The shape theorem for the frog model with random initial configuration

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Abstract

We prove a shape theorem for a growing set of simple random walks on $\mathbb{Z}^d$, known as frog model. The dynamics of this process is described as follows: There are active particles, which perform independent discrete time SRWs, and sleeping particles, which do not move. When a sleeping particle is hit by an active particle, the former becomes active as well. Initially, a random number of particles is placed into each site. At time 0 all particles are sleeping, except for those placed at the origin. We prove that the set of all sites visited by active particles, rescaled by the elapsed time, converges to a compact convex set.

Keywords: frog model, shape theorem, simple random walk

1 Introduction

In this note we study a discrete time particle system in $\mathbb{Z}^d$ named frog model. In this model there are active particles, which move as independent simple random walks (SRWs) on $\mathbb{Z}^d$, and sleeping particles, which do not move until activated. At time zero there is a random number $\eta(x)$ of particles at each site x of the lattice, where $\{\eta(x), x \in \mathbb{Z}^d\}$ are i.i.d., and all the particles are sleeping except for those that might be placed at the origin. Those active particles start to perform a discrete time SRW. From then on when an active particle jumps on a sleeping particle, the latter wakes up and starts jumping independently, also performing a SRW. If the origin is initially occupied then the number of active particles grows to infinity as active particles jump on sites that have not been visited before, awakening the particles that are sitting there. Let us underline that the active particles do not interact with each other and there is no "one-particle-per-site" rule.

The frog model can be viewed as a model for describing information spreading. The original idea is that every active particle has some information and it shares that information with a sleeping particle at the time the former jumps on the latter. Particles that have the information move freely helping in the process of spreading information. The model that we deal with in this paper is a discrete-time version of that proposed by R. Durrett (1996, private communication), who also suggested the term "frog model".

The first published result on this model is due to Telcs, Wormald [11], where it was referred to as the "egg model". They proved that, starting from the one-particle-per-site initial configuration, the origin will be visited infinitely often a.s. Popov [10] proved that the last result holds in dimension $d \geq 3$ for the initial configuration with a sleeping particle (or "egg") at each $x \neq 0$ with probability $\alpha/\|x\|^2$, α being a large positive constant. In Alves et al. [2] a modification of the present model was studied from the point of view of extinction and survival. The difference of the model of [2] from the model of this paper is that in the former active particles may disappear on each step. Recently A. Ramirez and V. Sidoravicius communicated to us that they are working on a continuous-time analog of this model, and that they have proved some results such as shape theorem and convergence to the product of Poissons.

In Alves et al. [1] it was proved that, starting from the one-particle-per-site initial configuration, the set of the original positions of all active particles, rescaled by the elapsed time, converges to a nonempty compact convex set. In the present paper we generalize the main result of [1] to the case of random initial configuration. It turns out that for the case when the initial configuration contains empty sites, this generalization is nontrivial.

Now we define the model in a formal way. Let $(\mathbb{N}^{\mathbb{Z}^d}, \mathcal{B}_1, \nu)$ be a probability space, where $\mathcal{B}_1$ is the product sigma algebra and ν is the translation invariant product measure determined by the distribution of $\{\eta(x) : x \in \mathbb{Z}^d\}$.

For each $\omega \in \mathbb{N}^{\mathbb{Z}^d}$ let $\{(S^x_{n,k}(\omega))_{n\in\mathbb{N}}, x \in \mathbb{Z}^d, 1 \le k \le \omega(x)\}$ be the independent simple random walks which are executed by the particles in ω when they are activated. We define $S^x_{0,k}(\omega) = x$, for all $x \in \mathbb{Z}^d$ and $1 \le k \le \omega(x)$. Denote by Ω^ω the path space of the trajectories of the random walks starting from the initial configuration ω and by $\mathbf{P}_\omega$ the corresponding path space measure. Let $\mathbf{P}$ be the measure on $\Omega = \prod_{\omega\in\mathbb{N}^{\mathbb{Z}^d}}(\omega \times \Omega^\omega)$ obtained by taking the base measure on $\mathbb{N}^{\mathbb{Z}^d}$ to be the product measure ν and the conditional measure $\mathbf{P}[\,\cdot \mid \omega] = \mathbf{P}_\omega$. For each $\omega \in \mathbb{N}^{\mathbb{Z}^d}$, let

$$t(x, z)(\omega) = \min\{n : S^x_{n,k}(\omega) = z \text{ for some } k, 1 \le k \le \omega(x)\}$$

and

$$T(x, z)(\omega) = \inf\Big\{\sum_{i=1}^{m} t(x_{i-1}, x_i)(\omega)\Big\}, \tag{1.1}$$

where the infimum is taken over all the finite sequences $x = x_0, x_1, \ldots, x_m = z$. Note that $t(x, z)(\omega) = T(x, z)(\omega) = \infty$ when $\omega(x) = 0$. Note also that for $d \ge 3$, $t(x, z)(\omega) = \infty$ with positive probability even when $\omega(x) \ge 1$.

Let us define the set of sites which were visited by active particles up to time n, provided that initially the active particles were only in x. Namely,

$$\xi^x_n(\omega) = \{y \in \mathbb{Z}^d : T(x, y)(\omega) \le n\}.$$

We are mostly concerned with $\xi_n := \xi^0_n$ and its asymptotic behaviour. In order to analyze that behaviour, define

$$\bar{\xi}^x_n(\omega) = \{y + (-1/2, 1/2]^d : y \in \xi^x_n(\omega)\} \subset \mathbb{R}^d,$$

and $\bar{\xi}_n := \bar{\xi}^0_n$.

The main result of this paper is the following

Theorem 1.1 *For any dimension $d \ge 1$ there is a nonempty convex set $\mathcal{A} = \mathcal{A}(d, \nu) \subset \mathbb{R}^d$ such that for ν-almost all initial configurations ω, conditioned on $\{\eta(0) \ge 1\}$, we have for any $0 < \varepsilon < 1$*

$$(1 - \varepsilon)\mathcal{A} \subset \frac{\bar{\xi}_n}{n} \subset (1 + \varepsilon)\mathcal{A}$$

for all n large enough $\mathbf{P}_\omega$-a.s.

Note that, although Theorem 1.1 establishes the existence of the asymptotic shape $\mathcal{A}$, it is difficult to identify exactly this shape. Of course, $\mathcal{A}$ is symmetric and $\mathcal{A} \subset \mathfrak{D}$, where, denoting $\|x\|_1 = |x^{(1)}| + \cdots + |x^{(d)}|$,

$$\mathfrak{D} = \{x \in \mathbb{R}^d : \|x\|_1 \leq 1\}.$$

Also, note that if the initial configuration is augmented (i.e. some new particles are added), then the asymptotic shape (when it exists) augments as well. In the paper [1] it was shown that if the initial configuration is constructed by adding m particles to each site and m is large enough, then the limiting shape $\mathcal{A}$ contains some pieces of the boundary of $\mathfrak{D}$ (a "flat edge" result). Now we show that if the distribution of η is heavy-tailed enough, then the limiting shape $\mathcal{A}$ coincides with $\mathfrak{D}$ (a "full diamond" result).

Theorem 1.2 *Suppose that for some positive $\delta < d$ and for all n large enough we have*

$$\mathbf{P}[\eta(x) \geq n] \geq (\log n)^{-\delta}. \tag{1.2}$$

Then, Theorem 1.1 is verified with $\mathcal{A} = \mathfrak{D}$.

2 Proofs

Proof of Theorem 1.1.
Step 1. First, we state a few preparatory results, which concern mainly the tails of random variables $T(\cdot,\cdot)$. Let us begin by recalling a technical fact from [1].

Lemma 2.1 *Suppose that $\omega(x) = 1$ for all $x \in \mathbb{Z}^d$. For all $d \geq 1$ and $x_0 \in \mathbb{Z}^d$ there exist positive finite constants $\alpha_1 = \alpha_1(x_0, d)$ and $\beta_1 = \beta_1(d)$ such that*

$$\mathbf{P}[T(0, x_0) \geq m] \leq \alpha_1 \exp\{-m^{\beta_1}\}$$

for all m.

Proof. Here we give only the main ideas of the proof, as it was given in full detail in [1] (Theorem 3.2).
1. The case $d \geq 4$. Pick $n \geq \|x_0\|^2$, where $\|\cdot\|$ is the Euclidean norm, and fix some ε such that $0 < \varepsilon < \frac{1}{2(d-2)}$. Define for $1 \leq i \leq \lfloor d/2 \rfloor$ the sets $\mathcal{D}_{i,\varepsilon}^{(n)} := \{x \in \mathbb{Z}^d : \|x\| \leq in^{1/2+\varepsilon}\}$.

1.1. Consider the trajectory of the initial particle until time n. With overwhelming probability it stays in $\mathcal{D}^{(n)}_{1,\varepsilon}$ all that time and awakens at least $O(n^{1-\varepsilon})$ sleeping particles. To see why the last claim is true, divide the time interval $[0, n]$ into n^{ε} disjoint subintervals of size $n^{1-\varepsilon}$. During a fixed subinterval, the expected size of the corresponding subrange is of order $n^{1-\varepsilon}$ (this can be seen by considering all the sites y which are at distance at most $\sqrt{n^{1-\varepsilon}}$ from the position of the initial particle at the beginning of the time subinterval, estimating the probability that y is hit and summing over y; or just by applying directly the known results about the size of the range of SRW, see e.g. Hughes [6], pp. 333, 338). As the size of the subrange is certainly not greater than $n^{1-\varepsilon}$, with probability bounded away from 0 it will be $O(n^{1-\varepsilon})$ (this follows from the following fact: For any random variable X with $0 \leq X \leq a$ a.s. and $\mathbf{E}X \geq b$ it is true that $\mathbf{P}[X \geq b/2] \geq b/(2a)$). So, with overwhelming probability the size of at least one of the subranges will be of order $n^{1-\varepsilon}$.

1.2. Pick $n^{1-\varepsilon}$ particles in $\mathcal{D}^{(n)}_{1,\varepsilon}$ which were awakened by the initial particle by time n. Divide them into n^{ε} disjoint groups of size $n^{1-2\varepsilon}$. Observe the particles of a fixed group during $n^{1+2\varepsilon}$ time units after their activation. Let ζ be the number of sites in $\mathcal{D}^{(n)}_{2,\varepsilon} \setminus \mathcal{D}^{(n)}_{1,\varepsilon}$ visited by the particles of that group along the time interval mentioned above. Clearly, $\zeta \leq n^{1-2\varepsilon} \times n^{1+2\varepsilon} = n^2$ and the direct computation (for each $y \in \mathcal{D}^{(n)}_{2,\varepsilon} \setminus \mathcal{D}^{(n)}_{1,\varepsilon}$, using that the $n^{1-2\varepsilon}$ particles from the group are independent, we compute a lower bound on the probability that at least one particle of the group hits y, and then sum over y) shows that $\mathbf{E}\zeta = O(n^2)$, so with probability bounded away from 0, ζ is of order n^2. Considering now all the n^{ε} groups and using the independence again, we obtain that with overwhelming probability there will be $O(n^2)$ particles in $\mathcal{D}^{(n)}_{2,\varepsilon} \setminus \mathcal{D}^{(n)}_{1,\varepsilon}$ by time $n + n^{1+2\varepsilon}$.

1.3. Now, proceeding in the same spirit, we use those $O(n^2)$ particles from $\mathcal{D}^{(n)}_{2,\varepsilon} \setminus \mathcal{D}^{(n)}_{1,\varepsilon}$ to awaken $O(n^3)$ particles in $\mathcal{D}^{(n)}_{3,\varepsilon} \setminus \mathcal{D}^{(n)}_{2,\varepsilon}$, and so on, to get finally $O(n^{\lfloor d/2 \rfloor})$ active particles in $\mathcal{D}^{(n)}_{\lfloor d/2 \rfloor,\varepsilon} \setminus \mathcal{D}^{(n)}_{\lfloor d/2 \rfloor -1,\varepsilon}$ at time $n + O(n^{1+2\varepsilon})$ with overwhelming probability.

1.4. Considering those $O(n^{\lfloor d/2 \rfloor})$ particles in $\mathcal{D}^{(n)}_{\lfloor d/2 \rfloor,\varepsilon} \setminus \mathcal{D}^{(n)}_{\lfloor d/2 \rfloor -1,\varepsilon}$ and waiting $n^{1+2\varepsilon}$ units of time more, one gets that with overwhelming probability at least one of those particles will hit x_0 by time $m_d := n + O(n^{1+2\varepsilon})$.

2. The case $d = 3$. Again, let $n \geq \|x_0\|^2$. Take $\varepsilon < 1/4$ and consider the $n^{1-\varepsilon}$ particles in $\mathcal{D}^{(n)}_{1,\varepsilon}$ awakened by the initial particle until time n. Now, as in the last step of the argument for $d \geq 4$, until time $m_3 := n + O(n^{1+2\varepsilon})$

with overwhelming probability at least one of those particles will hit x_0.

3. The case $d = 2$. This case is treated analogously to the case $d = 3$. That is, first, dividing the time interval $[0, n]$ into $n^{1/2}$ subintervals of size $n^{1/2}$ (i.e., taking $\varepsilon = 1/2$) and using the fact that the expected size of the range of SRW grows like $O(n/\log n)$, we prove that with large probability the original particle will awaken $O(n^{1/2}/\log n)$ sleeping particles in the ball of radius n until the moment n, where $n \geq \|x_0\|^2$. Considering the independent random walks of those particles until time $m_2 := n+O(n^2)$, we get the result.

4. The case $d = 1$. This was not treated in Theorem 3.2 of [1], but anyway it is quite analogous to the cases $d = 2, 3$. First, by time n we will have $n^{1/4}$ active particles situated not farther than n from the origin. Then, waiting until $m_1 := n + n^2$ one gets the result. $\square$

For $d \geq 3$ denote $\varepsilon_d = (6(d-2))^{-1}$; clearly, when doing the proof of Lemma 2.1 in dimension $d \geq 3$, one may fix $\varepsilon = \varepsilon_d$. From the above proof one can deduce that there exist deterministic constants h_d (which depend only on dimension) such that $m_d \leq h_d n^{1+2\varepsilon_d}$, $d \geq 3$, $m_d \leq h_d n^2$, $d = 1, 2$. This means that, in order to obtain an upper bound on $\mathbf{P}[T(0, x_0) \geq m]$, one should follow the steps of the proof of Lemma 2.1 with $n = (m/h_d)^{1/(1+2\varepsilon)}$, $d \geq 3$, or $n = (m/h_d)^{1/2}$, $d = 1, 2$. Keeping this in mind and denoting

$$\mathcal{R}_\omega(B) = \frac{1}{|B|} \sum_{x \in B} \mathbf{1}_{\{\omega(x) \geq 1\}},$$

where ω is the initial configuration and B is a finite subset of $\mathbb{Z}^d$, consider the following

Definition 2.1 *Let $p_1 = \mathbf{P}[\eta(0) \geq 1]$, and m be any positive integer. A fixed initial configuration ω is called m-good if*

- *$d \geq 4$ and*
  - *for any ball B of radius $n_d(m)^{(1-\varepsilon_d)/2}$ which is fully inside $\mathcal{D}_{1,\varepsilon_d}^{(n_d(m))}$ we have $\mathcal{R}_\omega(B) \geq p_1/2$;*
  - *we have $\mathcal{R}_\omega(\mathcal{D}_{i,\varepsilon_d}^{(n_d(m))} \setminus \mathcal{D}_{i-1,\varepsilon_d}^{(n_d(m))}) \geq p_1/2$ for all $i = 2, \ldots, \lfloor d/2 \rfloor$;*
- *$d = 3$, and for any ball B of radius $n_3(m)^{(1-\varepsilon_3)/2}$ which is fully inside $\mathcal{D}_{1,\varepsilon_3}^{(n_3(m))}$ we have $\mathcal{R}_\omega(B) \geq p_1/2$;*

- $d = 1, 2$, *and for any ball* B *of radius* $n_d(m)^{1/4}$ *situated not farther than* $n_d(m)$ *from the origin, we have* $\mathcal{R}_\omega(B) \geq p_1/2$,

where $n_d(m) = (m/h_d)^{1/(1+2\varepsilon_d)}$, $d \geq 3$, *or* $n_d(m) = (m/h_d)^{1/2}$, $d = 1, 2$, *and the notation* $\mathcal{D}^{(n)}_{i,\varepsilon_d}$ *is from the proof of Lemma 2.1.*

Lemma 2.2 *For all* $d \geq 1$ *and* $x_0 \in \mathbb{Z}^d$ *there exist positive finite constants* $\alpha_2 = \alpha_2(d, p_1)$ *and* $\beta_2 = \beta_2(d, p_1)$ *such that if* ω *is* m*-good and* $\|x_0\|^2 \leq n_d(m)$ *(the notation* $n_d(m)$ *is from Definition 2.1), then*

$$\mathbf{P}_\omega[T(0, x_0) \geq m] \leq \alpha_2 \exp\{-m^{\beta_2}\}.$$

Proof. From Definition 2.1 it follows that, by following the steps of the proof of Lemma 2.1 one can prove Lemma 2.2. For example, in the part 1.1 the expected amount of sleeping particles activated during a fixed time subinterval differs from the expected size of the subrange only by a constant factor (depending only on p_1 and d) when ω is m-good. In the part 1.2, $\mathbf{E}\zeta$ will be of the same order for m-good ω and for one-particle-per-site ω. In general, each time that one is computing the expected number of newly awakened particles in the proof of Lemma 2.1, considering m-good initial configuration instead of one-particle-per-site initial configuration costs only a constant factor. Another observation, which is crucial for the subsequent discussion, comes after examining the proof of Lemma 2.1: As long as $\|x_0\|^2 \leq n$, the estimates that one gets on $\mathbf{P}[T(0, x_0) > m_d]$ are uniform in x_0. □

Define $\hat{n}_d(m)$ to be equal to $n_d(m)^{(1-\varepsilon(d))/3}$, $d \geq 3$, and to $n_d(m)^{1/8}$, $d = 1, 2$.

Lemma 2.3 *For all* $d \geq 1$ *there exist positive finite constants* $\alpha_3 = \alpha_3(d, p_1)$ *and* $\beta_3 = \beta_3(d, p_1)$ *such that*

$$\mathbf{P}[\eta \textit{ is } m\textit{-good} \mid \eta(x_1) = \cdots = \eta(x_{k_0}) = 0] \geq 1 - \alpha_3 \exp\{-m^{\beta_3}\}$$

for any fixed collection of sites $x_1, \ldots, x_{k_0} \in \mathbb{Z}^d$, $k_0 \leq \hat{n}_d(m)^d$.

Proof. Note that $\hat{n}_d(m)^d$ is small relative to the sizes of sets which were considered in Definition 2.1. As $\eta(x)$ are i.i.d., it is straightforward to prove this fact by using the large deviations technique. □

In the sequel we will need the following basic fact which is stated without proof here.

Lemma 2.4 *Let $(X_i, i \geq 1)$ be a sequence of nonnegative random variables, and there exist φ_n, $n = 1, 2, \ldots$, such that* $\sup_i \mathbf{P}[X_i \geq n] \leq \varphi_n$ *and* $\sum_n \varphi_n < \infty$. *Then $X_n/n \to 0$ a.s.*

Step 2. Now the goal is to verify the conditions of Kingman-Liggett subadditive ergodic theorem [7, 8] for the sequence of random variables $Y(m, n)$ defined below.

For a fixed $x \in \mathbb{Z}^d$ and an $\omega \in \mathbb{N}^{\mathbb{Z}^d}$ satisfying the condition $\omega(0) \geq 1$ define a sequence of positive integers $\{v_k^x\}_{k=0}^{\infty}$ as follows:

$$\begin{aligned} v_0^x &= 0, \\ v_{k+1}^x &= \min\{n > v_k^x : \omega(nx) \geq 1\}. \end{aligned}$$

In words, $v_k^x x$, $k = 0, 1, 2, \ldots$, are the sites on the ray $\{nx, n \geq 0\}$ which are occupied in the initial configuration. Clearly, for any $x \in \mathbb{Z}^d$ and $i = 1, 2, \ldots$ it holds that $\mathbf{P}[v_i^x - v_{i-1}^x = k] = p_1(1 - p_1)^{k-1}$ (p_1 is from Definition 2.1). Now, for $m, n \geq 0$ consider the collection of random variables

$$Y(m, n) = T(v_m^x x, v_n^x x).$$

It is important to observe that the random variables $\{T(x, y) : x, y \in \mathbb{Z}^d\}$ are subadditive in the sense that for *all* ω,

$$T(x, z) \leq T(x, y) + T(y, z) \text{ for all } x, y, z \in \mathbb{Z}^d. \tag{2.1}$$

Here is the explanation. Fix the initial configuration ω. If the site y was empty in the initial configuration, then $T(y, z) = \infty$ and (2.1) follows. Now, suppose that $\omega(y) \geq 1$. If site z is reached before site y, then (2.1) is evident. If that does not happen, recall that the random variables $T(y, z)$, $y, z \in \mathbb{Z}^d$ are constructed using the same collection of the random variables $S_{n,k}^x$, i.e., each particle follows the same trajectory as soon as it wakes up. So the process departuring from only site y awakened (the one which gives the passage time $T(y, z)$) is coupled with the original process (i.e., that started from x), and for the latter one may have other particles awakened at time $T(x, y)$ besides that from y. Consequently, from (1.1) it follows that $T(x, z) - T(x, y)$, which is the remaining time to reach site z for the original process, is less than or equal to $T(y, z)$, thus proving (2.1). The equation (2.1) shows that for *all* initial configurations ω,

$$Y(m, k) \leq Y(m, n) + Y(n, k) \text{ for all } m, n, k \in \mathbb{N}. \tag{2.2}$$

Now, let us verify the conditions of Kingman-Liggett subadditive ergodic theorem for the random variables $Y(m,n)$.

Since ν is a product measure we can easily establish the following result: Given nonnegative integer numbers $m_1, m_2, \ldots, m_n, p$, the joint distributions of random variables $\{Y(m_1,m_2),\ldots,Y(m_{n-1},m_n)\}$ and $\{Y(m_1+p,m_2+p),\ldots,Y(m_{n-1}+p,m_n+p)\}$ are equal. From this we obtain the stationarity of the sequence $\{Y((n-1)k,nk)\}_{n\in\mathbb{N}}$ for each $k\in\mathbb{N}$. Ergodicity of the sequence follows from the observation that the events $\{T(v^x_{n_1k}x, v^x_{(n_1+1)k}x)\le a\}$ and $\{T(v^x_{n_2k}x, v^x_{(n_2+1)k}x)\le b\}$ are independent provided $a+b\le \|(n_1-n_2)kx\|_1$.

Next we prove that $\mathbf{E}T(0,v^x_1x)<\infty$. Using Lemmas 2.2 and 2.3, for y such that $\|y\|\le \hat n_d(m)^{1/2}$ (note that $\hat n_d(m)<n_d(m)$) and for any collection of sites $z_1,\ldots,z_{k_0}\in\mathbb{Z}^d$, $k_0\le \hat n_d(m)^d$, we have, denoting $\mathbf{P}^*[\,\cdot\,]=\mathbf{P}[\,\cdot\mid \eta(z_1)=\cdots=\eta(z_{k_0})=0]$,

$$\begin{aligned}\mathbf{P}^*[T(0,y)\ge m] &\le \mathbf{P}^*[T(0,y)\ge m\mid \eta \text{ is } m\text{-good}]+\mathbf{P}^*[\eta \text{ is not } m\text{-good}]\\ &\le \alpha_2\exp\{-m^{\beta_2}\}+\alpha_3\exp\{-m^{\beta_3}\}. \end{aligned}\tag{2.3}$$

Using (2.3), we get

$$\begin{aligned}\mathbf{P}[T(0,v^x_1x)\ge m] &= \sum_{k=1}^{\infty}\mathbf{P}[v^x_1=k]\mathbf{P}[T(0,kx)\ge m\mid v^x_1=k]\\ &\le \sum_{k\le\|x\|^{-1}\hat n_d(m)^{1/2}}\mathbf{P}[v^x_1=k]\mathbf{P}[T(0,kx)\ge m\mid v^x_1=k]\\ &\quad+\sum_{k>\|x\|^{-1}\hat n_d(m)^{1/2}}p_1(1-p_1)^{k-1}\\ &\le \alpha_2\exp\{-m^{\beta_2}\}+\alpha_3\exp\{-m^{\beta_3}\}\\ &\quad+(1-p_1)^{\|x\|^{-1}\hat n_d(m)^{1/2}}, \end{aligned}\tag{2.4}$$

so $\mathbf{E}Y(0,1)=\sum_{m\ge1}\mathbf{P}[T(0,v^x_1x)\ge m]<\infty$.

Thus, we have verified the conditions of Kingman-Liggett subadditive ergodic theorem for the sequence $\{Y(m,n)=T(v^x_mx,v^x_nx)\}$. Therefore, one gets that there exists $\mu'(x)$ such that

$$\lim_{n\to\infty}\frac{Y(0,n)}{n}=\mu'(x)<\infty\qquad \mathbf{P}\text{-a.s.}\tag{2.5}$$

Step 3. Now the goal is to pass from the random sequence $(v^x_kx, k\ge 0)$ to the whole ray $(kx, k\ge0)$.

Denote $\mu(x) := p_1\mu'(x)$. Let $n \in \mathbb{N}$ satisfy $v^x_{k(n)-1} \le n < v^x_{k(n)}$. Then, by the subadditivity,

$$\frac{T(0,nx)}{n} \le \frac{T(0,v^x_{k(n)}x)}{k(n)}\frac{k(n)}{n} + \frac{T(v^x_{k(n)}x,nx)}{n}. \tag{2.6}$$

Note that, since ν is a product measure, $v^x_{k(n)} - n$ has the same distribution as v^x_1. This means that the upper bound on $\mathbf{P}[T(v^x_{k(n)}x, nx) \ge m]$ is also given by (2.4), so by Lemma 2.4 we have that $n^{-1}T(v^x_{k(n)}x, nx) \to 0$, $\mathbf{P}$-a.s. Together with (2.5) and the fact that $k(n)/n \to p_1$ $\mathbf{P}$-a.s., this shows that one gets from (2.6) that

$$\limsup_{n\to\infty} \frac{T(0,nx)}{n} \le \mu(x) \qquad \mathbf{P}\text{-a.s.} \tag{2.7}$$

Now, let us prove that

$$\liminf_{n\to\infty} \frac{T(0,nx)}{n} \ge \mu(x) \qquad \mathbf{P}\text{-a.s.} \tag{2.8}$$

For a fixed site $y \in \mathbb{Z}^d$, $y \neq 0$, let U_y be a random variable defined in the following way. At the moment $T(0,y)$ consider the active particle situated in y (if there are several such particles, choose one of them by randomization). Let $Z^y_k, k \ge T(0,y)$ be the subsequent walk of this particle. We define $\tau = \min\{k : \eta(Z^y_k) \ge 1\}$ and so $U_y := \tau - T(0,y)$ is the time that the particle needs to travel from y to some site that initially contained sleeping particles. Clearly, $U_y = 0$ corresponds to the case $\eta(y) \ge 1$. Now, let $\hat\zeta_n = Z^{nx}_{T(0,nx)+U_{nx}}$; note that $\eta(\hat\zeta_n) \ge 1$. By subadditivity one can write

$$T(0, v^x_{k(n)}x) \le T(0,nx) + U_{nx} + T(\hat\zeta_n, v^x_{k(n)}x). \tag{2.9}$$

To proceed, we need to get some good bounds on the tails of U_{nx} and $T(\hat\zeta_n, v^x_{k(n)}x)$. Note that there exists a positive constant β which is not dependent on the dimension, such that $\mathbf{P}[|\{Z^{nx}_j, 0 \le j \le k^{1/2}\}| \ge k^{1/4}] \ge \beta$, for all $k \ge 1$. This shows that

$$\mathbf{P}[|\{Z^{nx}_j, 0 \le j \le k\}| \ge k^{1/4}] \ge 1 - (1-\beta)^{k^{1/2}}.$$

So, as any site in $\{Z^{nx}_j, 0 \le j \le k\}$ initially contained a sleeping particle with probability p_1, it holds that

$$\mathbf{P}[U_{nx} \ge k] \le (1-\beta)^{k^{1/2}} + (1-p_1)^{k^{1/4}}. \tag{2.10}$$

Now, as $\|nx - \hat{\zeta}_n\| \leq U_n$, (2.10) implies that

$$\begin{aligned}
\mathbf{P}[\|v_{k(n)}^x x - \hat{\zeta}_n\| \geq k] &\leq \mathbf{P}[\|v_{k(n)}^x x - nx\| \geq k/2] + \mathbf{P}[\|nx - \hat{\zeta}_n\| \geq k/2] \\
&\leq (1-p_1)^{k/2} + (1-\beta)^{(k/2)^{1/2}} + (1-p_1)^{(k/2)^{1/4}} \\
&=: \psi(k),
\end{aligned}$$

and, proceeding similarly to (2.4), we get that

$$\mathbf{P}[T(\hat{\zeta}_n, v_{k(n)}^x x) \geq m] < \alpha_2 \exp\{-m^{\beta_2}\} + \alpha_3 \exp\{-m^{\beta_3}\} + \psi(\|x\|^{-1}\hat{n}_d(m)^{1/2}). \tag{2.11}$$

Thus, by (2.10) and (2.11), dividing (2.9) by n and applying (2.5) and Lemma 2.4, we get (2.8). Now, from (2.7) and (2.8) we finally obtain that, for all $x \in \mathbb{Z}^d$, there exists $\mu(x) \geq 0$ such that

$$\lim_{n\to\infty} \frac{T(0,nx)}{n} \to \mu(x) \qquad \mathbf{P}\text{-a.s.} \tag{2.12}$$

Notice that the equation (2.12) is already enough to get the proof of Theorem 1.1 in dimension 1 with $\mathcal{A} = [-(\mu(1))^{-1}, (\mu(1))^{-1}]$. So, from now on we shall concentrate on the case $d \geq 2$.

Step 4. To proceed with the proof of Theorem 1.1, one has to assure that ξ_n grows *at least* linearly. For $y \in \mathbb{R}^d$ and $a > 0$ denote $D(y,a) = \{x \in \mathbb{R}^d : \|x - y\|_1 \leq a\}$.

Lemma 2.5 *For all $d \geq 2$ there exist constants $0 < \delta < 1$, $\alpha_4 > 0$, $\beta_4 > 0$, which depend only on the dimension, such that*

$$\mathbf{P}[D(x, n\delta) \subset \bar{\xi}_{n+T(0,x)}] \geq 1 - \alpha_4 \exp\{-n^{\beta_4}\}$$

for all n and x, conditioned on $\{\eta(0) \geq 1\}$.

Proof. The proof of this fact follows the spirit of the proof of Lemmas 4.2 and 4.3 of [1]. The difficulty that arises here is that, for ω such that $\omega(x) = 0$, $T(x,y)(\omega) = \infty$ for any y, so a direct application of the method of [1] does not work. To get around that difficulty, we use the construction similar to that used in Step 3. Recall the definition of random variable U_y from Step 3, and define for $y \neq 0$, $T^*(y,z) = U_y + T(\hat{\zeta}^y, z)$, where, similarly to Step 3, $\hat{\zeta}^y$ is the site with $\eta(\hat{\zeta}^y) \geq 1$ at which the active particle which was in y at time $T(0,y)$ arrived after U_y steps. Now, let y be such that $\|x - y\|_1 = n$, and let $x = y_0, y_1, \dots, y_n = y$ be a path connecting x to y such that for all i, $\|y_i - y_{i-1}\|_1 = 1$; note that $\|x - y_k\|_1 = k$, $k = 0, \dots, n$. Denote $Y_i = T^*(y_{i-1}, y_i)$. From the above construction it follows that

$$T(0,y) - T(0,x) \le \sum_{i=1}^{n} Y_i = \sum_{i=1}^{n^{1/2}} \sigma_i,$$

where

$$\sigma_i = \sum_{j: i + jn^{1/2} \le n} Y_{i+jn^{1/2}},$$

$i = 1, \dots, n^{1/2}$. Analogously to (2.10) and (2.11), it is not difficult to get that for some constants $\alpha_5, \beta_5 > 0$ and for all m

$$\mathbf{P}[Y_i \ge m] \le \alpha_5 \exp(-m^{\beta_5}), \tag{2.13}$$

uniformly in x, y and paths connecting them. Consider the event $B = \{Y_i < n^{1/2}/2, i = 1, \dots, n\}$; clearly, from (2.13) it follows that for some $\alpha_6, \beta_6 > 0$, $\mathbf{P}[B] \ge 1 - \alpha_6 \exp(-n^{\beta_6})$. Note that, if $|i - j| \ge n^{1/2}$, then the random variables $Y_i \mathbf{1}_B$ and $Y_j \mathbf{1}_B$ are independent, since if the event B occurs, then the random variables Y_i and Y_j depend on disjoint sets of random walks. So, when B occurs, each σ_i is a sum of independent random variables. Although Y_i are not identically distributed and we cannot guarantee the existence of moment generating function for Y_i, the condition (2.13) allows us, by applying Theorem 1.1 of Nagaev [9], to obtain that there exists $\delta_0 > 0$ such that $\sigma_i \le n^{1/2}/\delta_0$ with probability at least subexponentially high. This shows that, with overwhelming probability, $\sum_{i=1}^{n} Y_i \le n/\delta_0$. Thus, if y is at distance n from x, with overwhelming probability by time $T(0,x) + n/\delta_0$ it will be visited. Now, if $d \ge 2$ and $0 < \|x - y\|_1 < n$, then there exists $z \in \mathbb{Z}^d$ such that $\|z - x\|_1 = n$ and $\|z - y\|_1$ equals n or $n+1$. Since $T^*(x,y) \le T^*(x,z) + T^*(z,y)$, the result follows with $\delta = \delta_0/2$. $\square$

Step 5. The next step is to prove that $\mu(\cdot)$ can be extended to $\mathbb{R}^d$ in such a way that μ is a norm in $\mathbb{R}^d$. Let us extend the definition of $T(x,y)$ to the whole $\mathbb{R}^d \times \mathbb{R}^d$ by defining

$$T(x,y) = \min\{n : y \in \bar{\xi}_n^{x_0}\},$$

where $x_0 \in \mathbb{Z}^d$ is such that $x \in (-1/2, 1/2]^d + x_0$. From the fact $T(0, nx) \ge n\|x\|_1$ it follows that $\mu(x) \ge \|x\|_1$ for all $x \in \mathbb{Z}^d$.

Lemma 2.6 *For any $a \in \mathbb{R}_+$, $x, y \in \mathbb{Z}^d$ we have*

$$\mu(ax) := \lim_{n\to\infty} \frac{T(0, anx)}{n} = a\mu(x) \qquad \mathbf{P}\text{-}a.s., \tag{2.14}$$

and

$$\mu(x+y) \le \mu(x) + \mu(y). \tag{2.15}$$

Proof. First, the proof of (2.14) basically repeats what was done on Step 3, so we omit it. Let us turn to the proof of (2.15). Clearly, instead of (2.15) it is sufficient to prove that for any $x, y \in \mathbb{Z}^d$

$$\mu'(x+y) \le \mu'(x) + \mu'(y) \tag{2.16}$$

with $\mu'(\cdot)$ defined by (2.5).

For fixed $n \ge 1$ define a random sequence $s_{k,n}$, $k = 0, 1, 2, \dots$, in the following way:

$$\begin{aligned} s_{0,n} &= 0, \\ s_{k+1,n} &= \min\{m > s_{k,n} : \eta(my + v_n^x x) \ge 1\}, \end{aligned}$$

and let $z_{k,n} = s_{k,n} y + v_n^x x$. By the subadditivity,

$$T(0, v_n^{x+y}(x+y)) \le T(0, v_n^x x) + T(v_n^x x, z_{n,n}) + T(z_{n,n}, v_n^{x+y}(x+y)). \tag{2.17}$$

We have

$$\begin{aligned} v_n^{x+y}(x+y) - z_{n,n} &= x \sum_{i=1}^n ((v_i^{x+y} - v_{i-1}^{x+y}) - (v_i^x - v_{i-1}^x)) \\ &\quad + y \sum_{i=1}^n ((v_i^{x+y} - v_{i-1}^{x+y}) - (s_{i,n} - s_{i-1,n})). \end{aligned}$$

Fix arbitrary $\varepsilon > 0$. Now, $(v_i^{x+y} - v_{i-1}^{x+y})$, $(v_i^x - v_{i-1}^x)$, $(s_{i,n} - s_{i-1,n})$, $i = 1, \dots, n$, are i.i.d. random variables, geometrically distributed with parameter $(1-p_1)$. So, in the right-hand side of the above display we have two sums of i.i.d. random variables satisfying the Cramer condition and with zero mean. So, by the Large Deviation Principle, with exponentially high probability $\|v_n^{x+y}(x+y) - z_{n,n}\| \le \varepsilon(\|x\| + \|y\|)n$. By Lemma 2.5, with subexponentially high probability, $T(z_{n,n}, v_n^{x+y}(x+y)) \le \varepsilon\delta^{-1}(\|x\| + \|y\|)n$ (the site $z_{n,n}$ is random, so, to see that Lemma 2.5 is applicable here, note that $T(z_{n,n}, v_n^{x+y}(x+y))$ has the same distribution as $T(0,u)$, where u is a random site constructed as follows: First, let u' be the n-th occupied site on the ray along $(-y)$, then, u'' is the n-th occupied site on the ray along $(-x)$ beginning in u', and finally, u is the n-th occupied site on the ray along $x+y$ beginning in u''). Note that $T(v_n^x x, z_{n,n})$ has the same distribution as $T(0, v_n^y y)$, so $n^{-1}T(v_n^x x, z_{n,n}) \to \mu'(y)$ at least in probability. Dividing (2.17) by n and taking the limit in probability, we get

$$\mu'(x+y) \le \mu'(x) + \mu'(y) + \varepsilon\delta^{-1}(\|x\| + \|y\|);$$

when $\varepsilon \to 0$, we obtain (2.16). □

Now, let us show that $\mu(x)$ can be extended to the whole $\mathbb{R}^d$ in such a way that (2.12) still holds. This is done in a standard way: First, if $x \in \mathbb{Q}^d$ (i.e., all the coordinates of x are rational), we define $\mu(x) := \mu(mx)/m$, where m is the smallest positive integer such that $mx \in \mathbb{Z}^d$ (the fact that (2.12) still holds for $x \in \mathbb{Q}^d$ follows from (2.14)). Then, by using the fact that μ is a norm in $\mathbb{Q}^d$ (this follows from Lemma 2.6), it is extended to $\mathbb{R}^d$. To prove that (2.12) holds for any $x \in \mathbb{R}^d$, one can proceed as follows: By the subadditivity and Lemma 2.5, if $y \in \mathbb{Q}^d$, then $\limsup_{n\to\infty} n^{-1}T(0,nx) \le \mu(y) + \delta^{-1}\|x-y\|$, and $\liminf_{n\to\infty} n^{-1}T(0,nx) \ge \mu(y) - \delta^{-1}\|x-y\|$. Approximating x by vectors with rational coordinates, one gets the result.

Step 6. Now everything is ready to finish the proof of Theorem 1.1.

Let $\mathcal{A} := \{x \in \mathbb{R}^d : \mu(x) \le 1\}$. The following argument is standard (see e.g. [3, 4, 5]), we keep it to preserve the self-containedness of the paper.

Denote $\varepsilon' = (1-\varepsilon)^{-1} - 1$, and $\varepsilon'' = 1 - (1+\varepsilon)^{-1}$. To prove Theorem 1.1, it is enough to prove that $n\mathcal{A} \subset \bar{\xi}_{(1+\varepsilon')n}$ and $\bar{\xi}_{(1-\varepsilon'')n} \subset n\mathcal{A}$ for all n large enough, $\mathbf{P}$-a.s. (and so $\mathbf{P}_\omega$-a.s. for ν-almost all ω).

Choose a finite set $F := \{x_1, \ldots, x_k\} \subset \mathcal{A}$ such that $\mu(x_i) < 1$ for $i = 1, \ldots, k$, and (with δ from Lemma 2.5) $\mathcal{A} \subset \cup_{i=1}^k D(x_i, \varepsilon'\delta)$. Notice that (2.12) implies that $nF \subset \bar{\xi}_n$ for all n large enough, $\mathbf{P}$-a.s. Now, Lemma 2.5 and Borel-Cantelli imply that $\mathbf{P}$-a.s. for all n large enough we have $D(nx_i, n\varepsilon'\delta) \subset \bar{\xi}^{nx_i}_{n\varepsilon'}$, for all $i = 1, 2, \ldots, k$. So $n\mathcal{A} \subset \bar{\xi}_{(1+\varepsilon')n}$ and this part of the proof is done.

Now, let us choose $G := \{y_1, \ldots, y_k\} \subset 2\mathcal{A} \setminus \mathcal{A}$ in such a way that $2\mathcal{A} \setminus \mathcal{A} \subset \cup_{i=1}^k D(y_i, \varepsilon''\delta)$. Analogously, we get that $nG \cap \bar{\xi}_n = \emptyset$ for all n large enough $\mathbf{P}$-a.s., and that for all n large enough, if $\bar{\xi}_{(1-\varepsilon'')n} \cap (2\mathcal{A} \setminus \mathcal{A}) \neq \emptyset$, then

$\bar{\xi}_n \cap nG \neq \emptyset$. This shows that $\bar{\xi}_{(1-\varepsilon'')n} \subset n\mathcal{A}$ for all n large enough, $\mathbf{P}$-a.s., and so concludes the proof of Theorem 1.1. □

Proof of Theorem 1.2. Denote

$$\mathfrak{D}_n = \{x \in \mathbb{Z}^d : \|x\|_1 \leq n\}.$$

Choose $\theta < 1$ such that $\delta < \theta d$. Start the process and wait until the moment n^θ. By Lemma 2.5 there exist C_1, γ_0, such that with probability at least $1 - \exp\{n^{\gamma_0}\}$ all the frogs which were initially in the ball of radius $C_1 n^\theta$ centered in 0 will be awake. Clearly, the inequality (1.2) implies that

$$\mathbf{P}[\eta(x) \leq (4d)^n] \leq 1 - (\log 4d)^{-\delta} n^{-\delta}.$$

As the number of particles in that ball is of order $n^{\theta d}$, one gets that with probability at least $1 - \exp\{-C_2 n^{\theta d - \delta}\}$ at time n^θ one will have at least one activated site x with $\eta(x) \geq (4d)^n$ and $\|x\| \leq C_1 n^\theta$. Note the following simple fact: If x contains at least $(2d)^n$ active particles and $\|x - y\|_1 \leq n$, then until time n with probability bounded away from 0 at least one of those particles will hit y. Using this fact, as x really contains at least 2^n groups of $(2d)^n$ particles, we get that with overwhelming probability all the particles in the diamond $\mathfrak{D}_{n-n^\theta}$ will be awake at time $n^\theta + n$, which completes the proof of Theorem 1.2. □

3 Remarks about continuous time

A continuous-time version of the frog model can also be considered. Here we would like to remark that in the continuous-time context and for the case of bounded η, Theorem 1.1 also holds and its proof can be obtained by following the steps of our proof for the discrete case. The difficulty that arises is that, for continuous time, it is not evident that $\mu(x)$ (defined by (2.12)) is strictly positive for $x \neq 0$, i.e., we must rule out the possibility that the continuous-time frog model grows faster than linearly. To overcome that difficulty, note the following fact (compare with Lemma 9 on page 16 of Chapter 1 of [4]): there exist a positive number β such that, being $\|x\|_1 \geq \beta n$, $\mathbf{P}[T(0,x) < n]$ is exponentially small in n. This fact in turn follows from a domination of the frog model by branching random walk. So, we conclude that for a.s. bounded η our method works well in the continuous-time context too.

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