A note on transience vs. recurrence for branching random walk in random environment

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Abstract

In this note we consider a branching random walk in random environment on $\mathbb{Z}^d$ where particles perform independent simple random walks and branch, according to a given offspring distribution, at a random subset of sites whose density tends to zero at infinity. Given that initially one particle starts at the origin, we identify the *critical* rate of decay of the density of the branching sites separating transience from recurrence, i.e., the progeny hits the origin with probability < 1 resp. $= 1$. We show that for $d \geq 3$ there is a *dichotomy* in the critical rate of decay, depending on whether the mean offspring at a branching site is above or below a certain value related to the return probability of the simple random walk. The dichotomy marks a transition from local to global behavior in the progeny that hits the origin.

We also consider the situation where the branching sites occur in two or more types, with different offspring distributions, and show that the classification is more subtle due to a possible interplay between the types.

This note is part of a series of papers by the second author and various co-authors investigating the problem of transience vs. recurrence for random motions in random media.

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1 Introduction and results

1.1 Two colors. Let

$$p = \{p(x): x \in \mathbb{Z}^d\} \tag{1.1}$$

be a field of numbers satisfying $0 \leq p(x) < 1$ for all $x \in \mathbb{Z}^d$. Site x is colored *green* with probability $p(x)$ and *red* with probability $1 - p(x)$, independently of the other sites. In this way a random coloring of $\mathbb{Z}^d$ is obtained. Start with one particle at the origin. This particle performs a simple random walk on $\mathbb{Z}^d$ until it hits a green site. Upon hitting this green site the particle branches according to offspring distribution

$$q = \{q_i: i \geq 0\}, \tag{1.2}$$

i.e., with probability q_i the particle is replaced by i new particles ($0 \leq q_i \leq 1$ for $i \geq 0$, $\sum_{i\geq 0} q_i = 1$). Each new particle proceeds to perform a simple random walk, until it hits a next green site and branches according to the same offspring distribution, etc. There is no branching at the red sites. At any time, all particles walk and branch independently. The coloring is kept *fixed* ('quenched problem'). Throughout the paper we use the symbols P, E to denote probability and expectation on the joint probability space for the random coloring, the random walk and the random branching.

Let

$$\begin{array}{lcl} \eta_n(x) & = & \text{number of particles at site } x \text{ at time } n \\ \eta_n & = & \{\eta_n(x): x \in \mathbb{Z}^d\}. \end{array} \tag{1.3}$$

Then $(\eta_n)_{n\geq 0}$ is a Markov chain, with state space the finite subsets of $\mathbb{N}^{\mathbb{Z}^d}$, whose transition probabilities are easily written out in terms of q and the fixed coloring. This Markov chain is an example of a *branching random walk in random environment* (BRWRE). The goal of this note is to find necessary and sufficient conditions on p, q for $(\eta_n)_{n\geq 0}$ to be transient vs. recurrent for almost all colorings.

Definition 1 BRWRE is said to be recurrent when

$$P\{\eta_n(0) \neq 0 \text{ for some } n \geq 1\} = 1 \tag{1.4}$$

and transient otherwise.

By Kolmogorov's zero-one law, recurrence either holds for almost all colorings or for almost no coloring.

1.2 Criteria for transience vs. recurrence. It is obvious that BRWRE is transient when $q_0 > 0$ or when $\sum_{i\geq 1} iq_i \leq 1$, $q_1 \neq 1$. In both these cases the population has a strictly positive probability of dying out. It is further obvious that BRWRE is recurrent when $q_0 = 0$ and $d = 1, 2$. In that case no particle dies and each particle returns to the origin with probability 1. Therefore we shall henceforth assume that

$$q_0 = 0, \quad \sum_{i\geq 1} iq_i > 1, \quad d \geq 3. \tag{1.5}$$

Subject to (1.5), transience vs. recurrence only depends on the behavior of p at infinity. It is intuitively clear that BRWRE is transient when $p(x) \to 0$ fast enough as $\|x\| \to \infty$ and recurrent when slow enough. In fact, each particle alone performs a simple random walk, which is transient because $d \geq 3$. To have at least one particle in the progeny hit the origin, the population must grow fast enough.

It turns out that the critical rate of decay of p separating transience from recurrence exhibits a *dichotomy* depending on the value of $\sum_{i\geq 1} iq_i$. To state the result, let us abbreviate

$$\mu = \sum_{i\geq 1} iq_i, \quad \mu_c = 1/F, \tag{1.6}$$

where $0 < F = F(d) < 1$ denotes the probability that the simple random walk in dimension $d \geq 3$ returns to the origin (e.g. $F(3) \approx 0.34$).

Theorem 1 *Let $\mu > \mu_c$. Suppose that p satisfies the regularity condition*

$$\sup_{x\in\mathbb{Z}^d} \sup_{y\in\mathbb{Z}^d:\ c_1\|x\|\leq\|y\|\leq c_2\|x\|} \frac{p(x)}{p(y)} < \infty \qquad \text{for some } 0 < c_1 < c_2 < \infty. \tag{1.7}$$

Then for almost all colorings BRWRE is

$$\begin{aligned} &\textit{transient if} \ \sum_{x\neq 0} \|x\|^{2-d} p(x) < \infty \\ &\textit{recurrent if} \ \sum_{x\neq 0} \|x\|^{2-d} p(x) = \infty. \end{aligned} \tag{1.8}$$

Theorem 2 *Let $1 < \mu < \mu_c$. Then there exists an $0 < \alpha_c < \infty$ such that for almost all colorings BRWRE is*

$$\begin{aligned} &\textit{transient if} \ \limsup_{\|x\|\to\infty} \|x\|^2 p(x) < \alpha_c \\ &\textit{recurrent if} \ \liminf_{\|x\|\to\infty} \|x\|^2 p(x) > \alpha_c. \end{aligned} \tag{1.9}$$

The proofs of Theorems 1–2 are given in Sections 2–4. The key point to note is the difference in criteria on the tail behavior of p. For radially symmetric p, (1.8) reduces to the integral test $\int_0^\infty rp(r)dr < \infty$ resp. $= \infty$, while (1.9) reduces to $r^2 p(r) < \alpha_c$ resp. $r^2 p(r) > \alpha_c$ for large r.

The proof of Theorem 2 shows that $\alpha_c = \alpha_c(d,q)$ satisfies $\lim_{\mu\downarrow 1}\alpha_c = \infty$ and $\lim_{\mu\uparrow\mu_c}\alpha_c = 0$, where $\mu = \mu(q)$. It would be interesting to identify α_c explicitly, but this remains open (see Conjecture 1 below). We believe that α_c actually depends only on μ, and not of the full distribution q, provided the tail of q is tempered. We do not know what happens in the critical case $\lim_{\|x\|\to\infty}\|x\|^2 p(x) = \alpha_c$.

The dichotomy expressed by Theorems 1–2 marks a transition from local to global behavior in the progeny that hits the origin. Indeed, it turns out that if $\mu > \mu_c$, then recurrence occurs precisely when there is a *single* green site somewhere producing an infinite offspring, while if $1 < \mu < \mu_c$, then recurrence requires the progeny that hits the origin to come from *infinitely* many green sites.

We note in passing that if $q_0 > 0$ and $\mu > 1$, then *conditioned on survival* the results are the same as in Theorems 1–2 (as can be seen from the proofs in Section 2–4).

1.3 Three colors. Consider the model with three colors: site x is colored *green*, *blue* or *red* with probabilities $p_g(x)$, $p_b(x)$ resp. $1 - p_g(x) - p_b(x)$, independently of the other sites. There is no branching at red sites. At green and blue sites the offspring distribution (mean offspring) is q_g resp. q_b (μ_g resp. μ_b). Moreover, in analogy with (1.5), particles never die and $d \geq 3$.

If $\mu_g, \mu_b > \mu_c$, then the classification is the same as in Theorem 1 with p replaced by $p_g + p_b$. Similarly, if $1 < \mu_g, \mu_b < \mu_c$, then the classification is the same as in Theorem 2 with p replaced by $p_g + p_b$ and with $\alpha_c = \alpha_c(d, q_g, q_b)$. We shall therefore consider a mixed case, namely

$$\begin{array}{lll} 1 < \mu_g < \mu_c & \lim\limits_{\|x\|\to\infty} \|x\|^2 p_g(x) = \alpha & 0 < \alpha < \alpha_c(d, q_g) \\ \mu_b > \mu_c & \lim\limits_{\|x\|\to\infty} \|x\|^{2+\varepsilon} p_b(x) \to 1 & \varepsilon > 0. \end{array} \tag{1.10}$$

For this choice, BRWRE with the green sites alone or with the blue sites alone is transient. However, the interplay between green and blue can overcome transience.

Theorem 3 *For α small enough there exists an $0 < \varepsilon_c < \infty$ such that for almost all colorings BRWRE is*

$$\begin{array}{ll} \textit{transient if} & \varepsilon > \varepsilon_c \\ \textit{recurrent if} & 0 < \varepsilon < \varepsilon_c. \end{array} \tag{1.11}$$

The proof of Theorem 3 is given in Section 5 and shows that $\varepsilon_c = \varepsilon_c(d, q_g, q_b, \alpha)$. Again, it would be interesting to identify this quantity, which remains open. We believe that Theorem 3 actually holds for all $0 < \alpha < \alpha_c(d, q_g)$, but we are not able to prove this.

1.4 Many colors. Suppose there are n types of sites with branching. Site x has color i with probability $p_i(x)$, and the mean offspring at a site with color i is μ_i ($i = 1, \ldots, n$). As before, site x is colored red with probability $1 - \sum_{i=1}^{n} p_i(x)$, all colors are independent, and there is no branching at red sites.

Suppose that $\mu_i \neq \mu_c$ for $i = 1, \ldots, n$. Then the classification is very similar to the case of three colors ($n = 2$). Namely, transience vs. recurrence does not change if we group all colors i with $\mu_i < \mu_c$ into green and all colors j with $\mu_j > \mu_c$ into blue. In particular, the analogue of Theorem 3 holds after we set $p_g(x) = \sum_{i:\ \mu_i<\mu_c} p_i(x)$ and $p_b(x) = \sum_{j:\ \mu_j>\mu_c} p_j(x)$.

1.5 Motivation. Though mathematically challenging, the question addressed in this paper is obviously a bit singular. As stated in the abstract, we primarily view this note as part of a series of papers attempting to classify transience vs. recurrence for random motions in random media. Applications of the above model may for instance be found in chemistry, with the particles playing the role of a *reactant* and the coloring the role of a *catalyst*.

Problems concerning branching random walk in random environment have been treated in the literature, though far from extensively. For a recent overview we refer the reader to [4]. Most papers focus on the situation where both the initial particle configuration and the random environment are stationary ergodic random fields.

1.6 A conjecture. We close this introduction with a conjecture for the α_c in Theorem 2 due to S. Volkov.

Conjecture 1 *Let $1 < \mu < \mu_c$ and assume that q has finite variance. Then*

$$\alpha_c = \bar{\alpha}_c \frac{\mu_c - \mu}{\mu_c(\mu - 1)} \qquad \textit{with} \qquad \bar{\alpha}_c = \frac{(d-2)^2}{8d}. \tag{1.12}$$

In Section 6 we will give a heuristic argument in support of this conjecture. The main idea is to make a link with the situation where the coloring is updated at each unit of time ('annealed problem'). In fact, $\bar{\alpha}_c$ will be seen to play a role analogous to α_c.

2 Proof of Theorem 1

Note that if with positive probability the initial particle escapes the green set, then BRWRE is transient. Now, according to [3] Theorem 4.1 and [6] Remark 4, subject to (1.7) we have

$$\sum_{x \neq 0} \|x\|^{2-d} p(x) = \infty \Longleftrightarrow \text{the green set is hit a.s.} \tag{2.1}$$

Hence, all that we need to do is show that, when $\mu > 1/F$, if the green set is hit a.s. then the progeny hits the origin a.s.

Assign index 0 to the initial particle. Let x_1 be the first green point hit by this particle, $x_2 \neq x_1$ the second green point, etc. Since the green set is trapping, the sequence $x_1, x_2, \ldots$ is a.s. infinite. Assign index k to all particles that are generated by the initial particle at the point x_k. Suppose we allow the particles with index k to generate their offsprings *only* at the point x_k, and suppose we assign these offsprings also index k. Then, clearly, we obtain a process that is less recurrent than the original BRWRE. Therefore it suffices to prove that this process is recurrent.

For each k, a particle with index k escapes the point x_k forever with probability $1-F$, and returns to x_k and generates i offsprings there with probability $q_i F$. Consequently, we have a Galton-Watson process at x_k, with the mean number of particles per offspring equal to $\sum_{i\geq 1} i q_i F = \mu F > 1$, implying a positive probability of not dying out. Since the Galton-Watson processes for different k are independent, a.s. at least one of them will not die out. Consequently, a.s. at least one green point is visited infinitely often, and from irreducibility of the simple random walk it follows that a.s. the origin is visited infinitely often. ∎

3 Two elementary lemmas

In this section we formulate two elementary lemmas that will be needed in the course of the proof of Theorem 2.

3.1 Hitting of spheres. Let $(\xi_n)_{n\geq 0}$ be a simple random walk on $\mathbb{Z}^d$. For $A \subseteq \mathbb{Z}^d$ and $x \in \mathbb{Z}^d$, define

$$p_A(x) = P\{\exists n \geq 1\colon\ \xi_n \in A \mid \xi_0 = x\}. \tag{3.1}$$

For $A, B \subseteq \mathbb{Z}^d$ and $x \in \mathbb{Z}^d$, with $A \cap B = \emptyset$ and $p_{A\cup B}(x) = 1$, define

$$p_{A,B}(x) = P\{\exists n \geq 1\colon\ \xi_n \in A,\ \xi_m \notin B\ \forall 1 \leq m < n \mid \xi_0 = x\}. \tag{3.2}$$

Lemma 1 *For $d \geq 3$ and for any $a > 1$*

$$\lim_{h\uparrow 1} \liminf_{r\to\infty} \min_{x\in S(r)} p_{S(hr)}(x) \quad = \quad 1 \tag{3.3}$$

$$\limsup_{r\to\infty} \max_{x\in S(r)} p_{S(a^{-1}r),S(ar)}(x) \quad < \quad \frac{1}{2}, \tag{3.4}$$

where

$$S(r) = \{x \in \mathbb{Z}^d\colon r \leq \|x\| < r+1\}. \tag{3.5}$$

Proof. The proof is a straightforward calculation, which we include to make our exposition self-contained (see also [5] Lemma 4.3 and Lemma 5.1).
1. Let $Z_n = \|\xi_n\|$. It is easily shown that for $\|x\| \to \infty$

$$\begin{aligned} E(Z_{n+1} - Z_n \mid \xi_n = x) &= \frac{d-1}{2d}\frac{1}{\|x\|} + O(\|x\|^{-2}) \\ E((Z_{n+1} - Z_n)^2 \mid \xi_n = x) &= \frac{1}{d} + O(\|x\|^{-1}). \end{aligned} \tag{3.6}$$

To prove (3.4), we note that for all $\delta > d-2$ there exists an $r_0 < \infty$ such that $Z_n^{-\delta}$ is a supermartingale on $\{r \geq r_0\}$, i.e.,

$$E(Z_{n+1}^{-\delta} - Z_n^{-\delta} \mid Z_n = r) \leq 0 \text{ for } r \geq r_0. \tag{3.7}$$

Indeed, putting $\Delta_n = Z_{n+1} - Z_n$ and using (3.6), we have

$$\begin{aligned} E(Z_{n+1}^{-\delta} - Z_n^{-\delta} \mid Z_n = r) &= r^{-\delta} E\Big(\Big(1 + \tfrac{\Delta_n}{Z_n}\Big)^{-\delta} - 1 \mid Z_n = r\Big) \\ &= r^{-\delta} E\Big(-\delta \tfrac{\Delta_n}{Z_n} + \tfrac{\delta(\delta+1)}{2}\tfrac{\Delta_n^2}{Z_n^2} \mid Z_n = r\Big) + O(r^{-\delta-3}) \\ &= \Big[-\tfrac{\delta(d-1)}{2d} + \tfrac{\delta(\delta+1)}{2d}\Big] r^{-\delta-2} + O(r^{-\delta-3}), \end{aligned} \tag{3.8}$$

which is less than 0 for r large when $\delta > d-2$.
2. For $H > 1 > h > 0$ and for r such that $hr \geq r_0$, define

$$\begin{aligned} \tau_{h,r} &= \min\{n \geq 1 \colon Z_n \leq hr \mid Z_0 = r\} \\ \tau_{H,r} &= \min\{n \geq 1 \colon Z_n \geq Hr \mid Z_0 = r\} \\ p(h,H,r) &= P\{\tau_{h,r} < \tau_{H,r}\}. \end{aligned} \tag{3.9}$$

Put $\tau = \min\{\tau_{h,r}, \tau_{H,r}\}$. Since $E\tau < \infty$, we can use (3.7) to get

$$\begin{aligned} r^{-\delta} &= E(Z_0^{-\delta} \mid Z_0 = r) \\ &\geq E(Z_\tau^{-\delta} \mid Z_0 = r) \\ &\geq p(h,H,r)(hr)^{-\delta} + [1 - p(h,H,r)](Hr+1)^{-\delta} \\ &= (Hr+1)^{-\delta} + p(h,H,r)[(hr)^{-\delta} - (Hr+1)^{-\delta}], \end{aligned} \tag{3.10}$$

so

$$p(h,H,r) \leq \frac{r^{-\delta} - (Hr+1)^{-\delta}}{(hr)^{-\delta} - (Hr+1)^{-\delta}}. \tag{3.11}$$

But if $\|x\| = r$, then $p_{S(a^{-1}r),S(ar)}(x) = p(a^{-1}, a, r)$. So $\max_{x \in S(r)} p_{S(a^{-1}r),S(ar)}(x)$ $\leq [r^{-\delta} - (ar+1)^{-\delta}]/[(a^{-1}r)^{-\delta} - (ar+1)^{-\delta}]$ for $a^{-1}r \geq r_0$. Now let $r \to \infty$ to obtain the upper bound $1/(a^\delta + 1) < 1/2$.
3. To prove (3.3), we note that for all $\delta > d$ there exists an $r_0 < \infty$ such that Z_n^δ is a submartingale on $\{r \geq r_0\}$, i.e.,

$$E(Z_{n+1}^\delta - Z_n^\delta \mid Z_n^\delta = r) \geq 0 \text{ for } r \geq r_0. \tag{3.12}$$

Using the same argument as above, we find that

$$p(h, H, r) \geq \frac{r^\delta - (Hr+1)^\delta}{(hr)^\delta - (Hr+1)^\delta}. \tag{3.13}$$

By letting $H \to \infty$ and using that $\lim_{H\to\infty} p(h, H, r) = P\{\tau_{h,r} < \infty\}$, we get $P\{\tau_{h,r} < \infty\} \geq h^\delta$. But if $\|x\| = r$, then $p_{S(hr)}(x) = P\{\tau_{h,r} < \infty\}$. So $\min_{x\in S(r)} p_{S(hr)}(x) \geq h^\delta$ for $hr \geq r_0$. Now let $r \to \infty$ and $h \uparrow 1$. ∎

3.2 Hitting of sets and the Green's function. For $x \in \mathbb{Z}^d$ and a finite $A \subseteq \mathbb{Z}^d$, define

$$M_A(x) = \sum_{y\in A} g(x, y), \tag{3.14}$$

where

$$g(x, y) = \sum_{n\geq 0} P(\xi_n = y \mid \xi_0 = x) \tag{3.15}$$

denotes the Green's function of the simple random walk.

Lemma 2 *(a) If $x \in A$, then*

$$\frac{M_A(x) - 1}{\max_{y\in A} M_A(y)} \leq p_A(x) \leq \frac{M_A(x) - 1}{\min_{y\in A} M_A(y)}. \tag{3.16}$$

(b) If $x \notin A$, then the same bounds hold without the -1 in the numerators.

Proof. (a) By (3.14–3.15) we have

$$M_A(x) = E(|\{n \geq 0 \colon \xi_n \in A \mid \xi_0 = x\}|). \tag{3.17}$$

Putting $\tau_A = \min\{n \geq 1 \colon \xi_n \in A\}$, we may write for $x \in A$

$$M_A(x) - 1 = \sum_{y\in A} P\{\tau_A < \infty, \xi_{\tau_A} = y \mid \xi_0 = x\} M_A(y). \tag{3.18}$$

Since $p_A(x) = \sum_{y\in A} P\{\tau_A < \infty, \xi_{\tau_A} = y \mid \xi_0 = x\}$ by (3.1), the claim follows.
(b) For $x \notin A$, remove the -1 from the l.h.s. of (3.18). ∎

4 Proof of Theorem 2

We shall prove that, when $1 < \mu < 1/F$,

$$\begin{array}{lll} p(x) \geq \alpha\|x\|^{-2} \text{ for large } \|x\| \text{ and large } \alpha & \Longrightarrow & \text{BRWRE recurrent} \\ p(x) \leq \alpha\|x\|^{-2} \text{ for large } \|x\| \text{ and small } \alpha & \Longrightarrow & \text{BRWRE transient.} \end{array} \tag{4.1}$$

This will imply the claim made in Theorem 2, because transience vs. recurrence only depends on the behavior of p at infinity and is a monotone property: if $p' \leq p$, then BRWRE with p' is at least as transient as BRWRE with p, and vice versa for recurrent. The latter can be easily checked with the help of a coupling argument.

4.1 Large α

Let G denote the set of green points. We shall need the following:

Proposition 1 *Suppose there exist an $\varepsilon > 0$ and an infinite sequence of finite disjoint sets $A_1, A_2, \ldots \subseteq G$ with the following property:*

$$\inf_{i\geq 1} \min_{x\in A_i} p_{A_i}(x) > \frac{1}{\mu} + \varepsilon. \tag{4.2}$$

Suppose also that the set $\cup_{i\geq 1} A_i$ is trapping, i.e., at least one particle hits this set with probability 1. Then BRWRE with G as green set is recurrent.

Proof. The proof is analogous to that of Theorem 1 in Section 2. It is not difficult to see that if the set $\cup_{i\geq 1} A_i$ is trapping, then a.s. at least one particle hits an infinite number of sets $A_{i_1}, A_{i_2}, \ldots$ Then, in the proof of Theorem 1, instead of site x_1 we take A_{i_1}, instead of site x_2 we take A_{i_2}, etc. The rest is the same. ∎

Let us now start with the proof of the first claim in (4.1).

1. For $n \geq 0$, define

$$\begin{aligned} W_n &= \{x \in \mathbb{Z}^d \colon 2^n \leq \|x\| < 2^{n+1}\} \\ G_n &= W_n \cap G. \end{aligned} \tag{4.3}$$

We shall prove that, for α large enough, the infinite sequence G_n, $G_{n+1}, \ldots$ satisfies (4.2) for n large enough a.s. Because $p(x) \geq \alpha/\|x\|^2$ for large $\|x\|$, we have $\sum_{x\neq 0} \|x\|^{2-d} p(x) = \infty$, and so by (2.1) the set $\cup_{i\geq n} G_i$ is trapping a.s. for any $n \geq 0$. Hence we get recurrence via Proposition 1.

2. Abbreviate $m = 2^{n+1}$. For $h \in (1/2, 1)$, define the sphere

$$S(hm) = \{x \in \mathbb{Z}^d \colon hm \leq \|x\| < hm + 1\}. \tag{4.4}$$

From Lemma 1 it follows that for any $\delta > 0$ there exists an h close to 1 such that

$$\min_{x\in W_n} p_{S(hm)}(x) > 1 - \delta \text{ for } n \text{ large enough.} \tag{4.5}$$

(Note that $p_{S(hm)}(x) = 1$ when $\|x\| \leq hm$.) Therefore, to prove recurrence via Proposition 1, it suffices to show that for any $\delta > 0$ there exists an α large enough such that

$$\min_{y\in S(hm)} p_{G_n}(y) > 1 - \delta \text{ for } n \text{ large enough a.s.} \tag{4.6}$$

Indeed, because $G_n \subseteq W_n$, (4.5–4.6) combine to give

$$\min_{x\in G_n} p_{G_n}(x) \geq \min_{x\in W_n} p_{G_n}(x) > (1-\delta)^2 \text{ for } n \text{ large enough a.s.,} \tag{4.7}$$

so by picking δ such that $(1-\delta)^2 > 1/\mu$ (use that $\mu > 1$), we get what was claimed in 1.

3. Pick any $y \in S(hm)$. If $A' \subset A$, then $p_{A'}(y) \le p_A(y)$. So it suffices to prove (4.6) for a smaller set than G_n. Define

$$\begin{array}{rcl} U_n^h(y) & = & \{x \in \mathbb{Z}^d : \|x-y\| \le (1-h)m\} \\ G_n'(y) & = & G_n \cap U_n^h(y). \end{array} \tag{4.8}$$

For some $\beta > 0$ (which will be chosen later) we consider a partition of $\mathbb{Z}^d$ into cubes of size $\beta m^{2/d}$:

$$K_{i_1\ldots i_d} = \mathbb{Z}^d \cap \beta m^{2/d}\{[i_1-1, i_1) \times \cdots \times [i_d-1, i_d)\}. \tag{4.9}$$

Consider now those cubes $K_{i_1\ldots i_d}$ that lie fully inside the ball $U_n^h(y)$. The number of such cubes is equal to $l_n \sim C_1 m^{d-2}$ $(n \to \infty)$, with $C_1 = C_1(d, \beta, h)$. Denote these cubes by $K_1', \ldots, K_{l_n}'$. Let ν_i be the number of green points that lie in the cube K_i'. Clearly, because $p(x) \ge \alpha/\|x\|^2$ we have

$$\begin{array}{rcl} P\{\nu_i = 0\} & \le & \displaystyle\prod_{x \in K_i'} \left(1 - \frac{\alpha}{\|x\|^2}\right) \\ & \le & \displaystyle\left(1 - \frac{\alpha}{(2h-1)^2 m^2}\right)^{\beta^d m^2} \\ & \to & \displaystyle\exp\left\{-\frac{\alpha\beta^d}{(2h-1)^2}\right\} \qquad (n \to \infty), \end{array} \tag{4.10}$$

where in the second inequality we use that $x \in U_n^h(y)$ and $y \in S(hm)$ imply that $\|x\| \ge (2h-1)m$.

4. Define a subset $G_n''(y) \subseteq G_n'(y)$ in the following way. For $1 \le i \le l_n$ with $\nu_i > 0$ we pick an arbitrary green point x_i' from K_i', and put

$$G_n''(y) = \bigcup_{1 \le i \le l_n:\ \nu_i > 0} \{x_i'\}, \tag{4.11}$$

i.e., we simply remove some points from the green set $G_n'(y)$ until each cube K_i' contains 0 or 1 green point. Note that from (4.10) and the strong law of large numbers it follows that for any $\varepsilon > 0$ the proportion of cubes containing no green point is at most

$$\theta := \exp\left\{-\frac{\alpha\beta^d}{(2h-1)^2}\right\} + \varepsilon \tag{4.12}$$

for n large enough a.s.

5. It is well known that (see [8] Section 26)

$$g(x,y) \sim \frac{K_d}{\|x-y\|^{d-2}} \qquad (\|x-y\| \to \infty). \tag{4.13}$$

Since in the set $G''_n(y)$ the green points are far from each other (typically at a distance of order $m^{2/d}$), we shall pretend in our calculations that $g(x,y) = 1 \wedge [K_d/\|x-y\|^{d-2}]$. The reader can easily check that the error committed is negligible. We want to apply Lemma 2. To that end we derive upper and lower bounds on $M_{G''_n(y)}(x)$ for $x \in G''_n(y)$.

6. First, without loss of generality we may assume that $x \in K'_1 \cap G''_n(y)$. Since $G''_n(y)$ has not more than 1 green point in each cube K'_i, we see that for n large enough a.s.

$$\begin{aligned} M_{G''_n(y)}(x) &= \sum_{z \in G''_n(y)} g(x,z) \\ &\lesssim g(0,0) + \sum_{i=2}^{l_n} \frac{1}{|K'_i|} \sum_{z \in K'_i} g(z,x) \\ &\leq g(0,0) + \tfrac{1}{\beta^d m^2} \sum_{z \in U^h_n(y)} g(x,z) \\ &\leq g(0,0) + \tfrac{1}{\beta^d m^2} \sum_{\|z\| \leq (1-h)m} 1 \wedge \tfrac{K_d}{\|z-(x-y)\|^{d-2}}. \end{aligned} \tag{4.14}$$

Here, in the first inequality we use that the Green's function is essentially constant on each K'_i. Clearly, the r.h.s. of (4.14) is maximal when $x = y$, so for n large enough a.s.

$$\begin{aligned} -g(0,0) + \max\nolimits_{x \in G''_n(y)} M_{G''_n(y)}(x) &\lesssim \tfrac{1}{\beta^d m^2} \sum_{\|z\| \leq (1-h)m} 1 \wedge \tfrac{K_d}{\|z\|^{d-2}} \\ &\to \tfrac{C_2(1-h)^2}{\beta^d} =: \eta \quad (n \to \infty), \end{aligned} \tag{4.15}$$

with $C_2 = C_2(d)$. Next, the worst case for a lower bound occurs when all the cubes K'_i with 0 green points are grouped around the point y. Since the proportion of such cubes is at most θ, we can assume that they all lie in the ball with center y and radius $\sim \theta^{1/d}(1-h)m$ $(n \to \infty)$. Thus we have for n large enough a.s., in the same spirit as (4.14),

$$\begin{aligned} -g(0,0) + \min_{y \in S(hm)} M_{G''_n(y)}(x) &\gtrsim \tfrac{1}{\beta^d m^2} \sum_{\theta^{1/d}(1-h)m \leq \|z\| \leq (1-h)m} 1 \wedge \tfrac{K_d}{\|z\|^{d-2}} \\ &\to \eta(1-\theta^{2/d}) \quad (n \to \infty) \end{aligned} \tag{4.16}$$

7. Combining (4.15–4.16) and applying Lemma 2(a), we get

$$\min_{y \in S(hm)} p_{G''_n(y)}(x) \gtrsim 1 - \frac{1}{g(0,0)+\eta}[\eta\theta^{2/d} + 1] \quad \text{for } n \text{ large enough a.s.} \tag{4.17}$$

We see from (4.12), (4.15) and (4.17) that for fixed h it is possible to choose α large enough and β, ε small enough such that $\min_{y \in S(hm)} p_{G''_n(y)}(x)$ is arbitrarily close to 1 for n large enough a.s. Hence we have proved (4.6) and so the proof of the first claim in (4.1) is complete. ∎

4.2 Small α

Let G again denote the set of green points, and let W_n and G_n be the sets defined by (4.3). We again use the abbreviation $m = 2^{n+1}$. Let

$$H_R(x) = \{y \in \mathbb{Z}^d\colon\ \|y - x\| \leq R\}. \tag{4.18}$$

The key tool in our proof will be the following proposition, which plays a role analogous to Proposition 1.

Proposition 2 *Suppose that $p(x) \leq \alpha/\|x\|^2$ for large $\|x\|$.*
(a) For any $\varepsilon > 0$ there exists an $\alpha = \alpha(\varepsilon) > 0$ such that

$$\max_{x \in G_n} p_{G_n}(x) \leq F + \varepsilon \quad \textit{for } n \textit{ large enough a.s.} \tag{4.19}$$

(b) For any $\varepsilon > 0$ there exists an $\alpha = \alpha(\varepsilon) > 0$ and a $\delta = \delta(\varepsilon) > 0$ such that

$$\max_{x \notin G_n\colon\ H_{m^\delta}(x) \cap G_n = \emptyset} p_{G_n}(x) \leq \varepsilon \quad \textit{for } n \textit{ large enough a.s.} \tag{4.20}$$

Proof. We begin with the proof of Part (a).
1. Let $K_{i_1 \ldots i_d}$ be the cubes defined in (4.9). Consider those cubes that have a nonempty intersection with W_n. The number of such cubes is equal to $l_n \sim C_1 m^{d-2}$ $(n \to \infty)$, with $C_1 = C_1(d, \beta)$. Denote these cubes by $K'_1, \ldots, K'_{l_n}$. As before, let ν_i denote the number of green points in the cube K'_i. It is easily seen that ν_i is stochastically smaller than a Poisson random variable with parameter λ_n satisfying

$$\lambda_n \sim \frac{\alpha}{m^2}\beta^d m^2 = \alpha\beta^d \quad (n \to \infty). \tag{4.21}$$

Write

$$\theta_k = \frac{1}{l_n}\sum_{i=1}^{l_n} 1_{\{\nu_i = k\}} \tag{4.22}$$

to denote the proportion of cubes carrying exactly k green points.
2. We again want to apply Lemma 2(a). To that end we pick any $x \in G_n$ and try to bound $M_{G_n}(x)$ from above. Clearly

$$M_{G_n}(x) = \sum_{y \in G_n} g(x, y) = g(0, 0) + \sum_{k \geq 1} D_k(x), \tag{4.23}$$

where

$$D_k(x) = \sum_{i=1}^{l_n} \sum_{y \in G_n \cap K'_i\colon\ \nu_i = k} g(x, y). \tag{4.24}$$

To derive an upper bound for $D_k(x)$, we need some preparations.
3. Because $p(x) \le \alpha/\|x\|^2$ for large $\|x\|$, we have for any $0 < \delta < 1$

$$P\{|G \cap H_{\|x\|^\delta}(x)| \ge 2\} \lesssim C_2 \|x\|^{d\delta} \Big(\frac{\alpha}{\|x\|^2}\Big)^2 = C_2 \alpha^2 \frac{1}{\|x\|^{4-d\delta}} \tag{4.25}$$

for $\|x\|$ large enough a.s., with $C_2 = C_2(d)$. If $d = 3$, then for $0 < \delta < 1/3$ the r.h.s. of (4.25) is summable and by Borel-Cantelli we get that a.s. for n large enough there are no $x, y \in G_n$ such that $\|x - y\| \le m^\delta$. However, if $d \ge 4$, then there is no $\delta > 0$ for which the r.h.s. is summable. For this case Borel-Cantelli even gives that G contains an infinite number of nearest-neighbor pairs (the probability of a pair at x is of order $\|x\|^{-4}$). Nevertheless the following holds for any $d \ge 3$:

Proposition 3 *Let $0 < \delta < 1/d$ and $G^* = \{x \in G\colon\ G \cap [H_{\|x\|^\delta}(x) \setminus \{0\}] \neq \emptyset\}$. If α is small enough, then BRWRE with G as green set resp. $G \setminus G^*$ as green set are either both recurrent or both transient.*

Proof. The proof is postponed to Section 5.3. ∎

4. Before we proceed, let us explain why we need Proposition 3. We would like to show that $\max_{x \in G_n} M_{G_n}(x) - g(0,0)$ can be made arbitrarily small by picking α small, as this would allow us to apply Lemma 2(a). However, for $d \ge 4$ this is not possible, because G contains infinitely many nearest-neighbor pairs. Therefore we must remove some sites from G, without however affecting transience or recurrence. In essence, what Proposition 3 does is show that we can remove the set G^* because this is non-trapping not only for a *single* particle generated in $G \setminus G^*$ but for *all* particles generated in $G \setminus G^*$.
5. Let $G_n^* = W_n \cap (G \setminus G^*)$. Similarly as in Section 4.1, we note that the Green's function is essentially constant on each cube K_i', except the one containing x (which we may denote as K_1'). Therefore it follows from Borel-Cantelli and (4.21–4.22) that for any $\varepsilon > 0$ and for n large enough a.s.

$$\theta_k = 0 \quad (k > m^\varepsilon). \tag{4.26}$$

So, using (4.26) and Proposition 3, we have

$$\sum_{y \in K_1' \cap G_n^*} g(x,y) \le m^\varepsilon \Big[1 \wedge \frac{K_d}{m^{\delta(d-2)}}\Big], \tag{4.27}$$

which tends to zero as $n \to \infty$ when $\varepsilon < \delta(d-2)$. We therefore see that we can neglect all the green points that are in the same cube as x. This allows us to estimate $D_k(x)$. From now on we assume that $x \in G \setminus G^*$.
6. The worst case for the upper bound for $D_k(x)$ occurs when all the cubes K_i'

with $\nu_i = k$ are grouped around x. Since the number of such cubes is $l_n\theta_k \sim C_1 m^{d-2}\theta_k$, we therefore obtain, in the same spirit as the estimate in (4.14),

$$D_k(x) \lesssim k\frac{1}{\beta^d m^2} \sum_{\|z\| \le C_3[\beta^d m^2 C_1 m^{d-2}\theta_k]^{1/d}} 1 \wedge \frac{K_d}{\|z\|^{d-2}} \le C_4 k\theta_k^{2/d}, \tag{4.28}$$

with $C_3 = C_3(d)$, $C_4 = C_4(d)$. So, to estimate $M_{G_n^*}(x)$ in (4.23) we have to estimate the quantity (recall (4.26))

$$\psi_m = \sum_{k=1}^{m^\varepsilon} k\theta_k^{2/d}. \tag{4.29}$$

7. For $\gamma > 0$ and $k \ge 1$, define

$$p_k(\gamma) = P\Big\{k\theta_k^{2/d} > \frac{\gamma}{k^2}\Big\}. \tag{4.30}$$

To obtain a bound for $p_k(\gamma)$, we need the following elementary lemma.

Lemma 3 *Let ζ_i be i.i.d. random variables with $P\{\zeta_i = 1\} = p$ and $P\{\zeta_i = 0\} = 1 - p$. Then for any $0 < p < a < 1$*

$$P\Big\{\frac{1}{n}\sum_{i=1}^{n}\zeta_i \ge a\Big\} \le \exp\{-nH(a,p)\} \text{ for all } n \ge 1, \tag{4.31}$$

where

$$H(a,p) = a\log\frac{a}{p} + (1-a)\log\frac{1-a}{1-p} > 0. \tag{4.32}$$

Proof. See e.g. [7] p. 68. ∎

Using Lemma 3 and (4.21), we obtain

$$\begin{aligned} p_k(\gamma) &= P\Big\{\theta_k > \frac{\gamma^{d/2}}{k^{3d/2}}\Big\} \\ &= P\Big\{\frac{1}{l_n}\sum_{i=1}^{l_n} 1_{\{\nu_i = k\}} > \frac{\gamma^{d/2}}{k^{3d/2}}\Big\} \\ &\le \exp\Big\{-l_n H\Big(\frac{\gamma^{d/2}}{k^{3d/2}}, \frac{\lambda_n^k e^{-\lambda_n}}{k!}\Big)\Big\}. \end{aligned} \tag{4.33}$$

8. Next we note the following facts:

1. Since $\lambda_n \to 0$ as $n \to \infty$ and $\alpha \to 0$ (recall (4.21)), we can, for any $\gamma > 0$, take α small enough such that $\frac{\gamma^{d/2}}{k^{3d/2}} > \frac{\lambda_n^k e^{-\lambda_n}}{k!}$ for all $k \ge 1$ for n large enough.

2. $H(\frac{\gamma^{d/2}}{k^{3d/2}}, \frac{\lambda_n^k e^{-\lambda_n}}{k!}) \sim \gamma^{d/2} k^{-(3d-2)/2} \log k$ as $k \to \infty$.

Thus we obtain, recalling that $l_n \sim C_1 m^{d-2}$,

$$\max_{1 \le k \le m^\varepsilon} p_k(\gamma) \le \exp\{-C_5 m^{d-2-\frac{(3d-2)\varepsilon}{2}} \log m^\varepsilon\}, \tag{4.34}$$

with $C_5 = C_5(d, \beta, L, \gamma)$. Using Borel-Cantelli, we therefore see that for n large enough a.s. we have $k\theta_k^{2/d} \le \frac{\gamma}{k^2}$ for all $1 \le k \le m^\varepsilon$, so

$$\psi_m \le \sum_{k=1}^{m^\varepsilon} \frac{\gamma}{k^2} \le \frac{\pi^2 \gamma}{6} \text{ for } n \text{ large enough a.s.} \tag{4.35}$$

9. We can now complete the proof of Part (a) by choosing γ small enough. Indeed, via (4.23–4.29) and (4.35) we then get $M_{G_n^*}(x) \le g(0,0) + \varepsilon$ for any $\varepsilon > 0$ for n large enough a.s. But, trivially, $M_{G_n^*}(x) \ge g(0,0)$, and so Lemma 2(a) gives (recall that $x \in G_n^*$ is arbitrary)

$$\max_{x \in G_n^*} p_{G_n^*}(x) \le \frac{g(0,0) - 1 + \varepsilon}{g(0,0)} \text{ for } n \text{ large enough a.s.} \tag{4.36}$$

Since $F = 1 - 1/g(0,0)$, this proves the claim in Part (a).
10. Part (b) is proved just like Part (a), except that in (4.36) instead of Lemma 2(a) we use Lemma 2(b). Indeed, the constant $g(0,0)$ drops out of the r.h.s. of (4.23), and the rest of the proof may be left intact. ∎

Let us now start the proof of the second claim in (4.1).
1. Let

$$S_n = S(2^n) = \{x \in \mathbb{Z}^d : 2^n \le \|x\| < 2^n + 1\}, \tag{4.37}$$

which is the inner layer of W_n. A possible approach is to observe particles only when they are in one of the spheres $S_n, n \ge 0$. However, then a complication arises: there may be some points in G that lie close to S_n, namely, within distance $2^{\delta n - 1}$, where δ is from Proposition 3. To avoid this, we modify the set S_n. Pick any $x \in G$ such that $S_n \cap H_{2^{\delta n-1}}(x) \ne \emptyset$. Remove from S_n all the points that lie inside $H_{2^{\delta n-1}}(x)$ and add to S_n all the points that lie on $W_n \cap \partial H_{2^{\delta n-1}}(x)$, i.e., the part of the surface of $H_{2^{\delta n-1}}(x)$ sticking out of S_n. Repeat this procedure for different $x \in G$. The result is a new set S_n' that looks like the sphere S_n but has 'bubbles sticking out'. Clearly, the distance between G and S_n' is at least $2^{\delta n-1}$. We shall observe particles only when they are in one of the sets $S_n', n \ge 0$.
2. Pick any $n \ge 0$ and $x \in S_n'$. Suppose that at time 0 we place one particle at x and no particles at other sites. We introduce a *delayed* BRWRE defined as follows. At times $1, 2, \ldots, \tau_n$ the set $S_{n-1}' \cup S_{n+1}'$ is delaying, i.e., all particles upon entering this set are 'frozen' until time τ_n. Here we define τ_n to be the first time when there are no particles in the interior of the 'ring' between S_{n-1}' and

S'_{n+1}. After that we independently repeat this procedure with all the offsprings of the initial particle, starting from S'_{n-1} resp. S'_{n+1}, etc. Now, it can be easily seen that if $\tau_n < \infty$ a.s. for all n, then this delayed BRWRE is equivalent to the original BRWRE from the point of view of transience vs. recurrence.
3. We next prove that there exists an α small enough such that indeed $\tau_n < \infty$ a.s. for all n. Recall the definition of G^*. Let G'_n be the intersection of $G \setminus G^*$ with the 'ring' between S'_n and S'_{n+1}. Using Proposition 2(a), we see that we can choose α small enough such that a.s. for n large enough

$$\mu \max_{x \in G'_n} p_{G'_n}(x) < 1 \tag{4.38}$$

(use that $\mu < \mu_c = 1/F$). Since particles are frozen upon entering $S'_{n-1} \cup S'_{n+1}$, it is obvious that $P\{\tau_n < \infty\}$ is larger than or equal to the probability of extinction of a Galton-Watson process with the mean number of particles per offspring equal to $\mu \max_{x \in G'_n} p_{G'_n}(x)$. By (4.38) we therefore have $P\{\tau_n < \infty\} = 1$.
4. Thus it now suffices to prove that for α small enough the delayed BRWRE is transient. This will be done via comparison with a one-dimensional BRWRE. Let $\eta_n(x)$ be the total number of particles generated in the interior of the 'ring' between S'_{n-1} and S'_{n+1} up to time τ_n (not including the initial particle). Define the distribution function

$$h_n(k) = \max_{x \in S'_n} P\{\eta_n(x) \geq k\} \qquad (k \geq 0). \tag{4.39}$$

Define also $v_n = \max_{x \in S'_n} p_{S'_{n-1}}(x)$. Then the delayed BRWRE is at least as transient as the BRWRE on $\mathbb{N}$ defined as follows. If there is a particle in n, then this particle jumps to $n-1$ with probability v_n and to $n+1$ with probability $1 - v_n$. (Since in the d–dimensional process we do not know where this particle generates its offsprings, we must consider the worse scenario: all the offspring is absorbed in S'_{n-1}. Afterwards we add ζ_n particles to the point $n-1$, where the random variable ζ_n is distributed according h_n. We repeat this procedure for all particles independently, etc.
5. From Lemma 1 it follows that for all n large enough $v_n < 1/2 - \varepsilon$ for some $\varepsilon > 0$. Proposition 2(b) implies that for any $\varepsilon > 0$ we can choose α small enough such that $E(\zeta_n) < \varepsilon$ for all n. Indeed, since the distance between G and S'_n is at least $2^{\delta n - 1}$, Proposition 2(b) gives us that the probability of hitting a green point is small. Therefore the mean size of the total offspring generated by the Galton-Watson processes mentioned below (4.38) is finite and bounded uniformly in n. From this we get that the mean size of the progeny of the initial particle up to the moment τ_n is small. Indeed, let

$$k_0 = \max_n \max_{x \in G'_n} E\{\text{size of progeny} \mid \text{initial particle starts at } x\}. \tag{4.40}$$

Since the probability of hitting a green point is small, say less than ε_0, the mean size of the progeny is less than $\varepsilon_0 k_0$, and this can be made arbitrarily small by picking α small enough (recall Proposition 2(b)).

6. To complete the proof we compare the above BRWRE on $\mathbb{N}$ with a *spatially homogeneous* BRW on $\mathbb{N}$. Suppose that each particle produces a mean offspring r_{-1} at the left neighbor and a mean offspring r_{+1} at the right neighbor, while site 0 is absorbing.

Lemma 4 *There exists an $\varepsilon > 0$ small enough such that: if $r_{-1} < v + \varepsilon < 1/2$ and $r_{+1} = 1 - v > 1/2$, then spatially homogeneous BRWRE on $\mathbb{N}$ is transient.*

Proof. Let $f(x) = \lambda^x$ $(x \geq 0)$, where λ is the smallest positive root of the equation $(v + \varepsilon) - \lambda + (1 - v)\lambda^2 = 0$. If $v < 1/2$, then there exists an $\varepsilon > 0$ such that this equation has two real roots that are both strictly positive and strictly less than 1. Hence,

$$\sum_{i=-1,+1} r_i f(x + i) \leq f(x) \text{ for all } x > 0. \tag{4.41}$$

The claim now follows from [2] Theorems 2.1 and 2.2. ∎

7. If we now consider the quantities $r_{\pm 1}$ for the spatially homogeneous BRW on $\mathbb{N}$, and choose α small enough, then we can make ε arbitrarily small while keeping v fixed. Indeed, since $r_{-1} = v + \varepsilon$ and $r_{+1} = 1 - v$, by applying Lemma 4 we see that the spatially homogeneous BRW is transient. Hence the delayed BRWRE is transient, which finishes the proof of the second claim in (4.1). ∎

We conclude this section with a remark. Suppose that BRWRE is transient. Is it true that for almost all colorings the number of particles hitting 0 is finite a.s.? The answer is no.

To see why, consider first the case $\mu > \mu_c$. The set G, although non-trapping, is hit with a positive probability, and so the Galton-Watson processes in the proof of Theorem 1 survive with a positive probability. Hence, for almost all colorings the number of particles hitting 0 is *infinite with a positive probability.*

Next consider the case $1 < \mu < \mu_c$. For $d = 3$ the number of hittings of 0 is *finite with probability 1 only with some positive probability w.r.t. the coloring.* For $d \geq 4$, on the other hand, there exists a constant $1 < \mu(d) < \mu_c$ such that: if $1 < \mu < \mu(d)$ then the situation is the same as for $d = 3$, while if $\mu > \mu(d)$ then the situation is the same as for $\mu > \mu_c$. Indeed, the probability that there are k green points grouped together around x is of order $\|x\|^{-2k}$. So if $2k \leq d$, then there is an infinite number of such groups in $\mathbb{Z}^d$. Let $k_0 = \lfloor d/2 \rfloor$, and put (recall (3.1))

$$F_d := \max_{x_1,\dots,x_{k_0} \in \mathbb{Z}^d} \min_{i=1,\dots,k_0} p_{\{x_1,\dots,x_{k_0}\}}(x_i). \tag{4.42}$$

Trivially, $F_d \geq F$, and the inequality is strict for $d \geq 4$. It is not difficult to prove that $\mu(d) = 1/F_d$.

5 Proof of Theorem 3

Recall (1.10). We shall prove that BRWRE is

$$\begin{array}{l}\text{recurrent for fixed } \alpha \text{ and small } \varepsilon \\ \text{transient for fixed } \varepsilon \text{ and small } \alpha.\end{array} \tag{5.1}$$

This will imply the claim made in Theorem 3 by monotonicity: the more branching sites there are, the more recurrent BRWRE is.

5.1 Small ε

1. To prove the recurrence it suffices to prove that the blue set B is trapping w.r.t. the entire progeny. For that we need some preparatory lemmas. Denote by u_n the number of particles generated in the set $G_n = W_n \cap G$ (not including the initial particle).

Lemma 5 *$Eu_n > \gamma^n$ for some $\gamma > 1$ and n large enough. Moreover, $u_n > (\gamma - \varepsilon)^n$ for any $\varepsilon > 0$ and n large enough a.s.*

Proof. From the argument in Section 4.1 it follows that any particle entering W_n produces a mean offspring that is uniformly larger than 1 for all n. From this the claim follows. More precisely, from Section 4.1 we see that each particle entering W_n produces some random number of offsprings, say κ, which can be minorized by some random variable κ_n, the distribution of which is independent of n and satisfies $E\kappa_n > \gamma > 1$. From this and the strong law of large numbers the claim follows. ∎

2. We shall estimate from above the probability that a particle never enters B when it starts from some $x \in G_n$. Denoting $B_n = W_n \cap B$, we see that this probability is less than or equal to the probability that the particle never enters B_n, which is equal to $1 - p_{B_n}(x)$ with $x \in G_n$. Pick $\varepsilon, \varepsilon' > 0$. Similarly as in Section 4.1, we partition W_n into cubes K_i' of size $m^{(2+\varepsilon+\varepsilon')/d}$. The number of such cubes is equal to $l_n \sim C_1 m^{d-2-\varepsilon-\varepsilon'}$ $(n \to \infty)$, with $C_1 = C_1(d)$. The probability that there are no blue sites in a given cube K_i' is easily seen to be of order $\exp(-m^{\varepsilon'})$. Therefore, Borel-Cantelli gives us that in all but finitely many cubes we have at least one blue site. As before, we remove some blue sites until every cube contains exactly one blue site. Then a calculation similar to (4.14–4.16) gives

$$\min_{x \notin B_n} M_{B_n}(x) \geq \frac{C_2}{m^{\varepsilon+\varepsilon'}} \quad \text{for } n \text{ large enough a.s.,} \tag{5.2}$$

with $C_2 = C_2(\alpha, \varepsilon, \varepsilon')$.
3. Next we apply Lemma 2(b). The denominator in the lower bound in Lemma

2(b) is at least of order 1, since trivially $\min_{x\in B_n} M_{B_n}(x) \geq g(0,0)$. Since there is not more than one blue point in each cube, a calculation similar to (4.14–4.16) gives us that $\max_{x\in B_n} M_{B_n}(x) \leq g(0,0)+C_3$, with $C_3 = C_3(\alpha,\varepsilon,\varepsilon')$. So Lemma 2(b) gives

$$\min_{x\notin B_n} p_{B_n}(x) \geq \frac{C_4}{m^{\varepsilon+\varepsilon'}} \geq \frac{C_4}{2^{(n+1)(\varepsilon+\varepsilon')}}, \tag{5.3}$$

with $C_4 = C_4(\alpha,\varepsilon,\varepsilon')$.

4. Pick ε and ε' such that $2^{\varepsilon+\varepsilon'} < \gamma$. Number the particles in the order in which they are born (particles never die because of (1.5)). Then

$$\sum_{n=1}^{\infty}\sum_{i=1}^{u_n} P\{i\text{-th particle ever enters } B_n\} = \infty \qquad \text{a.s.} \tag{5.4}$$

by the second part of Lemma 5. Thus, by Borel-Cantelli, we see that the set B is visited infinitely often. An argument similar to that in the proof of Theorem 1 now gives recurrence because $\mu_b > \mu_c$. ∎

5.2 Small α

Let us explain why we have postponed the proof of Proposition 3 to the end of our paper. As discussed earlier, for $d \geq 4$ we have an infinite number of nearest-neighbor pairs of green sites a.s. Let (x,y) be any such a pair, and let

$$F' := P\{\xi_n = x \text{ or } y \text{ for some } n>0 \mid \xi_0 = x\}, \tag{5.5}$$

where $(\xi_n)_{n\geq 0}$ is a simple random walk in $\mathbb{Z}^d$. Clearly, F' does not depend on (x,y), and $F' > F$. Suppose that $1/F' < \mu_g < 1/F$. If a particle hits some nearest-neighbor pair of green sites, then with a positive probability it produces an infinity progeny there (cf. the proof of Theorem 1). So, to prove transience, it is necessary to prove that the set of all nearest-neighbor pairs of green sites is not trapping w.r.t. the entire progeny. In fact, we even need to prove that the set G^* (which contains all the nearest-neighbor pairs) from Proposition 3 is not trapping w.r.t. to the entire progeny. However, since the density of G^* around x is less than $\|x\|^{-2}$, it turns out that this problem is very similar to the problem of proving that the blue set is not trapping.

Thus, we have to prove that $B\cup G^*$ is not trapping w.r.t. all the particles generated in $G\setminus G^*$. In the present section we shall prove that B is not trapping. In the next section we shall prove that G^* is not trapping. The reason for choosing this order is that the latter requires some artificial construction. Thus, throughout this section we shall assume that only the points in $G\setminus G^*$ are green.

1. Recall the definition of u_n from Section 5.1.

Lemma 6 *$Eu_n < \gamma^n$ for some $\gamma > 1$ and n large enough. Moreover, $u_n < (\gamma+\varepsilon)^n$ for any $\varepsilon > 0$ and n large enough a.s. Here γ can be made arbitrarily close to 1 by picking α small enough.*

Proof. At the end of Section 4.2 we saw that our BRWRE can be majorized by a one-dimensional spatially homogeneous BRW that is transient. For the latter it is not difficult to get the claim of Lemma 6. One way of doing this is the following. Each particle alone performs a transient homogeneous random walk $(\zeta_n)_{n\geq 0}$ on $\mathbb{Z}_+$. Therefore there exists an $a > 0$ such that $\zeta_n > an$ a.s. for n large enough, and

$$P\{\zeta_n \leq an\} \leq e^{-bn} \tag{5.6}$$

for some $b > 0$. Since the BRW is spatially homogeneous, the total number of particles at time n is at most $e^{b'n}$ a.s. for n large enough, where $b' > 0$ can be made arbitrary small by picking α small enough. So, we choose α so small that $b' < b$, and we get from (5.6) that the coordinate of the leftmost particle of the progeny at time n is $\geq an$ a.s. for n large enough. Thus, the total number of particles that ever see sites $< an$ is at most $e^{b'n}$. From this the claim of Lemma 6 follows. ∎

2. We want to show that the blue set is not trapping with respect to the entire progeny. The main idea to achieve this is to keep our model *quenched* w.r.t. the green sites, but to make it *annealed* w.r.t. the blue sites. More precisely, first we color x green with probability $p_g(x)$, independently for all x, and the resulting green set remains fixed forever. After that, if x is not green, then at each unit of time we color it blue with probability $p'(x)$ and red with probability $1-p'(x)$, independently of the others sites and of the colors at previous times, where

$$p'(x) = \frac{p_b(x)}{1-p_g(x)}. \tag{5.7}$$

It is easily shown (see e.g. [3] Section 5) that the annealed blue set is more trapping than the quenched blue set. We note that $p'(x) \sim p_b(x)$ for large $\|x\|$.
3. Let N_n be the number of particles at time n, and let $(\xi_n^1, \ldots, \xi_n^{N_n})$ denote their positions. Let τ_i be the moment of birth of the i–th particle. Define

$$R_i = P\{i\text{–th particle never sees blue}\} = \prod_{n=\tau_i}^{\infty} [1-p'(\xi_n^i)]. \tag{5.8}$$

Then

$$\begin{aligned} &P\{\text{none of the particles ever sees blue}\} \\ &= \textstyle\prod_{n=1}^{\infty} \prod_{x:\ \xi_n^i = x \text{ for some } i=1,\ldots,N_n} [1-p'(x)] \\ &\geq \textstyle\prod_{n=1}^{\infty} \prod_{i=1}^{N_n} [1-p'(\xi_n^i)] \\ &= \textstyle\prod_{i=1}^{\infty} R_i. \end{aligned} \tag{5.9}$$

So we need to prove that $\prod_{i=1}^{\infty} R_i > 0$, i.e., $\sum_{i=1}^{\infty}(1-R_i) < \infty$.

4. We try to estimate $1-R_i$ from above when the i–th particle is born somewhere in W_n. Following [3] Lemma 2.1, we note that the probability of trapping by the annealed blue set is equal to the probability of hitting 0 for the random walk that jumps from x directly to 0 with probability $p'(x)$ and from x to nearest-neighbor sites with probability $1-p'(x)$. Let $(\zeta_n)_{n\geq 0}$ denote this random walk.

Lemma 7 *Let ε be as in (1.10). There exist constants $\kappa, K > 0$ such that $(f(\zeta_n))_{n\geq 0}$ is a supermartingale on $\{\|x\| > K\}$, where*

$$f(x) = \frac{1}{(\|x\|+\kappa)^{\varepsilon}}. \tag{5.10}$$

Proof. See [3]. Use that $p'(x) \sim p_b(x) \sim 1/\|x\|^{2+\varepsilon}$. ∎

5. Let $\pi(x) = P\{\exists n \geq 0\colon\ \zeta_n = 0 \mid \zeta_0 = x\}$ (i.e., the probability of trapping of the original random walk by the annealed blue set). Denote by $\pi_K(x)$ the probability of hitting the ball $H_K(0)$, and by $\pi_{K,M}(x)$ the probability of hitting the sphere $\partial H_M(0)$ before hitting the ball $H_K(0)$ $(M > K)$. Obviously, $\pi(x) \leq \pi_K(x)$. Make the ball and the sphere absorbing. Then $f(\zeta_n) \to f_\infty$ $(n\to\infty)$ and, using Lemma 7, we get that for large $\|x\|$

$$f(x) \geq Ef_\infty \geq \pi_{K,M}(x)f(K) + [1-\pi_{K,M}(x)]f(M), \tag{5.11}$$

or

$$\pi_{K,M}(x) \leq \frac{f(x)-f(M)}{f(K)-f(M)}. \tag{5.12}$$

By letting $M\to\infty$ we get

$$\pi(x) \leq \pi_K(s) \leq \frac{f(x)}{f(K)} \sim (K+\kappa)^{\varepsilon}\|x\|^{-\varepsilon}. \tag{5.13}$$

6. Finally, we apply Lemma 6, choosing α small enough to ensure that $\gamma < 2^{\varepsilon}$. We have

$$\sum_{i=1}^{\infty}(1-R_i) = \sum_{n=1}^{\infty}\sum_{i=1}^{u_n} P\{i\text{-th particle ever sees blue}\} < \infty \qquad \text{a.s.}, \tag{5.14}$$

which completes the proof. ∎

5.3 Proof of Proposition 3

We need to prove that with positive probability none of the particles generated in $G\backslash G^*$ hits G^*. The idea is the same as in the proof of Theorem 3: we majorize the quenched problem by the annealed one.

1. We begin by defining a random set G^{**} as follows: for each $x \in \mathbb{Z}^d$, with probability

$$p_x := \frac{C_2 \alpha^2}{\|x\|^{4-d\delta}} \tag{5.15}$$

we color green the whole ball $H_{4\|x\|^\delta}(x)$, independently of the other sites. Note that, by (4.25), p_x is greater than or equal to the probability of the event $\{|G \cap H_{\|x\|^\delta}(x)| \geq 2\}$.

2. The following lemma will be needed.

Lemma 8 *$(G \setminus G^*) \cup G^{**}$ is stochastically larger than G (except for a finite neighborhood of the origin).*

Proof. For $x \in \mathbb{Z}^d$, let $A_x = \{|G \cap H_{\|x\|^\delta}(x)| \geq 2\}$. By (4.25) we have

$$P(A_x) \lesssim p_x. \tag{5.16}$$

We want to show that the set G^{**} is stochastically larger than G^*.

Let $\eta_x = 1_{A_x}$. Then $\{\eta_x\colon x \in \mathbb{Z}^d\}$ is a field of *dependent* Bernoulli random variables, of which we know the distribution on all finite sets. We can *reconstruct* the set G in the following way: (i) construct a realization of the field $\{\eta_x\}$; (ii) given this realization, construct G, keeping in mind that it has to correspond to the field $\{\eta_x\}$.

Consider also a field of *independent* Bernoulli random variables $\{\eta'_x\colon x \in \mathbb{Z}^d\}$, with $P\{\eta'_x = 1\} = p_x$. Let

$$\begin{aligned} U &= \{x \in \mathbb{Z}^d\colon \eta_x = 1\} \\ U' &= \{x \in \mathbb{Z}^d\colon \eta'_x = 1 \text{ or } \exists y \text{ such that } \eta'_y = 1 \text{ and } H_{\|x\|^\delta}(x) \subset H_{4\|y\|^\delta}(y)\}. \end{aligned} \tag{5.17}$$

Clearly, it is sufficient to show that U' is stochastically larger than U.

Note that if $y \notin H_{4\|x\|^\delta}(x)$, then $H_{\|x\|^\delta}(x) \cap H_{\|y\|^\delta}(y) = \emptyset$, so in this case the random variables η_x and η_y are independent. We construct a coupling such that $U' \supset U$. To that end, we let $\{\zeta_x\colon x \in \mathbb{Z}^d\}$ be i.i.d. random variables, uniformly distributed on $[0,1]$, and we put $\eta'_x := 1_{\{\zeta_x < p_x\}}$. To construct the field $\{\eta_x\}$, we first enumerate all the sites: $\mathbb{Z}^d = \{x_1, x_2, \ldots\}$. Next we put $\eta_{x_1} = 1_{\{\zeta_{x_1} < p_{x_1}\}}$, so $\eta_{x_1} = \eta'_{x_1}$. If $\eta_{x_1} = 0$, then using that the random variables η_{x_1} and η_{x_2} are either independent or positively correlated, we have

$$p'_{x_2} := P\{\eta_{x_2} = 1 \mid \eta_{x_1} = 0\} \leq P\{\eta_{x_2} = 1\} = p_{x_2}, \tag{5.18}$$

so

$$\eta_{x_2} := 1_{\{\zeta_{x_2} < p'_{x_2}\}} \leq \eta'_{x_2}. \tag{5.19}$$

If $\eta_{x_1} = \eta'_{x_1} = 1$, then all sites y such that η_x and η_y are dependent already belong to U', so we can just exclude them from the sequence $\{x_1, x_2, \ldots\}$. We continue to construct the field $\{\eta_x\}$ in this way. At each step n we have

$$[U \cap \{x_1, x_2, \ldots, x_n\}] \subset [U' \cap \{x_1, x_2, \ldots, x_n\}]. \tag{5.20}$$

Hence, by induction, $U \subset U'$, and the proof of Lemma 8 is complete. ∎

3. Since every $x \in \mathbb{Z}^d$ belongs to $\sim C_2\|x\|^{d\delta}$ balls $H_{4\|y\|^\delta}(y)$, with $C_2 = C_2(d)$, we get by using (4.25) that

$$p''(x) := P\{x \in G^{**}\} \lesssim \frac{C_2^2\alpha^2}{\|x\|^{4-2d\delta}} \quad \text{for } \|x\| \text{ large enough.} \tag{5.21}$$

Because $0 < \delta < 1/d$, this implies that the density of G^{**} is $\lesssim C_2\|x\|^{-(2+\varepsilon)}$ for some $\varepsilon > 0$. Then, it is possible to follow the ideas in the proof of Theorem 3, with G^{**} taking over the role of the blue set, to complete the proof of Proposition 3. More precisely, using that G^{**} is independent of G, we can make the set G^{**} *annealed*. Then, using that the points of G^{**} are positively correlated, we can easily show that (5.9) also holds with $p'(x)$ is replaced by $p''(x)$. The rest of the argument is the same. ∎

6 Support for Conjecture 1

In this section we explain what is behind Conjecture 1. The argument is due to S. Volkov.

Transience vs. recurrence only depends on the behavior of $p(x)$ for large $\|x\|$. Because for large $\|x\|$ the density of green sites is small, most green sites will be isolated, i.e., far away from other green sites. Therefore it is reasonable to assume that the dominant part of the offspring comes from particles that never 'commute' between green sites. In other words, particles can return a number of times to the same green site, but once they hit another green site they never return.

In this approximation, the mean total offspring $\bar{\mu}$ per particle at a green site is given by the equation

$$\bar{\mu} = \mu[(1-F) + F\bar{\mu}], \tag{6.1}$$

because each particle has a probability F to return to the green site and generate its own offspring. So

$$\bar{\mu} = \frac{\mu(1-F)}{1-\mu F}. \tag{6.2}$$

Thus, the annealed equivalent of our quenched problem is one where the mean offspring at site x is

$$[1-p(x)] + \bar{\mu}p(x) = 1 + p(x)\frac{\mu - 1}{1-\mu F}. \tag{6.3}$$

In the case where $p(x) \sim \alpha\|x\|^{-2}$, this equals

$$1 + \frac{\bar{\alpha}}{\|x\|^2} + \text{h.o.} \qquad \text{with} \qquad \bar{\alpha} = \alpha\frac{\mu_c(\mu-1)}{\mu_c - \mu}, \tag{6.4}$$

where we recall that $\mu_c = 1/F$. A similar calculation shows that if the offspring distribution at a green site has finite variance, then the variance of the offspring at site x is $O(\|x\|^{-2})$. Next, in [5] Theorem 5.1 it is shown that, under these conditions, the annealed BRWRE is transient if $\bar{\alpha}$ is small and recurrent if $\bar{\alpha}$ is large. In fact, the calculations in [5] Section 4 suggest that there is a critical value $\bar{\alpha}_c$ separating transience from recurrence given by

$$\bar{\alpha}_c = \frac{(a_3)^2}{8a_2} \qquad \text{with} \qquad a_3 = 2a_1 - a_2, \tag{6.5}$$

where $a_1 = \frac{d-1}{2d}$ and $a_2 = \frac{1}{d}$ are the coefficients occurring in (3.6). If this were true, then $\bar{\alpha}_c = \frac{(d-2)^2}{8d}$ and Conjecture 1 would follow from the relation between α and $\bar{\alpha}$ in (6.4).

The above heuristic argument hinges on the assumption that most particles do not 'commute' between green sites. As explained in Steps 3 and 4 in the proof of Proposition 2 in Section 4.2, for $d \geq 4$ this point is actually a bit delicate. This is why a proof of Conjecture 1 is presently beyond our reach.

References

[1] F. Comets, M.V. Menshikov, S.Yu. Popov, Lyapunov functions for random walks and strings in random environment, to appear in *Ann. Prob.*

[2] F. Comets, M.V. Menshikov, S.Yu. Popov, One-dimensional branching random walk in random environment: a classification. *Markov Processes Relat. Fields* **4** (1998), 465–477.

[3] F. den Hollander, M.V. Menshikov, S.E. Volkov, Two problems about random walks in a random field of traps, *Markov Processes Relat. Fields* **1** (1995), 185–202.

[4] A. Klenke, A review on spatial catalytic branching, in: *International Conference on Stochastic Models, Carleton University, June 9–12, 1998, in honor of D. Dawson* (eds. L. Gorostiza and G. Ivanoff), Conference Proceedings Series of the Canadian Mathematical Society, 1999.

[5] M.V. Menshikov and S.E. Volkov, Branching Markov chains: qualitative characteristics, *Markov Processes Relat. Fields* **3** (1997), 1–18.

[6] R. Pemantle, S. Volkov, Markov chains in a field of traps, *J. Theor. Prob.* **11** (1998).

[7] A. Shiryaev, *Probability* (2nd. ed.), Springer, New York, 1989.

[8] F. Spitzer, *Principles of Random Walk* (2nd. ed.), Springer, New York, 1976.