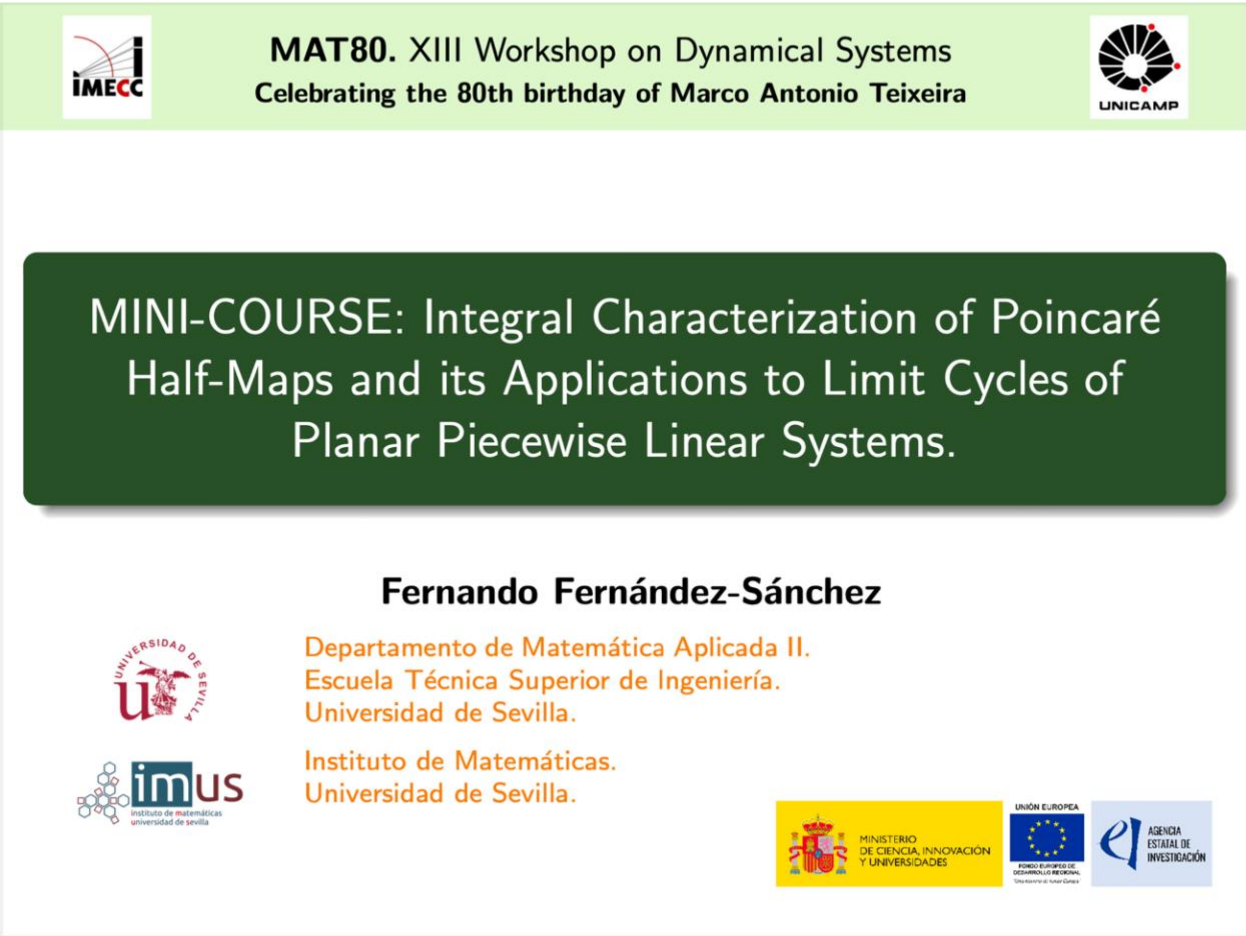
Book of Exercises



(Send the answers to fefesan@us.es)

What is an inverse integrating factor?

Let us consider the SDE

(S) $\left\{ \begin{array}{rcl} \dot{x} & = & f(x,y), \\ \dot{y} & = & g(x,y). \end{array} \right. \quad \left(\cdot = \frac{d}{dt} \right)$

The ODE can also be written as

(O) $g(x,y)dx - f(x,y)dy = 0.$

An inverse integrating factor (IIF) of system (S) in a region $\mathcal{U} \subset \mathbb{R}^2$ is a function $V : \mathcal{U} \to \mathbb{R}$ such that:

- $V \in C^1(\mathcal{U})$,
- V is not locally null,
- V satisfies the PDE

$$\nabla V(x,y) \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} = V(x,y)\,\mathrm{div} \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}.$$

Why the name IIF?

Exercise

- If V satisfies $\nabla V(x,y) \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} = V(x,y)\,\mathrm{div} \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$, then $1/V$ is an integrating factor for equation (O) on $\mathcal{U} \setminus V^{-1}(\{0\})$, that is, the equation $\frac{g(x,y)}{V(x,y)}dx - \frac{f(x,y)}{V(x,y)}dy = 0$ is exact on $\mathcal{U} \setminus V^{-1}(\{0\})$.

- Moreover, after the change of time $ds = V(x,y)dt$, the system $\left\{ \begin{array}{rcl} \dot{x} & = & f(x,y), \\ \dot{y} & = & g(x,y), \end{array} \right.$ can be written on $\mathcal{U} \setminus V^{-1}(\{0\})$ as the hamiltonian system $\left\{ \begin{array}{rcl} \frac{dx}{ds} & = & \frac{f(x,y)}{V(x,y)}, \\ \frac{dy}{ds} & = & \frac{g(x,y)}{V(x,y)}. \end{array} \right.$

Linear systems: Generalized Liénard canonical form

Exercise

Consider $\left\{ \begin{array}{rcl} \dot{x}_1 &=& m_{11}x_1 + m_{12}x_2 + b_1, \\ \dot{x}_2 &=& m_{21}x_1 + m_{22}x_2 + b_2, \end{array} \right.$ with Poincaré section $x_1 = 0$

- Prove that for $m_{12} = 0$, a Poincaré map cannot be defined.
- Try a linear change of variables $\left\{ \begin{array}{l} x = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3, \\ y = \beta_1 x_1 + \beta_2 x_2 + \beta_3, \end{array} \right.$ to transform the system into (LCF) $\left\{ \begin{array}{rcl} \dot{x} &=& Tx - y, \\ \dot{y} &=& Dx - a, \end{array} \right.$ and to keep the section fixed as $x = 0$.
  (Soln.: $x = x_1$, $y = m_{22}x_1 - m_{12}x_2 - b_1$).
- Check that for $D \neq 0$ there is one equilibrium at $(a/D, aT/D)$.
- Prove that the system is invariant to $(x, y, a) \leftrightarrow (-x, -y, -a)$.
- Study the flow on $x = 0$.

Inverse integrating factors: Linear systems in Liénard form

(LCF) $\left\{ \begin{array}{rcl} \dot{x} &=& Tx - y, \\ \dot{y} &=& Dx - a. \end{array} \right.$

Proposition: The set $\mathcal{V}$ of polynomial inverse integrating factors $V(x, y)$ of degree less or equal than two for system (LCF) is a finite dimensional vector space whose dimension depends on the parameters a, T and D. Concretely, the following bases $\mathcal{B}_i$ may be chosen:

- If $a^2 + D^2 \neq 0$ and
  - $T \neq 0$, then $\mathcal{B}_1 = \{D^2x^2 - DTxy + Dy^2 + a(T^2 - 2D)x - aTy + a^2\}$.
  - $T = 0$, then $\mathcal{B}_2 = \{1,\ Dx^2 + y^2 - 2ax\}$.
- If $a^2 + D^2 = 0$ and
  - $T \neq 0$, then $\mathcal{B}_3 = \{y^2 - Txy,\ y - Tx\}$.
  - $T = 0$, then $\mathcal{B}_4 = \{1,\ y,\ y^2\}$.

Inverse integrating factors: Linear systems in Liénard form

Exercise

Prove the Proposition.

Soln.: To do this, substitute the polynomial

$$V(x,y) = \sum_{0\le i+j\le 2} \alpha_{ij} x^i y^j$$

into the equation

$$\nabla V(x,y) \begin{pmatrix} Tx-y \\ Dx-a \end{pmatrix} = V(x,y)\,\mathrm{div}\begin{pmatrix} Tx-y \\ Dx-a \end{pmatrix}.$$

Then, solve the linear system of equations obtained from the equality of the coefficients of the corresponding terms and group the solutions in terms of a^2+D^2 and T.

Inverse integrating factors: Zero set

$$\boxed{V(x,y) = D^2x^2 - DTxy + Dy^2 + a(T^2-2D)x - aTy + a^2}$$

Proposition: The zero set $V^{-1}(\{0\})$ of function V is given by:

- For $D = 0$ (no equilibrium case) and
  - $T = 0$, then $V^{-1}(\{0\}) = \emptyset$.
  - $T \neq 0$, then $V^{-1}(\{0\}) = \{(x,y)\in\mathbb{R}^2 :\ T^2x - Ty + a = 0\}$.
- For $D \neq 0$ (equilibrium at $(x,y) = (a/D, aT/D)$) and
  - $T^2 - 4D > 0$, then $V^{-1}(\{0\}) = \{(x,y)\in\mathbb{R}^2 :\ 2D\left(x-\frac{a}{D}\right) = \left(T \pm \sqrt{T^2-4D}\right)\left(y - \frac{aT}{D}\right)\}$.
  - $T^2 - 4D = 0$, then $V^{-1}(\{0\}) = \{(x,y)\in\mathbb{R}^2 :\ 2D\left(x-\frac{a}{D}\right) = T\left(y - \frac{aT}{D}\right)\}$.
  - $T^2 - 4D < 0$, then $V^{-1}(\{0\}) = \{(a/D, aT/D)\}$.

A brief Comment. For $D \neq 0$ and $A = \begin{pmatrix} T & -1 \\ D & 0 \end{pmatrix}$,

$$V(x,y) = -D\ \det\left(A\begin{pmatrix} x-\frac{a}{D} \\ y-\frac{aT}{D}\end{pmatrix} \Bigg| \begin{pmatrix} x-\frac{a}{D} \\ y-\frac{aT}{D}\end{pmatrix}\right).$$

Exercise
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Inverse integrating factors: Some comments

$$\boxed{V(x,y) = D^2x^2 - DTxy + Dy^2 + a(T^2-2D)x - aTy + a^2}$$

- The level curves of the inverse integrating factor V are conic sections.
- In particular, let us assume that $4D - T^2 > 0$.
  - The level curves of V are ellipses whose center is the equilibrium.
  - The change $\begin{cases} x = X + \dfrac{a}{D}, \\ y = \alpha X + \beta Y + \dfrac{aT}{D}, \end{cases}$ for $\alpha = \dfrac{T}{2}$, $\beta = \dfrac{\sqrt{4D-T^2}}{2}$. transforms $V(x,y)$ into $\widetilde{V}(X,Y) = \beta^2(\alpha^2+\beta^2)(X^2+Y^2)$.

Exercise

$\longleftarrow$

  - $V(x,y)$ is a Lyapunov function:

$$\nabla V(x,y)\begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} = V(x,y)\,\mathrm{div}\begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} = TV(x,y).$$

Inverse integrating factors: Characteristic Polynomial

Exercise

Let $A = \begin{pmatrix} T & -1 \\ D & 0 \end{pmatrix}$ and p_A its characteristic polynomial.

- For $D(Dx-a) \neq 0$, it is $V(x,y) = \dfrac{(Dx-a)^2}{D}\, p_A\left(D\,\dfrac{Tx-y}{Dx-a}\right)$.
- Specifically, $V(0,y) = \dfrac{a^2}{D}\, p_A\left(D\,\dfrac{y}{a}\right)$.
- On the other hand, for $D=0$, it is $V(0,y) = y^2\, p_A\left(D\,\dfrac{a}{y}\right)$.

Construction of a suitable conservative vector field

For $V(x,y) = D^2x^2 - DTxy + Dy^2 + a(T^2 - 2D)x - aTy + a^2$, system (LCF) can be written as the hamiltonian system

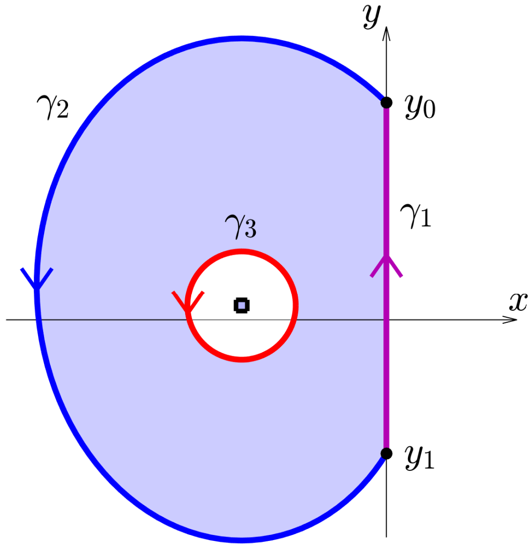
$$\begin{cases} \dfrac{dx}{ds} = \dfrac{Tx - y}{V(x,y)}, \\ \dfrac{dy}{ds} = \dfrac{Dx - a}{V(x,y)}, \end{cases}$$

on $\mathcal{U} \setminus V^{-1}(\{0\})$, where $ds = V(x,y)dt$.

Moreover, the vector field $G(x,y) = \left(-\dfrac{Dx-a}{V(x,y)}, \dfrac{Tx-y}{V(x,y)}\right)$ is

- conservative on any connected component of $\mathcal{U} \setminus V^{-1}(\{0\})$,
- orthogonal to the flow on $\mathcal{U} \setminus V^{-1}(\{0\})$.

The integral (Equilibrium in the interior of the curve)



Remind that $\vec{\gamma}_2$ is an orbit.
Let $\vec{\gamma} = \vec{\gamma}_1 \cup \vec{\gamma}_2$.
Let $\vec{\gamma}_3$ be a "small" curve around Eq.

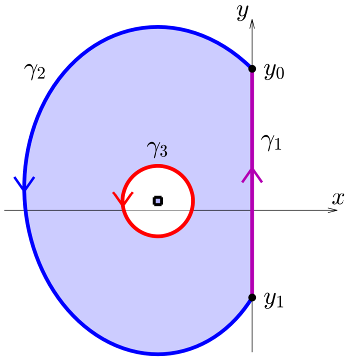
- $\oint_{\vec{\gamma}} G \cdot dr = \oint_{\vec{\gamma}_3} G \cdot dr$. (Cons.)
- $\int_{\vec{\gamma}_2} G \cdot dr = 0$. (Orth.)
- $\oint_{\vec{\gamma}_3} G \cdot dr = \frac{2\pi T}{D\sqrt{4D - T^2}}$.

Exercise
$\longleftarrow (*)$

Therefore
$\int_{\vec{\gamma}_1} G \cdot dr = \int_{y_1}^{y_0} \frac{-y}{V(0,y)} dy = \frac{2\pi T}{D\sqrt{4D-T^2}}$.

Finally $\boxed{\displaystyle\int_{y_1}^{y_0} \frac{-y}{Dy^2 - aTy + a^2} dy = \frac{2\pi T}{D\sqrt{4D - T^2}}}$.

Remark: The integral on $\vec{\gamma}_3$



A good choice for $\vec{\gamma}_3$ is

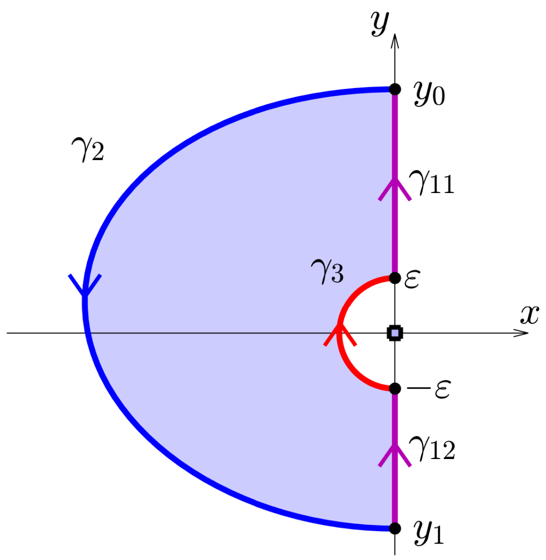
$$\vec{\gamma}_3 \equiv \begin{cases} x = 2\cos\theta + \dfrac{a}{D}, \\ y = T\cos\theta + \sqrt{4D - T^2}\sin\theta + \dfrac{aT}{D}. \end{cases}$$

where $\theta \in [0, 2\pi]$.

This is a positively oriented parameterization of the ellipse given by $V(x,y) = D(4D - T^2)$.

Therefore $\displaystyle\oint_{\vec{\gamma}_3} G \cdot dr = \frac{2\pi T}{D\sqrt{4D - T^2}}$.

The integral (Equilibrium on the Poincaré section)



Remind that $\vec{\gamma}_2$ is an orbit.
Let $\vec{\gamma} = \vec{\gamma}_{11} \cup \vec{\gamma}_3 \cup \vec{\gamma}_{12} \cup \vec{\gamma}_2$.
Let $\vec{\gamma}_3$ be a half-circle (r: ε, c: O).

- $\oint_{\vec{\gamma}} G \cdot dr = 0$. (Cons.)
- $\int_{\vec{\gamma}_2} G \cdot dr = 0$. (Orth.)
- $\int_{\vec{\gamma}_3} G \cdot dr = \frac{-\pi T}{D\sqrt{4D - T^2}}$.

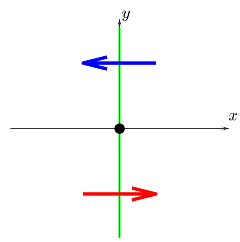
Exercise
←

Therefore
$\lim_{\varepsilon \searrow 0} \int_{\vec{\gamma}_{11} \cup \vec{\gamma}_{12}} G \cdot dr \overset{(*)}{=}$
$\text{PV}\left\{\int_{y_1}^{y_0} \frac{-y}{V(0,y)} dy\right\} = \frac{\pi T}{D\sqrt{4D - T^2}}$.
$(*)$ Cauchy Principal Value

Finally $$\boxed{\text{PV}\left\{\int_{y_1}^{y_0} \frac{-y}{Dy^2 - aTy + a^2} dy\right\} = \frac{\pi T}{D\sqrt{4D - T^2}}}.$$

Short summary: Liénard form, Inverse Integrating Factor

The linear system (Liénard canonical form):



$$\text{(LCF)} \quad \left\{ \begin{array}{rcl} \dot{x} &=& Tx - y \\ \dot{y} &=& Dx - a \end{array} \right. \qquad [[\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})]]$$

Poincaré section is $x = 0$. It is assumed that $D^2 + a^2 \neq 0$.
For $D \neq 0$ there is one equilibrium at $(a/D, aT/D)$.

The inverse integrating factor

$$\boxed{V(x,y) = D^2x^2 - DTxy + Dy^2 + a(T^2 - 2D)x - aTy + a^2}$$

The vector field $\mathbf{G}(x,y) = \left(-\frac{Dx - a}{V(x,y)}, \frac{Tx - y}{V(x,y)}\right) = \frac{\mathbf{F}(x,y)^{\perp}}{V(x,y)}$ is

- conservative on any connected component of $\mathbb{R}^2 \setminus V^{-1}(\{0\})$,
- orthogonal to the flow on $\mathbb{R}^2 \setminus V^{-1}(\{0\})$.

Obtaining the Flight time

Remind that if Φ is the flow of system and F is its vector field, then

$$V(\Phi(t;\mathbf{p})) = V(\mathbf{p}) \exp\left(\int_0^t \operatorname{div} F(\Phi(s;\mathbf{p}))\, ds\right).$$

- For system (LCF) it is $\operatorname{div} F(x,y) \equiv T$. Thus, $T\tau = \log\left(\frac{V(0,y_1)}{V(0,y_0)}\right)$.
- Moreover, if

$$\text{PV} \int_{y_1}^{y_0} \frac{-y}{Dy^2 - aTy + a^2} dy = \frac{k\pi T}{D\sqrt{4D - T^2}}, \quad k \in \{0, 1, 2\},$$

then

$$\log\left(\frac{V(0,y_1)}{V(0,y_0)}\right) = T\left(\frac{2k\pi}{\sqrt{4D - T^2}} + \int_{y_1}^{y_0} \frac{a}{V(0,y)} dy\right).$$

Exercise

↖

Solution of the exercise

- Case $D \cdot a \neq 0$:

$$k \in \{0,1,2\}, \qquad \frac{k\pi T}{D\sqrt{4D-T^2}} = \mathrm{PV}\int_{y_1}^{y_0} \frac{-y}{Dy^2 - aTy + a^2}dy =$$

$$= -\frac{1}{2}\int_{y_1}^{y_0} \frac{2Dy - aT}{Dy^2 - aTy + a^2}dy + \frac{1}{2}\int_{y_1}^{y_0} \frac{-aT}{Dy^2 - aTy + a^2}dy \quad \Longleftrightarrow$$

$$\Longleftrightarrow \quad \log\left(\frac{V(0,y_1)}{V(0,y_0)}\right) = T\left(\frac{2k\pi}{\sqrt{4D-T^2}} + \int_{y_1}^{y_0} \frac{a}{V(0,y)}dy\right)$$

- Case $D = 0,\ a \neq 0$: (Imply $k = 0$).
  - Case $T \neq 0$: $\displaystyle\int_{y_1}^{y_0} \frac{a}{-aTy + a^2}dy = \frac{-1}{T}\int_{y_1}^{y_0} \frac{-aT}{-aTy + a^2}dy$
  - Case $T = 0$: Trivial.

- Case $D \neq 0,\ a = 0$: (Imply $D > 0$ and $k = 1$).

$$\frac{\pi T}{D\sqrt{4D-T^2}} = \mathrm{PV}\int_{y_1}^{y_0} \frac{-1}{Dy}dy = \frac{1}{D}\log\left(\left|\frac{y_1}{y_0}\right|\right) = \frac{1}{2D}\log\left(\frac{y_1^2}{y_0^2}\right) \qquad \square$$

Generalized Liénard Form of a Piecewise Linear System

Exercise

Consider $\dot{\mathbf{x}} = \begin{cases} A_L\mathbf{x} + \mathbf{b}_L, & \text{if} \quad x_1 \leq 0, \\ A_R\mathbf{x} + \mathbf{b}_R, & \text{if} \quad x_1 \geq 0, \end{cases}$ where $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$,
$A_{L,R} = (a_{ij}^{L,R})_{2\times 2}$, $\mathbf{b}_{L,R} = (b_1^{L,R}, b_2^{L,R}) \in \mathbb{R}^2$.

1. Prove that $a_{12}^L a_{12}^R > 0$ is a neccesary condition for the existence of limit cycles.
2. Find a homeomorphism preserving the separation line $x = 0$, that transforms the system into the following Liénard canonical form

$$\begin{cases} \dot{x} = T_L x - y \\ \dot{y} = D_L x - a_L \end{cases} \text{for} \quad x < 0, \qquad \begin{cases} \dot{x} = T_R x - y + b \\ \dot{y} = D_R x - a_R \end{cases} \text{for} \quad x > 0,$$

where $a_L = a_{12}^L b_2^L - a_{22}^L b_1^L$, $a_R = a_{12}^L(a_{12}^R b_2^R - a_{22}^R b_1^R)/a_{12}^R$, $b = a_{12}^L b_1^R / a_{12}^R - b_1^L$, and T_L, T_R and D_L, D_R are the traces and determinants of the matrices A_L and A_R.

Generalized Liénard Form of a Piecewise Linear System

Hint:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a_{22}^L & -a_{12}^L \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} 0 \\ b_1^L \end{pmatrix}, \quad x_1 \leqslant 0,$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{a_{12}^R} \begin{pmatrix} a_{12}^L & 0 \\ a_{12}^L a_{22}^R & -a_{12}^L a_{12}^R \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} 0 \\ b_1^L \end{pmatrix}, \quad x_1 > 0.$$

E. Freire, E. Ponce, and F. Torres, Canonical discontinuous planar piecewise linear systems, SIAM J. Appl. Dyn. Syst., 11 (2012). [Prop 3.1]

Lum-Chua's conjecture: Liénard canonical form

Under the (necessary) condition $a_{12} \neq 0$ the linear change of variables $(x, y) = (x_1, a_{22}x_1 - a_{12}x_2 - b_1)$ transforms the system into the Liénard canonical form,

$$(S_L) \begin{cases} \dot{x} = T_L x - y \\ \dot{y} = D_L x - a \end{cases} \text{for} \quad x < 0, \quad (S_R) \begin{cases} \dot{x} = T_R x - y \\ \dot{y} = D_R x - a \end{cases} \text{for} \quad x \geqslant 0,$$

where $a = a_{12}b_2 - a_{22}b_1$.

Exercise

- Prove that no limit cycle exists for $T_L T_R \geqslant 0$ (Hint: Green's Theorem; See, for instance, E. Freire, E. Ponce, F. Rodrigo, F. Torres, Internat. J. Bifur. Chaos Appl. 6 Sci. Engrg. 8 (1998)).
- Prove that no limit cycle exists for $a = 0$ (Hint: The system is homogeneous).

Therefore, $T_L T_R < 0$ must be assumed for the existence of limit cycles.