One-dimensional Branching Random Walk in a Random Environment: a Classification

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Abstract

We consider a supercritical branching random walk on $\mathbf{Z}_+$ in a one-dimensional independent, homogeneous random environment, which enters both the branching mechanism and the step transition. We introduce an appropriate classification and we prove some transience and recurrence criteria.

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Introduction. In this paper we consider a random walk with branching but no death in a random environment. Similar models have been studied as infinite systems ([5] for instance), or continuous limits of large systems (e.g. [3, 4]). Instead we start here with one particle, which opens quite different perspectives. Except for the random, homogeneous branching, our model is the classical one-dimensional random walk in random environment, a well understood model at the level of classification (recurrence, transience, ...) [9]: we address here similar questions when branching occurs. Such questions for branching random walks (b.r.w.) are considered in [6, 7, 8], but

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without random environment; a martingale method (method of Lyapunov functions) was developed there, and also for the classical random walk in random environment [2]. The method of Lyapunov functions is our approach here again. Common wisdom suggests that statements of the form

$$\mathbf{P}\{\omega : \text{the b.r.w. in the medium } \omega \text{ is recurrent}\} = 0 \text{ or } 1$$

hold, they are part of our results below together with constructive criteria. Like in the classical branching models the behavior of the process depends on the expected number of offsprings, but on no other characteristics of the branching mechanism.

We define branching random walks in Section 1, then we introduce a natural classification for this model and we give some martingale criteria. The random environment is formally described in Section 3. The main results are formulated in Section 4, and the rest of the paper contains the proofs. We give various examples in Section 4, emphasizing in Example 2 the different behavior in random medium and in non-random one (homogeneous case or slightly inhomogeneous one).

1 Branching random walks

Let $\mathcal{L}$ be a time-homogeneous, irreducible discrete time Markov chain with a countable state space X and transition probabilities P_{xy}. For all $x \in X$ let $r_{(i)}(x)$, $i = 1, 2, \dots$ be an integer-valued distribution with finite first moment, i.e. a collection of positive numbers such that

$$\sum_{i=1}^{\infty} r_{(i)}(x) = 1, \qquad r(x) := \sum_{i=1}^{\infty} i r_{(i)}(x) < \infty \tag{1}$$

We define the branching random walk as follows. At time 0 we have one particle in x_0; when a particle is in x, it generates $i-1$ offsprings at x with probability $r_{(i)}(x)$. Particles never die. All these offsprings, and the particle that generated them, evolve later independently according to the Markov chain $\mathcal{L}$ and generate their offsprings as well. Clearly, this process is a Markov chain with a countable state space

$$X' = X \cup X^2 \cup X^3 \cup \dots$$

The state of the process at time t is a collection of coordinates $\tilde{x}(t) = \{x_1(t), x_2(t), \ldots, x_{N(t)}(t)\}$, where $x_i(t)$ is the location of ith particle and $N(t)$ is the total number of particles at time t.

2 Classification

Since particles do not die, except in the trivial case $N(t)$ tends to infinity as $t \to \infty$ almost surely, and the process, viewed as a Markov chain on X', is always transient. So we introduce a more appropriate definition of transience for the processes. We fix some point $0 \in X$, and we make it absorbing, i.e. we modify from now on $r_{(1)}(0) = 1$ and $P_{00} = 1$. Denoting by $N_x(t)$ the number of particles in state x at time t, we define the (possibly infinite) random variable ν as the a.s. increasing limit

$$\nu = \lim_{t \to \infty} N_0(t) \quad .$$

Definition 2.1 *We say that the b.r.w. is*

- *recurrent, if* $\mathsf{P}\{\nu > 0\} = 1$*;*
- *strongly recurrent, if it is recurrent and* $\mathsf{E}\nu < \infty$*;*
- *transient, if* $\mathsf{P}\{\nu > 0\} < 1$*;*
- *strongly transient, if* $\mathsf{P}\{\nu > 0\} < 1$ *and* $\mathsf{E}\nu < \infty$*.*

The position x_0 of the initial particle is not essential for this definition, at least inside the irreducible classes of the chain conditioned to stay in $X - \{0\}$. Therefore we will assume in addition that there is only one infinite irreducible class for the chain conditioned to $X - \{0\}$. This definition is not the only way to classify branching random walks, a different approach is developed in [8]. Theorems 2.1 and 2.2 give the (uncomplete) classification of the process. Recall $r(x)$ from (1).

Theorem 2.1 *For* $\mathsf{E}\nu$ *to be finite, it is necessary and sufficient that there exists a positive function* $f(x)$*,* $x \in X$*, such that for all* $x \neq 0$

$$\sum_y P_{xy} f(y) = \frac{f(x)}{r(x)} \, . \tag{2}$$

Theorem 2.2 *For* $\mathsf{E}\nu$ *to be finite, it is sufficient that there exists a positive function* $f(x)$*,* $x \in X$*, such that for all* $x \neq 0$

$$\sum_y P_{xy} f(y) \leq \frac{f(x)}{r(x)} \,. \tag{3}$$

Moreover, if $f(x) \to 0$ *as* $x \to \infty$*, then the b.r.w. is strongly transient. If* $f(x) \to \infty$ *as* $x \to \infty$*, then the b.r.w. is strongly recurrent.*

The equation (2) first appeared in [6], where it was used for classification of one-dimensional branching random walks. These two theorems in generalized forms can be found in [8]. To keep the paper self-contained we briefly sketch their proof and we refer to [8] for more detailed discussion.

Sketch of the proof of Theorems 2.1 *and* 2.2. First, let $\mathsf{E}\nu < \infty$. Take $f(x)$ to be equal to the mean number of particles absorbed in 0, provided that initially we had only one particle in x. Markov property implies that this function satisfies (2), thus proving the necessity in Theorem 2.1.

Supposing now that there exists a function $f(x)$ satisfying (3) we indicate how to prove that $\mathsf{E}\nu < \infty$. Consider a process $Q(t)$ defined as follows:

$$Q(t) = \sum_{i=1}^{N(t)} f(x_i(t)),$$

in the notations of Section 1. From (3) $Q(t)$ is a non-negative supermartingale and it converges almost surely as $t \to \infty$ to some random variable Q_∞. Using Fatou's lemma, one gets that

$$\mathsf{E}Q_\infty \leq \mathsf{E}Q(0) = f(x_0),$$

where x_0 is the position of the initial particle. On the other hand, for all t

$$Q(t) \geq N_0(t) f(0),$$

with the random variable $N_0(t)$. In the limit we have $\nu \leq Q_\infty / f(0)$, so

$$\mathsf{E}\nu \leq \frac{\mathsf{E}Q_\infty}{f(0)} \leq \frac{Q(0)}{f(0)} = \frac{f(x_0)}{f(0)} < \infty, \tag{4}$$

thus proving that (3) is sufficient for $\mathsf{E}\nu < \infty$.

When $f(x) \to 0$ we can take x_0 such that $f(x_0)/f(0) < 1$ and use (4) to obtain $\mathsf{P}\{\nu \geq 1\} < 1$.

Now, let $f(x) \to \infty$. Denote by τ the random moment when *all* the particles are in the absorbing state 0, i.e. when the b.r.w. completely stops. Suppose that with positive probability $\tau = \infty$: then at least one particle goes to infinity with positive probability, yielding a contradiction to $\mathsf{E}Q_\infty \leq \mathsf{E}Q(0) < \infty$. So $\tau < \infty$ a.s., and therefore the b.r.w. is recurrent. $\square$

3 Random environment

In this section we consider a branching random walk introduced in Section 1, but in a random environment. Fix some positive integer d — the maximal jump length, and define M to be the set of all infinite sequences $\xi = (p^\xi_{-d}, \ldots, p^\xi_d, r^\xi_{(1)}, r^\xi_{(2)}, \ldots)$ of non-negative numbers such that

$$\sum_{j=-d}^{d} p_j^\xi = 1 \ , \qquad \sum_{i=1}^{\infty} r^\xi_{(i)} = 1 \ , \qquad r^\xi := \sum_{i=1}^{\infty} i r^\xi_{(i)} < \infty \ ,$$

and which satisfy also the following

Condition P. $\sum_{i=1}^d p^\xi_{-i} > 0$ and $\sum_{i=1}^d p^\xi_i > 0$.

Let $\widetilde{Q}$ be some probability distribution on M such that the condition

Condition R. $\widetilde{Q}\{\xi \in M : \sum_{i \geq 1} i r^\xi_{(i)} = 1\} < 1$

holds. Under the product measure $\mathbf{P} = \widetilde{Q}^{\otimes \mathbf{Z}^+}$ the random sequence $\omega = \{\xi_1, \xi_2, \ldots\}$ is independent and identically distributed according to $\widetilde{Q}$. For any fixed realization of $\omega = \{\xi_x; x \geq 1\}$ we consider the branching random walk in the random environment ω, defined as the b.r.w. with state space $X = \{0, 1, 2, \ldots\}$, with $r_{(i)}(x) = r^{\xi_x}_{(i)}, i \geq 1$ and:

- if $x \geq d$, then $P_{x,x+j} = p_j^{\xi_x}$;
- if $0 < x < d$, then for j such that $x + j > 0$ we set again $P_{x,x+j} = p_j^{\xi_x}$, but

$$P_{x,0} = \sum_{j : x+j \leq 0} p_j^{\xi_x}.$$

We do not really need to define the dynamics starting at point 0 since we will make it absorbing in view of Definition 2.1. We are interested in properties of this b.r.w. which hold for $\mathbf{P}$-almost all ω. Note that the conditions P and R ensures the b.r.w. is a.s. nondegenerate.

4 Main results

We first need some definitions. For a sequence $\xi \in M$ let us define the characteristic polynomial,

$$h_\xi(\lambda) = \sum_{j=-d}^{d} p_j^\xi \lambda^j - \frac{1}{r^\xi},$$

which depends on $r_\cdot^\xi$ only through the mean $r^\xi = \sum_{i=1}^k i r_{(i)}^\xi$ as above. Let $\phi : M \to [0,1]^{2d+1} \times [1,\infty)$ be given by $\phi(\xi) = (p_{-d}^\xi, \dots, p_d^\xi, r^\xi)$, and $Q = \widetilde{Q} \circ \phi^{-1}$ the distribution of $\phi(\xi)$ on the range $\phi(M)$ of ϕ. Then, the set

$$K(\xi) = \{\lambda > 0 : h_\xi(\lambda) \leq 0\} \subset (0, +\infty)$$

depends only on $\phi(\xi)$ and a crucial ingredient will be the set

$$K_{\widetilde{Q}} = \bigcap_{\xi : \phi(\xi) \in \operatorname{supp} Q} K(\xi).$$

which depends only on the support of Q. Recall that we view Q as a distribution on $\phi(M)$, not on the whole $[0,1]^{2d+1} \times [1,\infty)$. In particular, $\operatorname{supp} Q$ is defined as the smallest closed (*relative* to $\phi(M)$) subset F of $\phi(M)$ such that $Q(F) = 1$; this implies that for all $(p_{-d}, \dots, p_d, r)$ belonging to the support of Q, the Condition P holds true. Our main results below deal with expectations and probabilities in a given environment, but for simplicity we will forget the dependence in ω in the notations $\mathsf{E}\nu, \mathsf{P}(\nu > 0), \dots$ Our criteria are given in terms of $K_{\widetilde{Q}}$, that is of $\operatorname{supp} Q$; in the light of branching theory in a fixed medium it is most natural that our criteria below involve Q but not $\widetilde{Q}$ itself.

Theorem 4.1 *Assume that Conditions R and P hold. Then* $K_{\widetilde{Q}} \neq \emptyset$ *implies* $\mathsf{E}\nu < \infty$ $\mathbf{P}$-*a.s.. Furthermore,*

i) *if* $K_{\widetilde{Q}} \subset [1, +\infty)$*, then the b.r.w. is strongly recurrent for* $\mathbf{P}$*-almost all environments* ω*;*

ii) *if* $K_{\widetilde{Q}} \subset (0, 1)$*, then* $\mathsf{P}\{\nu > 0\} < 1$ $\mathbf{P}$*-a.s., i.e. the b.r.w. is strongly transient for almost all environments.*

This classifies the behavior of the b.r.w. in the case $K_{\widetilde{Q}} \neq \emptyset$, as we will show in Lemma 6.1 below: if $K_{\widetilde{Q}} \neq \emptyset$, then we have either $K_{\widetilde{Q}} \subset (0, 1)$ or $K_{\widetilde{Q}} \subset (1, \infty)$. We now turn to the case when $K_{\widetilde{Q}} = \emptyset$.

Theorem 4.2 *Assume that Conditions R and P hold, that* $d = 1$ *and that* $K_{\widetilde{Q}} = \emptyset$*. Then* $\mathsf{E}\nu = \infty$ $\mathbf{P}$*-a.s.*

Remark. For the general b.r.w. introduced in Section 1, the property $\mathsf{E}\nu = \infty$ does not imply $\mathsf{P}\{\nu > 0\} = 1$. We give a counterexample after the proof of Theorem 4.3. Nevertheless, in the case of random (homogeneous) environment we have

Theorem 4.3 *Assume the condition P holds and* $\mathsf{E}\nu = \infty$ $\mathbf{P}$*-a.s. for the b.r.w. in a random environment (introduced in Section 3). Then we have* $\mathbf{P}$*-a.s.*

$$\mathsf{P}\{\nu > 0\} = 1, \tag{5}$$

i.e. the b.r.w. is recurrent for almost every environment.

Note that if $d = 1$, then these theorems give us the complete classification of the process. Let us now consider an important case, for which the criteria can be rewritten in a more explicit way.

Nearest neighbour b.r.w. in random environment with independent branching rates and step transitions. Let $d = 1$, $q := p_{-1}^{\xi} = 1 - p_1^{\xi}$, and assume that the r.v.'s $r = r^{\xi}$ and q are independent under Q. In this case

$$h_{\xi}(\lambda) = \lambda^{-1}\left[\lambda^2 + q(1 - \lambda^2) - \frac{\lambda}{r}\right]$$

so that $K(\xi) \neq \emptyset \iff 4q(1-q)r^2 < 1$, and then $K(\xi) \subset [1, +\infty)$ [resp., $K(\xi) \subset (0, 1]$] if $q \geq 1/2$ [resp., $q \leq 1/2$]. Observe that $h_{\xi}(\lambda)$ is increasing in r, and decreasing in q when $\lambda < 1$ but increasing in q when $\lambda > 1$. Therefore,

$$K(\xi') \cap (0, 1] \subset K(\xi) \cap (0, 1] \quad \text{if } q \leq q' \text{ and } r \leq r',$$

$$K(\xi') \cap [1, +\infty) \subset K(\xi) \cap [1, +\infty) \quad \text{if } q \geq q' \text{ and } r \leq r'.$$

For simplicity assume in addition that the support of the distribution of the r.v. q [resp., r] is some interval $[q^-, q^+] \subset [0, 1]$ [resp., $[r^-, r^+] \subset [1, \infty)$]. Then the support of the joint law of (q, r) is $[q^-, q^+] \times [r^-, r^+]$ by independence, and from our results we conclude that

- if $q^- > 1/2$ and $4q^-(1-q^-)(r^+)^2 < 1$ the b.r.w. is strongly recurrent
- if $q^+ < 1/2$ and $4q^+(1-q^+)(r^+)^2 < 1$ the b.r.w. is strongly transient
- otherwise the b.r.w. is recurrent (but not strongly)

under the Conditions R and P.

We give two precise examples to illustrate the specific character of b.r.w. in a random environment.

Example 1. Consider the classical random walk in random environment, i.e., the transitions from x may only occur to $x-1$ and $x+1$ with probabilities $q(x)$ and $1-q(x)$ for a sequence $q(x)$ of i.i.d.r.v.. Then the transience/recurrence of this r.w. in random environment depends on the sign of the expectation of $\log \frac{q(x)}{1-q(x)}$ [9]. When the branching is present (and satisfies the Condition R), the b.r.w. is always recurrent provided that $\mathbf{P}(q \geq 1/2) > 0$. Indeed this assumption rules out the second case above. Remarkably in this case, even though the original r.w. may have an *arbitrary large*, positive drift, any small amount of branching will make the b.r.w. recurrent. A pathwise picture is as follows: the condition $\mathbf{P}(q \geq 1/2) > 0$ means that large regions with non-positive drift occur in the environment, they slow down the walk, and the branching generates enough particles there to balance the large deviation for coming back to the origin.

Example 2. Let $d = 1$, and we have two types of states

1. $r_{(2)}(x) = 1$ (so $r(x) = 2$) and $p_{\pm 1}(x) = (2 \pm \sqrt{3})/4$;
2. $r_{(3)}(x) = 1$ (so $r(x) = 3$) and $p_{\pm 1}(x) = (3 \pm 2\sqrt{2})/6$.

In other words, the support of $\widetilde{Q}$ consists of two points:

$$\xi^{(1)} = \Big(\frac{2-\sqrt{3}}{4}, 0, \frac{2+\sqrt{3}}{4}, 0, 1, 0, \dots\Big), \quad \xi^{(2)} = \Big(\frac{3-2\sqrt{2}}{6}, 0, \frac{3+2\sqrt{2}}{6}, 0, 0, 1, 0, \dots\Big).$$

Then elementary calculations show that $K(\xi^{(1)}) = \{2-\sqrt{3}\}$, $K(\xi^{(2)}) = \{3-2\sqrt{2}\}$, so $K_{\widetilde{Q}} = K(\xi^{(1)}) \cap K(\xi^{(2)}) = \emptyset$.

If all the states are of the same type (1 or 2), then this spatially homogeneous b.r.w. is transient (see [6]). However, if we randomly "mix" these two types of states, i.e. $\widetilde{Q}\{\xi^{(1)}\} = 1 - \widetilde{Q}\{\xi^{(2)}\} \in (0,1)$, the b.r.w. in this random environment is recurrent (but not strongly recurrent).

5 Proof of Theorem 4.3

Proof of Theorem 4.3.

The main idea is to minorize the b.r.w. by some new process called *delayed* b.r.w. *with killing*. The new process being *less recurrent* than the initial process, we simply need to prove the new one *is* recurrent.

First we consider the original, irreducible b.r.w.; in particular 0 is no longer absorbing. In order to prove that the b.r.w. hits the state 0 it is sufficient from irreducibility to prove that there exists some state x which is visited infinitely often.

We now construct the delayed b.r.w. with killing, starting with one particle at site 1. Fix some arbitrary $\mu > 1$. Consider the set $U_1 = [l_1, l_1+d-1]$ for some arbitrary integer $l_1 > d$. Let τ_1 ($\tau_1 < \infty$ a.s.) be the first hitting time of U_1 and denote by $\eta_0^{(1)} > 0$ the number of particles which are in the set U_1 at time τ_1. Kill all the other particles. These $\eta_0^{(1)}$ remaining particles are allowed to move and to generate their offsprings with the following restriction: at the times $\tau_1+1, \ldots, \tau_1+k_1$ where the number k_1 will be chosen later, the set U_1 becomes absorbing (more precisely, delaying) after re-entering U_1 from the right: all the particles which jump to the right of U_1 and re-enter U_1 at that times become frozen up to time τ_1+k_1, i.e. they neither branch nor move.

Denote by $\eta_1^{(1)}$ the number of particles which are in the set U_1 at time τ_1+k_1. Since $\mathsf{E}\nu = \infty$ a.s. for any starting point for the ancestor, we can choose $k_1 = k_1(\omega)$ large enough ($k_1 < \infty$ a.s.) such that

$$\mathsf{E}(\eta_1^{(1)} \mid \mathcal{F}_{\tau_1}) > \mu\eta_0^{(1)} \tag{6}$$

where as usual $\mathcal{F}_{\tau_1}$ denotes the history of the process up to the stopping time τ_1. Indeed, a particle starting in U_1 has a positive probability to exit

U_1 jumping to the right from condition P, and then from our assumption, it has (for for almost every environment) an infinite expected number of descendents finally hitting U_1 from the right. Note that the estimate (6) does not depend on the locations of the $\eta_0^{(1)}$ particles inside U_1 at time τ_1. Then we repeat this procedure with these $\eta_1^{(1)}$ particles; the other ones — which do not exit U_1 first by the right or which are not in U_1 at time $\tau_1 + k_1$ — are called *escaped* particles, they are ignored for the moment but they will be used to construct the process $\eta_\cdot^{(2)}$ defined below. At time $\tau_1 + ik_1$ ($i = 0, 1, 2, \ldots$) we have $\eta_i^{(1)}$ particles lying in U_1, ignoring the ones which already escaped.

When $d = 1$ the process $\eta_\cdot^{(1)}$ is simply a Galton-Watson with mean number of offsprings larger than μ. But also in the general case, this non-Markovian process is related to a branching process. Recalling the exact location of the $\eta_i^{(1)}$ particles considered at time $\tau_1 + ik_1$, we denote by $\mathrm{Z}_i^{(1)}(x)$, $x = 1, \ldots, d$, the number of these particles located at site $l_1 + x - 1$ ($\sum_x \mathrm{Z}_i^{(1)}(x) = \eta_i^{(1)}$). Clearly the d-dimensional process $\mathrm{Z}_\cdot^{(1)}$ is a multi-type Galton-Watson process, and (6) shows it is supercritical ([1], Th.2, Ch.V). Therefore the probability of extinction θ_1 for the Galton-Watson process starting with one ancestor at the worst location x is strictly less than 1. In addition, θ_1 depends on the medium only through $(\xi_x, x \in V_1)$ with $V_1 = \{l_1, \ldots, l_1 + d(k_1 + 1)\}$. Finally, the process $\eta_\cdot^{(1)}$ dies out with probability smaller than θ_1.

Take now some $l_2 > l_1 + d(k_1 + 1)$, consider the set $U_2 = [l_2, l_2 + d - 1]$, the stopping time τ_2 ($\tau_2 < \infty$ a.s. if $\eta_\cdot^{(1)}$ dies out) when the first particles (a number $\eta_0^{(2)}$ of them) escaped from the process $\eta_\cdot^{(1)}$ enter the set U_2. In a similar manner we fix some $k_2 = k_2(\omega)$, finite with probability one, such that (6) holds for the process $\eta_\cdot^{(2)}$ constructed as above. This new process dies out with a probability strictly smaller than some random $\theta_2 < 1$ depending on ξ_x, $x \in V_2 = \{l_2, \ldots, l_2 + d(k_2 + 1)$; since V_1 and V_2 do not intersect, these variables are independent. The particles escaped from $\eta_\cdot^{(1)}$, but not in U_2 at time τ_2, will be killed at this time. Iterating the procedure we construct sequences of processes $\eta_\cdot^{(j)}$ and of independent variables $\theta_j < 1$, $j = 1, 2, \ldots$ such that $\mathsf{P}(\exists i : \eta_i^{(j)} = 0) < \theta_j$. Since the environment is stationary, the variables θ_j have the same distribution if we take the smallest possible k_j to ensure (6). From the law of large numbers we see that the processes $\eta_\cdot^{(j)}$ all finally die with probability $\prod_j \theta_j = 0$. Hence at least one of the processes $\eta^{(j)}$ will never stop, and at least one point is visited infinitely often by the delayed b.r.w. with killing. This clearly implies the same statement for the original b.r.w., and Theorem 4.3 is proved. □

To close this section we give a counterexample which shows that in general, $\mathsf{E}\nu = \infty$ does not imply $\mathsf{P}\{\nu > 0\} = 1$.

Example. Let $r_{(2)}(x) = 1$ (i.e. $r(x) = 2$) for all x, 0 be the absorbing state. We define the transition probabilities as follows: $p_{-1}(1) = p_1(1) = 1/8$, $p_0(1) = 3/4$, and $p_{\pm 1}(x) = (2 \pm \sqrt{3})/4$ for $x \geq 2$. This b.r.w. is transient, as spatially homogeneous b.r.w. with $r(x) = 2$ and $p_{\pm 1}(x) = (2 \pm \sqrt{3})/4$ is transient (see [7]). To prove that $\mathsf{E}\nu = \infty$, we denote by η_k the number of particles which are in the state 1 at moment k. It can be easily seen that η_k can be minorated by a Galton-Watson process with mean number of direct offsprings of the particle being equal to $3/2 > 1$. So with positive probability $\eta_k \to \infty$ as $k \to \infty$, and from this one gets that $\mathsf{E}\nu = \infty$.

In this example the inhomogeneity is present at the single point 1, though random (stationary) environments create an homogeneous landscape of inhomogeneities.

6 Proof of Theorems 4.1 and 4.2

Proof of Theorem 4.1. We start with a useful lemma.

Lemma 6.1 *Under Condition R, at least one of the two sets*

$$K_{\widetilde{Q}} \cap (0,1), \qquad K_{\widetilde{Q}} \cap (1,+\infty) \tag{7}$$

is empty.

Proof. Indeed, for $\lambda > 0$ we have $h''_\xi(\lambda) > 0$, so the equation $h_\xi(\lambda) = 0$ cannot have more than two solutions, so the set $K(\xi)$ is connected. Taking Condition R into account we see that with positive probability $h_\xi(1) = 1 - 1/r > 0$, so $K(\xi)$ cannot intersect both $(0,1)$ and $(1,\infty)$. This proves Lemma 6.1. □

Now we prove Theorem 4.1. For $\lambda_0 \in K_{\widetilde{Q}}$ the relation (3) holds with $f(x) = \lambda_0^x$, and therefore Theorem 4.1 follows directly from Theorem 2.2. □

Proof of Theorem 4.2. We show that no function $f(x)$ satisfying (2) can remain positive for all x. From Theorem 2.1 this implies the claim. We need the following

Lemma 6.2 *Assume $K_{\widetilde{Q}} = \emptyset$ and $d = 1$ in addition to Conditions R and P. Then there exist $\delta > 0$, $\sigma > 0$, a finite number of sets $G_l \subset M$, $l = 1, \ldots, n_0$, such that*

1. *For all $l = 1, \ldots, n_0$, $\widetilde{Q}(G_l) \geq \sigma$.*
2. *For any $\lambda_0 \in \mathbf{R}_+$ there exists l, $1 \leq l \leq n_0$, such that for all $\xi \in G_l$*
$$\frac{1}{p_1^\xi} h_\xi(\lambda_0) \geq \delta. \tag{8}$$

We denote this number l by $l(\lambda_0)$.

Proof. For $y \in \mathbf{R}^{2(d+1)}$ denote by $O_\varepsilon(y)$ the Euclidean ball with center y and radius ε. Take some arbitrary $\xi \in \phi^{-1}(\operatorname{supp} Q)$ and $\delta_1 > 0$. From Condition P we see that there exists $\lambda_1 > 0$ such that $\frac{1}{p_1^\xi} h_\xi(\lambda) > \delta_1$ for all $\lambda > \lambda_1$. Furthermore there exists by continuity $\varepsilon > 0$ such that

$$\frac{1}{p_1^{\xi'}} h_{\xi'}(\lambda) \geq \delta_1 \tag{9}$$

for all $\xi' \in \phi^{-1}(O_\varepsilon(\phi[\xi]))$ and $\lambda > \lambda_1$. Since $\phi(\xi) \in \operatorname{supp} Q$, we have $\sigma_1 := \widetilde{Q}(G_1) > 0$ with $G_1 = \phi^{-1}(O_\varepsilon(\phi[\xi]))$. In just the same way it can be proved that there exists $\lambda_2 > 0$ and $G_2 \subset M$ such that (8) holds for all $0 < \lambda < \lambda_2$ and all $\xi \in G_2$, and we let $\sigma_2 = \widetilde{Q}(G_2) > 0$. Now let $\lambda \in [\lambda_2, \lambda_1]$. Since $K_{\widetilde{Q}} = \emptyset$, there exists $\xi \in \phi^{-1}(\operatorname{supp} Q)$ and $\delta_\lambda > 0$ such that $\frac{1}{p_1^\xi} h_\xi(\lambda) > \delta_\lambda$. By continuity there exist two positive numbers ε and ε' such that

$$\frac{1}{p_1^{\xi'}} h_{\xi'}(\lambda') \geq \delta_\lambda$$

holds for all $\lambda' \in I_\lambda = (\lambda - \varepsilon, \lambda + \varepsilon)$ and for all $\xi' \in G_\lambda = \phi^{-1}(O_{\varepsilon'}(\phi[\xi]))$. The collection $\{I_\lambda\}$ is a covering of the segment $[\lambda_2, \lambda_1]$; let $\{I_{\lambda_3}, \ldots, I_{\lambda_{n_0}}\}$ be a finite subcovering. Letting $G_l = G_{\lambda_l}, \sigma = \min_l \widetilde{Q}(G_{\lambda_l}), \delta = \min_l \delta_{\lambda_l}$, we get the desired result. $\square$

We continue the proof of the theorem. Since (2) can be rewritten now as

$$p_{-1}(x) f(x-1) + p_0(x) f(x) + p_1(x) f(x+1) - \frac{f(x)}{r(x)} = 0, \tag{10}$$

the function $f(x)$ is completely determined by its values in any two nearest neighbour points $f(n)$ and $f(n+1)$. At this point we need another lemma.

Lemma 6.3 *Let f_1 and f_2 be two solutions of (10) such that $f_1(0) = f_2(0) > 0$ and $0 < f_1(1) < f_2(1)$. Let*

$$\hat{x} = \max\{x : f_1(x') > 0 \text{ for all } x' \leq x\}.$$

Then $f_1(x) < f_2(x)$ for all $x \leq \hat{x}$.

Proof. Denote $\lambda_n^{(1)} = f_1(n+1)/f_1(n)$ and $\lambda_n^{(2)} = f_2(n+1)/f_2(n)$ for $n = 0, \ldots, \hat{x} - 1$. For $i = 1, 2$ we have

$$p_{-1}(n)\frac{f_i(n)}{\lambda_{n-1}^{(i)}} + p_0(n)f_i(n) + p_1(n)f_i(n+1) - \frac{f_i(n)}{r(n)} = 0,$$

which can be written in the form

$$f_i(n+1) = \lambda_{n-1}^{(i)} f_i(n) - \frac{1}{p_1(n)} h_{\xi_n}(\lambda_{n-1}^{(i)}) f_i(n).$$

Since $\lambda_n^{(i)} = f_i(n+1)/f_i(n)$ we have

$$\lambda_n^{(i)} = \lambda_{n-1}^{(i)} - \frac{1}{p_1(n)} h_{\xi_n}(\lambda_{n-1}^{(i)}) \stackrel{\text{def}}{=} g(\lambda_{n-1}^{(i)}) \tag{11}$$

Using Condition P one can check that $g'(\lambda) > 0$, so the function $g(\lambda)$ increases on $\mathbf{R}_+^*$. Then, since $\lambda_0^{(1)} < \lambda_0^{(2)}$, we get by induction that $\lambda_x^{(1)} < \lambda_x^{(2)}$ for all n provided that $\lambda_n^{(1)}$ remains positive for $n < x$. Since $f_i(n+1) = f_i(0)\lambda_1 \ldots \lambda_n$, the lemma follows. □

We now finish the proof of Theorem 4.2. First, we have from (10)

$$f(2) = \frac{1}{p_1(1)}\Big(- p_{-1}(1)f(0) - p_0(1)f(1) + \frac{f(1)}{r(1)}\Big) \leq f(1)\frac{1}{r(1)p_1(1)}$$

Let $\lambda' > 0$ such that

$$\widetilde{Q}\{\xi : (r^{\xi}p_1^{\xi})^{-1} \leq \lambda'\} \geq \sigma, \tag{12}$$

with σ from Lemma 6.2. Denote $G_0 = \{\xi : (r^{\xi}p_1^{\xi})^{-1} \leq \lambda'\}$ and $\lambda_1 = f(2)/f(1)$. If $\xi_1 \in G_0$, then $\lambda_1 \leq \lambda'$.
We prove now that there exist a "configuration" $(s_1, \ldots, s_N)$, $s_i \in \{0, 1, \ldots, n_0\}$, such that if $\xi_i \in G_{s_i}$ for $i \in \{1, \ldots, N\}$, then for any initial conditions $f(0)$ and $f(1)$ the solution f of (10) becomes negative before at most N steps, i.e.

$$\forall f(0), f(1) > 0, \ \exists n_1 \leq N \ : \ f(n_1) < 0 \tag{13}$$

The number n_0 and sets $G_i \subset M$ were defined in Lemma 6.2. Note that N does not depend on the initial conditions $f(0)$ and $f(1)$, but n_1 does.

Let us construct the desired configuration. Taking $s_1 = 0$ we have $\lambda_1 \leq \lambda'$, and we see from Lemma 6.3 it is enough to check that the solution f of (10) with the particular initial conditions $f(0) > 0$ and $f(1) = \lambda' f(0)$ becomes negative before at most N steps.

If $s_2 = l(\lambda')$ we have for all $\xi \in G_{s_2}$

$$\frac{1}{p_1^\xi} h_\xi(\lambda') \geq \delta$$

from Lemma 6.2. Assuming that $\xi_2 \in G_{s_2}$, we obtain from (11)

$$\lambda_2 = \lambda' - \frac{1}{p_1(2)} h_{\xi_2}(\lambda') \leq \lambda' - \delta.$$

with the usual notation $\lambda_2 = f(3)/f(2)$.

We recursively define $s_{j+1} = l(\lambda_j)$ as long as $\lambda_j > 0$. If $\xi_j \in G_{s_j}$ for these j's then $\lambda_{j+1} \leq \lambda_j - \delta$ and therefore the function $f(x) = f(0)\lambda'\lambda_1 \ldots \lambda_{x-1}$ becomes negative after no more than $N = [\lambda'/\delta] + 1$ steps. This proves that (13) holds for the above configuration $(s_1, \ldots, s_N)$.

Since the random variables ξ_i are independent with $\mathbf{P}\{\xi_i \in G_l\} \geq \sigma > 0$, for almost all ω there exists some finite $j = j(\omega)$ such that $\xi_{i+j} \in G_{s_i}$, $i = 1, \ldots, N$. Then (13) implies that for $\mathbf{P}$-a.e. environment ξ, no positive function can satisfy (2). Theorem 4.2 is proved. $\square$

Remark. A finite sequence $\{u_i\}_{i=1,\ldots,l_0}$, $u_i \in M$ is called a *barrier*, if the condition $\xi_{j+i} = u_i$, $i = 1, \ldots, l_0$ for some j, implies that $\mathsf{E}\nu = \infty$ (the environment at all the other point can be arbitrary, only provided that the corresponding Markov chain is irreducible). In fact, we just have proved that if $u_i \in s_i$, $i = 0, \ldots, n_0$, then $\{u_i\}_{i=0,\ldots,n_0}$ is a barrier. One can notice that the sets V_i from the proof of Theorem 4.3 also are barriers in this sense.

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