{(d-2)(n_0+1)}$. Define $B_0 = \{x_1, \dots, x_M\}$, $D_0 = \emptyset$. We will make an attempt to construct a sequence of random sets $B_i, D_i \subset F_{n_0+i-1}$ such that $B_i \cap D_i = \emptyset$, $|B_i| = 2^{(d-2)i}M$, $|D_i| = 2^{(d-2)(i-1)}M$, $i \geq 1$, and all particles from B_i, D_i are activated by the particles from B_{i-1}. The construction is described as follows: Suppose that the sets B_j, D_j, $0 \leq j \leq i$ are *successfully* (we describe the meaning of this below) constructed. Abbreviate

$K = |B_i| = 2^{(d-2)i}M$ and $m = 2^{n_0+i+1}$. Note that $K \le m^{d-2}$ and

$$\frac{m}{4} \le \|x - y\| \le \frac{3m}{2} \tag{2.5}$$

for any $x \in B_i$, $y \in F_{n_0+i}$. For all $y \in F_{n_0+i}$ let ζ_y be the indicator of the following event:

$$\{\text{at least one active particle starting from } B_i \text{ eventually hits } y\}.$$

For $i \ge 0$ denote

$$G_i^{M,n_0} = \Big\{ \sum_{y \in F_{n_0+i}} \zeta_y \ge (2^{d-2}+1)K \Big\}$$

We call ith inductive step *successful* if the event G_i happens. In this case

$$|\{y \in F_{n_0+i} : \zeta_y = 1\}| \ge 2^{d-2}K + K,$$

so we can pick $2^{d-2}K + K$ elements from this set to form the sets B_{i+1}, D_{i+1} with $|B_{i+1}| = 2^{d-2}K$, $|D_{i+1}| = K$.

Before going further, let us make the following remark. Suppose that initially there were sleeping particles in all the points of two disjoint sets $\tilde{A}, \tilde{B}$, and that we know that for any $\tilde{x} \in \tilde{A}$ there exists $\tilde{y} \in \tilde{B}$ such that $\tilde{x}$ was activated by the particle originating from $\tilde{y}$. Then all the random walkers originating from $\tilde{A}$ are independent; this is justified by the fact that the only interaction permitted in this model is when an active particle hits a sleeping particle. Now, since by construction all the particles from B_i are activated by "external" particles (from B_{i-1}), for what follows one can suppose that all the random walkers starting from B_i are independent. Using (2.2)–(2.3) and (2.5), we have

$$\begin{aligned} \mathsf{E}\zeta_y = \mathsf{P}\{\zeta_y = 1\} &= 1 - \prod_{x \in B_i} (1 - q_x(y)) \\ &\ge 1 - \Big(1 - \frac{2\gamma_d'}{3m^{d-2}} + O(m^{-d})\Big)^K \ge \frac{C_1 K}{m^{d-2}} \end{aligned} \tag{2.6}$$

for some C_1.

For some $b > 0$ (which will be chosen later) we consider a partition of $\mathbb{Z}^d$ into cubes of size $bm^{2/d}$:

$$Z_{i_1 \ldots i_d} = \mathbb{Z}^d \cap bm^{2/d}\{[i_1 - 1, i_1) \times \cdots \times [i_d - 1, i_d)\}.$$

Consider now those cubes $Z_{i_1 \ldots i_d}$ with *even* coordinates $i_1, \ldots, i_d$ that lie fully inside the set W_{n_0+i}. The number of such cubes is equal to $L \simeq C_2 b^{-d} m^{d-2}$, with $C_2 = C_2(d)$. Denote these cubes by $Z'_1, \ldots, Z'_L$. Note that if $y \in Z'_{j_1}$, $z \in Z'_{j_2}$, $j_1 \neq j_2$, then $\|y - z\| \geq bm^{2/d}$. Let ν_j be the number of points from F_{n_0+i} that lie in the cube Z'_j. Because $p(x) \geq \alpha/\|x\|^2$ and $\|x\| \leq m$, we have

$$\begin{aligned}
\mathbf{P}\{\nu_i = 0\} &\leq \prod_{x \in Z'_i} \left(1 - \frac{\alpha}{\|x\|^2}\right) \\
&\leq \left(1 - \frac{\alpha}{m^2}\right)^{b^d m^2} \\
&\simeq \exp\{-\alpha b^d\}. \qquad (2.7)
\end{aligned}$$

Define a subset $F'_{n_0+i} \subset F_{n_0+i}$ in the following way. For $1 \leq j \leq L$ with $\nu_j \geq 1$ we pick an arbitrary point $x'_j \in F_{n_0+i} \cap Z'_j$ and put

$$F'_{n_0+i} = \bigcup_{j=1}^{L} \{x'_j\},$$

i.e. we simply remove some points from F_{n_0+i} until each cube Z'_i contains 0 or 1 point from F_{n_0+i}. Note that from (2.7) and the strong law of large numbers it follows that for any $\varepsilon > 0$ the proportion of cubes containing no point from F_{n_0+i} is at most $\theta := \exp\{-\alpha b^d\} + \varepsilon$ for n_0 large enough $\mathbf{P}$-a.s. So, $L \geq |F'_{n_0+i}| \geq L' := (1 - \theta)L = C_3 m^{d-2}$, where

$$C_3 = C_2 b^{-d}(1 - \exp\{-\alpha b^d\} - \varepsilon)$$

can be made arbitrarily large by choosing ε small, α large, and $b = \alpha^{-1/d}$. Let us choose those parameters in such a way that $C_1 C_3 > 2^{d-2} + 1$.

Now, using (2.6) and Chebyshev inequality, we have

$$\begin{aligned}
&\mathsf{P}\Big\{\sum_{y\in F'_{n_0+i}}\zeta_y<(2^{d-2}+1)K\Big\}\\
&\qquad\le\ \mathsf{P}\Big\{\sum_{y\in F'_{n_0+i}}(\zeta_y-\mathsf{E}\zeta_y)<-(C_1C_3-2^{d-2}-1)K\Big\}\\
&\qquad\le\ \frac{C_4}{K^2}\Big(\sum_{y\in F'_{n_0+i}}\mathrm{Var}\,\zeta_y+\sum_{\substack{y,z\in F'_{n_0+i}\\ y\neq z}}\mathrm{cov}\,(\zeta_y,\zeta_z)\Big) \qquad (2.8)
\end{aligned}$$

for some $C_4>0$.

Analogously to (2.6) one gets that for any $y\in F'_{n_0+i}$

$$\mathsf{E}\zeta_y\le 1-\Big(1-\frac{4\gamma'_d}{m^{d-2}}+O(m^{-d})\Big)^K\le\frac{C_5K}{m^{d-2}}, \qquad (2.9)$$

so, by (2.9),

$$\mathrm{Var}\,\zeta_y\le\mathsf{E}\zeta_y^2=\mathsf{E}\zeta_y\le\frac{C_5K}{m^{d-2}}. \qquad (2.10)$$

Using (2.4), after elementary calculations one gets

$$\begin{aligned}
\mathrm{cov}\,(\zeta_y,\zeta_z)\ &=\ \mathsf{P}\{\zeta_y=0,\zeta_z=0\}-\mathsf{P}\{\zeta_y=0\}\mathsf{P}\{\zeta_z=0\}\\
&=\ \prod_{x\in B_i}\Big[1-(q_x(y)+q_x(z))\Big(1-\frac{\gamma'_d}{\|y-z\|^{d-2}}+O(\Phi)\Big)\Big]\\
&\qquad-\prod_{x\in B_i}(1-q_x(y))(1-q_x(z))\le\frac{C_6}{m^{d-2}\|y-z\|^{d-2}}. \qquad (2.11)
\end{aligned}$$

Note that if z,u are from the same cube, then $\|y-z\|^{d-2}\ge 2^{-(d-2)}\|y-u\|^{d-2}$, so (supposing without restricting of generality that y is from Z'_1)

$$\begin{aligned}
\sum_{z\in F_{n_0+i}}\frac{1}{\|y-z\|^{d-2}}\ &\le\ \sum_{k=2}^{L}\frac{1}{|Z'_k|}\sum_{u\in Z'_k}\frac{2^{d-2}}{\|y-u\|^{d-2}}\\
&\le\ \frac{2^{d-2}}{b^dm^2}\sum_{0<\|v\|\le 2m}\frac{1}{\|v\|^{d-2}}\simeq C_7b^{-d} \qquad (2.12)
\end{aligned}$$

with $C_7=C_7(d)$. Inserting (2.10), (2.11) and (2.12) into (2.8), one gets that

$$\mathsf{P}(G_i^{M,n_0}\mid G_0^{M,n_0},\dots,G_{i-1}^{M,n_0})\ge 1-\frac{C_8}{K}=1-\frac{C_8}{M2^{(d-2)i}},$$

with $C_8 = C_8(d)$, so

$$\mathsf{P}\Big(\bigcap_{i=0}^{\infty} G_i^{M,n_0}\Big) \geq 1 - \frac{C_8}{M}\sum_{i=0}^{\infty}\frac{1}{2^{(d-2)i}} = 1 - \frac{C_8}{M(1-2^{-(d-2)})}. \tag{2.13}$$

Consider now the sets D_i, $i \geq 1$. All the particles from there are independent and

$$\sum_{i=1}^{\infty}\sum_{x\in D_i} q_x(0) \geq \sum_{i=1}^{\infty}\frac{2^{(d-2)(i-1)}M}{2^{(d-2)(n_0+i+1)}} = \infty,$$

so, by Borel-Cantelli,

$$\mathsf{P}\Big(A \mid \bigcap_{i=0}^{\infty} G_i^{M,n_0}\Big) = 1. \tag{2.14}$$

Since F is trapping, M can be made arbitrarily large, so (2.13)–(2.14) imply that $\mathsf{P}(A) = 1$. $\square$

Proof of Theorem 1.3 (ii). First, let us show that $\mathsf{P}(A) < 1$. Theorem 3.1 (b) of [5] implies that, subject to finite changes of the environment, with positive probability the initial active particle will not activate anything. As such changes do not affect the fact $\mathsf{P}(A) < 1$, the proof follows.

Let us now prove that $\mathsf{P}(A) > 0$. This fact can be proved using ideas similar to those of the proof of Theorem 1.1 (ii), so we give only an outline of the proof without going into details.

1. Keep the notation F, W_n, V_n, F_n from the proof of Theorem 1.1 (ii). Note that the condition $\sigma < d-2$ together with Borel-Cantelli implies that there is an infinite number of sleeping particles in $\mathbb{Z}^d$. For any $U \subset \mathbb{Z}^d$ denote by $\mathcal{N}(U)$ the number of sleeping particles in U, i.e.

$$\mathcal{N}(U) = \sum_{x\in U\cap F} N(x).$$

2. Abbreviate $m = 2^{n+1}$. Divide the space into the cubes with side $bm^{(2+\sigma)/d}$ and consider those which lie in W_n and have even coordinates. Each cube has (at least) Poisson number of points from F and the number of cubes is of order $m^{d-2-\sigma}$. Define the set F_n' analogously; one can prove that

$$|F_n'| \geq C_1' m^{d-2-\sigma} \tag{2.15}$$

and

$$\mathcal{N}(F'_n) \geq C'_2 m^{d-2} \tag{2.16}$$

for all n large enough $\mathbf{P}$-a.s., and C'_2 can be made as large as we want by choosing β_2 large.

3. Choose β_2 such that $C'_2 > 2^{d-2}$ and suppose that for a given realization of the random environment (2.15)–(2.16) hold for all $n \geq n_0 - 1$. As $F'_n \subset W_n \subset V_{n+1}$, one has $\mathcal{N}(V_{n_0}) \geq 2^{(d-2)(n_0+1)}$. Suppose now that *all* the sleeping particles from V_{n_0} were activated by the initial particle (clearly, this happens with positive probability).

Denote $B_0 := F \cap V_{n_0}$, $D_0 := \emptyset$. We are going to construct a sequence of disjoint sets $B_i, D_i \subset F'_{n_0+i-1}$ such that $\mathcal{N}(B_i) \geq 2^{(d-2)(n_0+i+1)}$, $\mathcal{N}(D_i) \geq 2^{(d-2)(n_0+i)}$, and all the particles from B_i, D_i are activated by the particles from B_{i-1}. Now we need to show that *with positive probability* such construction will be successful.

4. To this end, suppose that the sets B_j, D_j, $0 \leq j \leq i$ are successfully constructed. Define (recall the definition of ζ_y)

$$\tilde{G}_i = \Big\{ \sum_{y \in F'_{n_0+i}} \zeta_y \geq \frac{(2^{d-2}+1)2^{(d-2-\sigma)(n_0+i+1)}}{\beta''_2 2^\sigma} \Big\}.$$

Since $N(x) \geq \beta''_2 2^{\sigma(n_0+i)}$ when $x \in F'_{n_0+i}$, we have that

$$\mathcal{N}(\{y \in F'_{n_0+i} : \zeta_y = 1\}) \geq (2^{d-2}+1)2^{(d-2)(n_0+i+1)},$$

so one can form the sets B_i, D_i when $\tilde{G}_i$ happens.

Analogously to (2.8)–(2.12), one can prove that

$$\mathsf{P}(\tilde{G}_i \mid \tilde{G}_0, \dots, \tilde{G}_{i-1}) \geq 1 - C'_3 2^{-(d-2)(n_0+i+1)},$$

so $\mathsf{P}(\bigcap_{i=0}^\infty \tilde{G}_i) > 0$. Considering now the sets D_i, one gets that $\mathsf{P}(A \mid \bigcap_{i=0}^\infty \tilde{G}_i) = 1$, so $\mathsf{P}(A) > 0$. $\square$

References

[1] F. Comets, M.V. Menshikov and S.Yu. Popov (1998) Lyapunov functions for random walks and strings in random environment. *Ann. Probab.* **26** (4), 1433–1445.

[2] F. Comets, M.V. Menshikov and S.Yu. Popov (1998) One-dimensional branching random walk in random environment: a classification. *Markov Processes Relat. Fields* **4** (4), 465–477.

[3] G.R. Grimmett, M.V. Menshikov and S.E. Volkov (1996) Random walks in random labyrinths. *Markov Processes Relat. Fields* **2** (1), 69–86.

[4] F. den Hollander, M.V. Menshikov and S.Yu. Popov (1999) A note on transience versus recurrence for a branching random walk in random environment. *J. Statist. Phys.* **95** (3/4), 587–614.

[5] F. den Hollander, M.V. Menshikov and S.E. Volkov (1995) Two problems about random walks in a random field of traps. *Markov Processes Relat. Fields* **1** (2), 185–202.

[6] Y. Hu and Z. Shi (1998) The limits of Sinai's simple random walk in random environment. *Ann. Probab.* **26** (4), 1477–1521.

[7] G.F. Lawler (1991) *Intersections of random walks.* Birkhäuser Boston.

[8] M.V. Menshikov and S.E. Volkov (1997) Branching Markov chains: qualitative characteristics. *Markov Processes Relat. Fields* **3** (2), 225–241.

[9] R. Pemantle and Y. Peres (1995) Critical random walk in random environment on trees. *Ann. Probab.* **23** (1), 105–140.

[10] F. Solomon (1975) Random walks in a random environment. *Ann. Probab.* **3** (1), 1–31.

[11] F. Spitzer (1976) *Principles of Random Walk.* 2nd ed. Springer, New York.