On the Convergence of the Primal Hybrid Finite Element Method on Quadrilateral Meshes

Giovanni Taraschi[a], Maicon R. Correa[a,*]

[a] *Universidade Estadual de Campinas, Instituto de Matemática, Estatística e Computacão Científica, Departamento de Matemática Aplicada, Rua Sergio Buarque de Holanda, 651 Barão Geraldo, Campinas, 13083-859, SP, Brazil*

Abstract

We consider the approximation of a second order elliptic equation (Darcy problem) by using the Primal Hybrid Finite Element Method on quadrilateral meshes. We present new results in terms of sufficient, and in some cases also necessary, conditions to obtain the optimal convergence rates on convex quadrilaterals obtained from bilinear mappings. Numerical experiments are performed to illustrate the theoretical results.

Keywords: Primal Hybrid Finite Element Method, Quadrilaterals, Darcy Problem, Second order elliptic equation, Lagrange Multipliers

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1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^2$, with a Lipschitz continuous boundary $\partial\Omega$. We consider the second order elliptic model problem

$$-\operatorname{div}(\mathcal{K}\,\nabla u) = f \qquad \text{in} \quad \Omega \tag{1.1a}$$

$$u = 0 \qquad \text{on} \quad \partial\Omega, \tag{1.1b}$$

where $f \in L^2(\Omega)$ is a given function and $\mathcal{K} = \mathcal{K}(\boldsymbol{x})$ is a symmetric and uniformly positive definite tensor, i.e., there exist two positive constants C_1 and C_2 such that

$$C_1\boldsymbol{\xi}^T\boldsymbol{\xi} \leqslant \boldsymbol{\xi}^T\,\mathcal{K}(\boldsymbol{x})\boldsymbol{\xi} \leqslant C_2\boldsymbol{\xi}^T\boldsymbol{\xi} \quad \forall\boldsymbol{\xi} \in \mathbb{R}^2,\, \boldsymbol{x} \in \bar{\Omega}. \tag{1.2}$$

Here we choose to treat homogeneous Dirichlet boundary conditions, but other boundary conditions could be imposed (see, e.g. [1]). Problem (1.1) can be associated to the problem of finding the pressure field u in the flow of an incompressible fluid in a rigid saturated heterogeneous porous medium, where the tensor $\mathcal{K}$ represents the permeability of the porous matrix divided by the fluid viscosity, in the so-called Darcy problem [2].

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*Corresponding author.

Email addresses: `gitaraschi@gmail.com` (Giovanni Taraschi), `maicon@ime.unicamp.br` (Maicon R. Correa)

The most classical finite element approach to solve (1.1) is the use of the H^1-conforming Galerkin method [3], where the finite element subspaces of $H^1(\Omega)$ are composed of functions that are continuous along the interelement boundaries. In [4] the authors introduced the Primal Hybrid Finite Element Method for solving (1.1), which is a non-conforming finite element method based on a primal hybrid variational principle. This principle allows for the relaxation of the interelement continuity of the finite element subspaces, through the introduction of new variables on the faces of the elements, which are related to Lagrange multipliers. In this case, the discrete solution no longer belongs to $H^1(\Omega)$ [4]. The first methods based on this principle, in the context of second order elliptic problems, were originally proposed in [5, 6]. Since then, variations of the Primal Hybrid Method and have been proposed and successfully applied in the solution of different problems, such as quasi-linear [7] and non-linear [8] elliptic problems, and parabolic problems [9, 10]. The primal hybrid principle is also at the core of the development of the multiscale hybrid-mixed methods [11, 12, 13]. Naturally, the ideas that support the hybridization process are also applicable for mixed [14, 1], dual-mixed [15, 16] and discontinuous Galerkin methods [17, 18]. When combined with stabilization techniques that circumvent the discrete inf-sup condition, the hybrid formulations provide a powerful tool for the development of higher-order schemes such as the hybrid high-order methods and the hybridizable discontinuous Galerkin methods [19, 18, 20, 1]. An extensive comparative study of the accuracy and computational performance of continuous, discontinuous, mixed, and stabilized primal hybrid finite element methods for second-order elliptic problems can be found, for example, in [21].

Here, we restrict ourselves to the original primal hybrid variational formulation of the model problem (1.1), without stabilization terms, which can be stated as: find the pair $(u, \lambda) \in X \times M$ such that

$$\sum_{K \in \mathcal{T}_h} \int_K (\mathcal{K} \nabla u) \cdot \nabla v \, dx + \sum_{K \in \mathcal{T}_h} \langle \lambda, v \rangle_{\partial K} = \int_\Omega f v \, dx \quad \forall\, v \in X, \tag{1.3a}$$

$$\sum_{K \in \mathcal{T}_h} \langle \mu, u \rangle_{\partial K} = 0 \qquad \forall\, \mu \in M. \tag{1.3b}$$

Here λ is the Lagrange multiplier, the spaces X and M are linked to a given regular partition $\mathcal{T}_h$ of Ω composed of non-overlapping convex subdomains K, and $\langle \cdot, \cdot \rangle_{\partial K}$ denotes the duality between $H^{-\frac{1}{2}}(\partial K)$ and $H^{\frac{1}{2}}(\partial K)$. More specifically, the spaces X and M are defined by

$$X(\mathcal{T}_h) = \left\{ v \in L^2(\Omega) \,:\, v|_K \in H^1(K)\,, \forall K \in \mathcal{T}_h \right\} \quad \text{and} \tag{1.4}$$

$$M(\mathcal{T}_h) = \left\{ \mu \in \prod_{K \in \mathcal{T}_h} H^{-\frac{1}{2}}(\partial K) \,:\, \exists\, \boldsymbol{q} \in H(\mathrm{div}, \Omega) \text{ s.t. } \boldsymbol{q} \cdot \boldsymbol{n}^{\partial K} = \mu \text{ on } \partial K\,, \forall K \in \mathcal{T}_h \right\}, \tag{1.5}$$

where $\boldsymbol{n}^{\partial K}$ represents the unit outward normal along ∂K. The well posedness of problem (1.3) is studied in [4], where the authors show that it has a unique solution (u, λ), with $u \in H^1_0(\Omega)$ and

$$\lambda = -(\mathcal{K} \nabla u) \cdot \boldsymbol{n}^{\partial K} \text{ on } \partial K, \quad \forall\, K \in \mathcal{T}_h, \tag{1.6}$$

linking the Lagrange multiplier to the flux (normal component of Darcy's velocity), which is an important quantity in several applications, usually evaluated through the use of mixed finite elements [22, 23, 24, 2].

Finite element approximations for the primal hybrid formulation (1.3) are based on the construction of finite dimensional subspaces $X_h \subset X$ and $M_h \subset M$, that must satisfy some compatibility conditions in order to provide stable and accurate solutions, which reduce the flexibility in constructing such subspaces. In [4] the authors present families of compatible spaces $X_h \subset X$ and $M_h \subset M$ on triangular and quadrilateral meshes, generated from affine mappings of standard reference elements $\hat{K}$. For the particular case of quadrilateral meshes, the use of affine mappings of the reference element (in general, the unit square) limits the domains that can be meshed, since it generates at most parallelograms [25].

Herein we discuss the approximation of the primal hybrid formulation (1.3) by finite element subspaces X_h generated by bilinear mappings, covering the more general case of meshes composed of convex quadrilaterals (Theorems 3.4 and 3.6). The proposed analysis makes use of results established in [25] and can be seen as a complement to the original analysis carried out in [4].

2. The primal hybrid finite element method on quadrilaterals

Let $\mathcal{T}_\mathrm{h}$ be a partition of Ω into convex quadrilaterals which satisfies the following regularity condition described in [26]. For each quadrilateral $K \in \mathcal{T}_\mathrm{h}$ we obtain four triangles by the four possible choices of three vertices of K. Denote by $T_i, 1 \leqslant i \leqslant 4$ each one of these triangles. Then we define

$$\rho_K = 2 \min_{1 \leqslant i \leqslant 4} \{ \text{ diameter of circle inscribed in } T_i \} \tag{2.1}$$

and

$$h_K = \text{ diameter of } K. \tag{2.2}$$

The shape constant of K is then defined as $\gamma_K = h_K/\rho_K$ and the shape constant γ of a mesh as the supremum of γ_K for $K \in \mathcal{T}_\mathrm{h}$. A family of meshes is called shape-regular if the shape-constant of its meshes can be uniformly bounded [23, 24]. The mesh parameter is defined as $h = \max_{K \in \mathcal{T}_\mathrm{h}} h_K$. Denoting by $\hat{K}$ the standard reference element, in our case the unit square $[0,1] \times [0,1]$, each geometrical element $K \in \mathcal{T}_\mathrm{h}$, is generated from an isomorphism $F_K : \hat{K} \to \mathbb{R}^2$ such that $K = F_K(\hat{K})$.

We denote by $P_r(\hat{K})$ the space of polynomials on $\hat{K}$ of total degree at most r, by $P_{r,s}(\hat{K})$ the space of polynomials on $\hat{K}$ of degree at most r in $\hat{x}_1$ and degree at most s in $\hat{x}_2$ and set $Q_r(\hat{K}) = P_{r,r}(\hat{K})$. We also denote by $E_m(\partial \hat{K})$ the following polynomial space over $\partial \hat{K}$

$$E_m(\partial \hat{K}) = \{\mu \in L^2(\partial \hat{K}) \, : \, \mu|_e \in P_m(e), \forall \, e \text{ (edges) of } \hat{K}\}$$

and by $T_m(\partial \hat{K})$ the subspace of $E_m(\partial \hat{K})$ composed of continuous functions. Finally we define the Serendipity spaces in $\hat{K}$ by

$$S_r(\hat{K}) := \mathrm{span}\{\hat{x}_1^i \hat{x}_2^j, \hat{x}_1^r \hat{x}_2, \hat{x}_1 \hat{x}_2^r \, : \, i + j \leqslant r\}, \tag{2.3}$$

as in [27]. Now, defining the finite dimensional spaces $\hat{U} \subset H^1(\hat{K})$ and $\hat{\Lambda} \subset L^2(\partial \hat{K})$, such that for $r \geqslant 1$ and $m \geqslant 0$

$$P_r(\hat{K}) \subset \hat{U}, \quad T_r(\partial \hat{K}) \subset \hat{U}|_{\partial \hat{K}} \tag{2.4a}$$

$$E_m(\partial \hat{K}) \subset \hat{\Lambda}, \quad \text{and} \quad \{\hat{\alpha}\hat{\mu} : \hat{\mu} \in \hat{\Lambda}\} \subset \hat{\Lambda} \; \forall \hat{\alpha} \in E_0(\partial \hat{K}), \tag{2.4b}$$

the spaces X_h and M_h are constructed as follows

$$X_h = \{v \in L^2(\Omega) \, : \, \forall\, K \in \mathcal{T}_\mathrm{h}\, ,\, v|_K \in U_K\},$$

$$M_h = \Bigg\{ \mu \in \prod_{K \in \mathcal{T}_\mathrm{h}} \Lambda_K \, : \, \mu|_{\partial K_1} + \mu|_{\partial K_2} = 0 \text{ on } K_1 \cap K_2\, , \text{ for every pair of adjacent elements } K_1, K_2 \in \mathcal{T}_\mathrm{h} \Big\}\, ,$$

with

$$U_K = \{v \in H^1(K) \, : \, v = \hat{v} \circ F_K^{-1}\, ,\, \hat{v} \in \hat{U}\}$$

and

$$\Lambda_K = \{\mu \in L^2(\partial K) \, : \, \mu = \hat{\mu} \circ F_K^{-1}\, ,\, \hat{\mu} \in \hat{\Lambda}\}.$$

The discrete problem is then defined by: Find the pair $(u_h, \lambda_h) \in X_h \times M_h$ such that

$$a(u_h, v) + b(v, \lambda_h) = \int_\Omega f v \,\mathrm{dx} \quad \forall\, v \in X_h, \tag{2.5a}$$

$$b(u_h, \mu) = 0 \qquad\qquad \forall\, \mu \in M_h \tag{2.5b}$$

where $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are the continuous bilinear forms on $X \times X$ and $X \times M$ defined, respectively, by

$$a(v, u) = \sum_{K \in \mathcal{T}_\mathrm{h}} \int_K (\mathcal{K}\, \nabla u) \cdot \nabla v \,\mathrm{dx} \quad \text{and} \quad b(v, \mu) = \sum_{K \in \mathcal{T}_\mathrm{h}} \langle \mu, v \rangle_{\partial K}. \tag{2.6}$$

Notice that, once determined the mappings F_K, the spaces X_h and M_h are completely described by the choice of $\hat{U}$ and $\hat{\Lambda}$. Thus, we also refer to the approximation space $X_h \times M_h$ by simply stating the adopted pair $(\hat{U}, \hat{\Lambda})$. From X_h and M_h, we introduce the space

$$V_h = \{v \in X_h \, : \, \forall\, \mu \in M_h,\, b(v, \mu) = 0\},$$

which can be seen as a nonconforming approximation for the space $H_0^1(\Omega)$ [4]. The following compatibility condition is presented in [4] as a necessary and sufficient condition for the existence and uniqueness of solution to (2.5)

$$\{\mu \in M_h \, : \, \forall v \in X_h,\, b(v, \mu) = 0\} = \{0\}. \tag{2.7}$$

Examples of compatible spaces $\hat{U}$ and $\hat{\Lambda}$, for the particular case of two-dimensional problems with quadrilateral meshes, are provided by the following lemma

Lemma 2.1 (Lemma 8 from [4])**.** *Let $m \geqslant 0$ be any non-negative integer. For some integer $r \geqslant m + 1$ consider the function*

$$\begin{aligned} v_0(\hat{x}_1, \hat{x}_2) &= [\hat{x}_1(1-\hat{x}_1) - \hat{x}_2(1-\hat{x}_2)] \\ &\quad [(\hat{x}_1(1-\hat{x}_1))^{\frac{r-1}{2}} + (\hat{x}_2(1-\hat{x}_2))^{\frac{r-1}{2}}], \text{ for } r \text{ odd} \\ v_0(\hat{x}_1, \hat{x}_2) &= [\hat{x}_1(1-\hat{x}_1) - \hat{x}_2(1-\hat{x}_2)] \\ &\quad (2\hat{x}_1 - 1)(2\hat{x}_2 - 1)[(\hat{x}_1(1-\hat{x}_1))^{\frac{r-2}{2}} + (\hat{x}_2(1-\hat{x}_2))^{\frac{r-2}{2}}], \text{ for } r \text{ even.} \end{aligned}$$

Denote by $Q_r^+(\hat{K})$ the space spanned by $Q_r(\hat{K})$ plus the function v_0, and by $S_r^+(\hat{K})$ the space spanned by $S_r(\hat{K})$ plus the same function v_0. It follows that the pairs $(Q_r^+(\hat{K}), E_m(\partial\hat{K}))$ and $(S_r^+(\hat{K}), E_m(\partial\hat{K}))$ are both compatible pairs and provide unique solution (u_h, λ_h) for the discrete problem (2.5)*.*

3. Convergence on quadrilateral meshes

In order to extend the convergence analysis of [4] and provide bounds for the errors $u - u_h$ and $\lambda - \lambda_h$ on bilinear meshes, we begin by recalling the norms

$$|||v|||_X = \left(\sum_{K \in \mathcal{T}_h} |||v|||_{1,K}^2 \right)^{1/2}, \tag{3.1}$$

with

$$|||v|||_{1,K} = (|v|_{1,K}^2 + h_K^{-2} \|v\|_{0,K}^2)^{1/2},$$

$$|||\mu|||_M = \sup_{v \in X} \frac{b(v, \mu)}{|||v|||_X}. \tag{3.2}$$

If $E_0(\partial \hat{K}) \subset \hat{\Lambda}$, the bilinear form $a(\cdot, \cdot)$ defines a norm over the space V_h, that we denote by

$$\|v\|_h = (a(v, v))^{1/2}. \tag{3.3}$$

For affine meshes and sufficient regular solutions, it is shown in [4] that if $\hat{\Lambda} = E_m(\partial \hat{K})$ and $S_r^+(\hat{K}) \subset \hat{U}$, with $r \geqslant m + 1$, then the following error bounds hold

$$\|u - u_h\|_h \leqslant C_1 h^l |u|_{l+1,\Omega}, \tag{3.4}$$

$$|||\lambda - \lambda_h|||_M \leqslant C_2 h^l |u|_{l+1,\Omega}, \tag{3.5}$$

where $l = \min\{r, m + 1\}$. These conditions do not guarantee optimal results for bilinear meshes, though. In this case, more general results about the approximation by quadrilateral finite elements need to be used. We begin by summarizing important results of [25] into the two following lemmas.

Lemma 3.1 (Theorems 3 and 4 from [25])**.** *Let F_K be any bilinear isomorphism of $\hat{K}$ onto a convex quadrilateral K, then*

$$P_r(K) \subset U_K \iff Q_r(\hat{K}) \subset \hat{U}.$$

Lemma 3.2 (Section 3 from [25])**.** *Let $\mathcal{T}_h$ be a regular bilinear mesh of Ω, then the bounds*

$$\inf_{v \in X_h} \|u - v\|_{0,\Omega} \leqslant C h^{r+1} |u|_{r+1,\Omega}, \ \forall u \in H^{r+1}(\Omega),$$

$$\inf_{v \in X_h} \|\nabla_h (u - v)\|_{0,\Omega} \leqslant C h^r |u|_{r+1,\Omega}, \ \forall u \in H^{r+1}(\Omega),$$

are satisfied if and only if $Q_r(\hat{K}) \subset \hat{U}$.

By applying these necessary and sufficient conditions on the Primal Hybrid Method, we have:

Lemma 3.3. *Assuming the same hypotheses of Lemma 3.2 and considering the space V_h constructed in Section 2, the estimate*

$$\inf_{v \in V_h} \|u - v\|_h \leqslant C h^r |u|_{r+1,\Omega}, \ \forall u \in H^{r+1}(\Omega) \cap H_0^1(\Omega), \tag{3.6}$$

holds if and only if $Q_r(\hat{K}) \subset \hat{U}$.

Proof. First we show that $Q_r(\hat{K}) \subset \hat{U}$ is sufficient to obtain (3.6). Let $W_h \subset H^1(\Omega)$ be the space

$$W_h = \{w_h \in C_0(\bar{\Omega}) \,:\, \forall\, K \in \mathcal{T}_\mathrm{h}, w_h|_K \in P_r(K), w_h|_{\partial\Omega} = 0\}.$$

Since W_h is composed of continuous functions, Lemma 3.1 implies that $W_h \subset V_h$. From (3.3) and (1.2) and using classical results from approximation theory follows that

$$\inf_{v \in V_h} \|u - v\|_h \leqslant C_2 \inf_{w \in W_h} |u - w|_{1,\Omega} \leqslant Ch^r |u|_{r+1,\Omega}.$$

On the other hand, suppose that estimate (3.6) holds. Since $V_h \subset X_h$, we have from (3.3) and (1.2) that

$$C_1 \inf_{v \in X_h} \|\nabla_h (u - v)\|_{0,\Omega} \leqslant \inf_{v \in V_h} \|u - v\|_h \leqslant Ch^r |u|_{r+1,\Omega},$$

and follows directly from Lemma 3.2 that $Q_r(\hat{K}) \subset \hat{U}$. □

Theorem 3.4. *Consider $\mathcal{T}_\mathrm{h}$ a regular bilinear partition of Ω and let X_h and M_h be the spaces constructed in Section 2 such that conditions (2.4) and (2.7) are satisfied. Assume that $u \in H^{l+1}(\Omega) \cap H^1_0(\Omega)$ with $l = \min\{s, m+1\}$. Then there exists a constant C independent of h such that*

$$\|u - u_h\|_h \leqslant Ch^l |u|_{l+1,\Omega}, \tag{3.7}$$

if and only if $Q_s(\hat{K}) \subset \hat{U}$.

Proof. From Theorem 3 of [4] the discrete solution u_h satisfies

$$\|u - u_h\|_h^2 = \left(\inf_{v_h \in V_h} \|u - v_h\|_h \right)^2 + \left(\inf_{\mu_h \in M_h} \sup_{v_h \in V_h} \frac{b(v_h, \lambda - \mu_h)}{\|v_h\|_h} \right)^2. \tag{3.8}$$

Since $V_h \subset X$, from Lemma 9 of [4], we have

$$\inf_{\mu_h \in M_h} \sup_{v_h \in V_h} \frac{b(v_h, \lambda - \mu_h)}{\|v_h\|_h} \leqslant Ch^l |u|_{l+1,\Omega}. \tag{3.9}$$

Them the desired result is obtained from (3.8), (3.9) and Lemma 3.3. □

Remark 3.5. Assuming that the domain Ω is convex, by an extension of the Nitsche technique is possible to obtain, from equation (3.7), the following estimate in $L^2(\Omega)$ [4]

$$\|u - u_h\|_{0,\Omega} \leqslant Ch^{l+1} |u|_{l+1,\Omega}.$$

Theorem 3.6. *Consider $\mathcal{T}_\mathrm{h}$ a regular bilinear partition of Ω and let X_h and M_h be compatible spaces constructed as in Section 2 such that conditions (2.4) and (2.7) are satisfied. Assume that $u \in H^{l+1}(\Omega) \cap H^1_0(\Omega)$ with $l = \min\{s, m+1\}$. Then $Q_s(\hat{K}) \subset \hat{U}$ implies that there is a constant C independent of h such that*

$$|||\lambda - \lambda_h|||_M \leqslant Ch^l |u|_{l+1,\Omega}. \tag{3.10}$$

Proof. From Theorem 3 and Lemma 10 of [4], there exists a constant $\alpha > 0$ such that

$$|||\lambda - \lambda_h|||_M \leqslant \frac{1}{\alpha}\|u - u_h\|_h + \left(1 + \frac{1}{\alpha}\right) \inf_{\mu_h \in M_h} |||\lambda - \mu_h|||_M. \tag{3.11}$$

From Lemma 9 of [4], (3.2) and by using that $\|v\|_h \leqslant C_2 |||v|||_X$ we obtain

$$\inf_{\mu_h \in M_h} |||\lambda - \mu_h|||_M \leqslant Ch^{m+1}|u|_{l+1,\Omega}. \tag{3.12}$$

Substituting (3.7) and (3.12) in (3.11) the proof is completed. □

Remark 3.7. In [4] Lemma 10 states the existence of $\alpha > 0$ satisfying (3.11) for the case of affine meshes. This result can be extend for the bilinear case, by using the bounds

$$\|DF_K\|_{\infty,\hat{K}} \leqslant C_1 h_K, \quad \|JF_K\|_{\infty,\hat{K}} \leqslant C_2 h_K^2, \tag{3.13}$$

$$\|DF_K^{-1}\|_{\infty,K} \leqslant C_3 \frac{h_K}{\rho_K^2}, \quad \|JF_K^{-1}\|_{\infty,K} \leqslant C_4 \frac{1}{\rho_K^2} \tag{3.14}$$

where DF_K is the Jacobian matrix of F_K and JF_K its determinant.

Remark 3.8. Defining the norm over the space $\prod_{K \in \mathcal{T}_h} L^2(\partial K)$

$$\|\mu\|_{M_h} = \left(\sum_{K \in \mathcal{T}_h} h_K \|\mu\|_{0,\partial K}^2 \right)^{1/2},$$

it is possible to show that (3.10) implies the following estimate [4]

$$\|\lambda - \lambda_h\|_{M_h} \leqslant Ch^l |u|_{l+1,\Omega}.$$

4. Numerical experiments

In this section we present convergence studies in order to check the conditions for optimal convergence given by Theorems 3.4 and 3.6, based on two test problems, with known analytic solutions.

4.1. First test problem

The first test problem is defined on the unit square $\Omega = (0,1)^2$ with $\mathcal{K} = I$, $f(\boldsymbol{x}) = 2\pi^2 \sin(\pi x)\sin(\pi y)$ and has exact solution $u(\boldsymbol{x}) = \sin(\pi x)\sin(\pi y)$. In the computations we considered homogeneous Dirichlet boundary conditions and the system (2.5) was solved using the compatible pairs (Lemma 2.1) $(Q_{m+1}^+(\hat{K}), E_m(\partial\hat{K}))$ and $(S_{m+1}^+(\hat{K}), E_m(\partial\hat{K}))$ with $m = 1, 2$. Two sequences of meshes were adopted, as shown in Figure 1. The first is a uniform mesh of $n \times n$ squares and the second is a mesh of $n \times n$ congruent trapezoids of base h and parallel vertical edges of size $0.75h$ and $1.25h$, as proposed in [25]. Note that the meshes in the first sequence are affine and in the second one are bilinear.

The errors and the rates of convergence are presented in Table 1. On square meshes, the results indicate convergence $\mathcal{O}(h^{m+2})$ for $\|u - u_h\|_{0,\Omega}$ and $\mathcal{O}(h^{m+1})$ for $\|\lambda - \lambda_h\|_{M_h}$, as

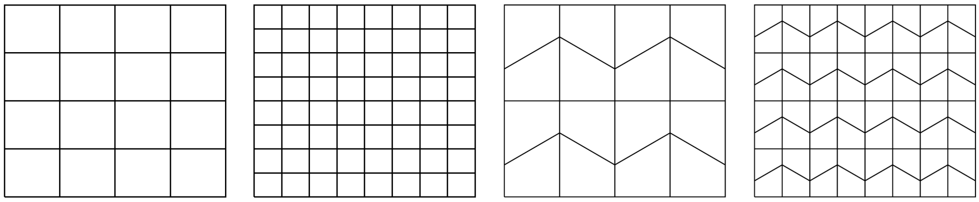

Figure 1: First test problem; sequences of square and trapezoidal meshes, with $n = 4$ and 8.

predicted by the theory. On trapezoidal meshes, the spaces $(Q^+_{m+1}(\hat{K}), E_m(\partial\hat{K}))$, $m = 1, 2$, achieved optimal convergence rates for both $\|u - u_h\|_{0,\Omega}$ and $\|\lambda - \lambda_h\|_{M_h}$, as predicted by Theorems 3.4 and 3.6, respectively. The results for the spaces $(S^+_{m+1}(\hat{K}), E_m(\partial\hat{K}))$, $m = 1, 2$, indicate a reduction on the convergence orders, since these spaces do not satisfy the requirements of Theorems 3.4 and 3.6. It is important to note that the quadratic convergence for $\|\lambda - \lambda_h\|_{M_h}$ with the space $(S^+_2(\hat{K}), E_1(\partial\hat{K}))$ does not contradict Theorem 3.6, since it provides only sufficient conditions.

Table 1: First test problem; errors and rates of convergence for different spaces.

	Square meshes				Trapezoidal meshes			
	$\|u-u_h\|_{0,\Omega}$		$\|\lambda-\lambda_h\|_{M_h}$		$\|u-u_h\|_{0,\Omega}$		$\|\lambda-\lambda_h\|_{M_h}$	
n	err.	rate	err.	rate	err.	rate	err.	rate
$(Q_2^+(\hat{K}), E_1(\partial\hat{K}))$								
8	2.5507e-04	2.97	2.9136e-02	1.85	3.5012e-04	2.96	3.8196e-02	1.87
16	3.2141e-05	2.99	7.54448-03	1.95	4.4162e-05	2.99	9.8923e-03	1.95
32	4.0270e-06	3.00	1.9061e-03	1.98	5.5353e-06	3.00	2.4988e-03	1.99
64	5.0369e-07	3.00	4.7788e-04	2.00	6.9247e-07	3.00	6.2640e-04	2.00
$(S_2^+(\hat{K}), E_1(\partial\hat{K}))$								
8	3.5012e-04	2.96	2.9136e-02	1.85	3.5832e-04	2.95	3.7889e-02	1.87
16	4.4162e-05	2.99	7.5448e-03	1.95	4.6780e-05	2.94	9.8153e-03	1.95
32	5.5353e-06	3.00	1.9061e-03	1.98	6.5986e-06	2.83	2.4838e-03	1.98
64	6.9247e-07	3.00	4.7788e-04	2.00	1.1202e-06	2.56	6.2726e-04	1.99
$(Q_3^+(\hat{K}), E_2(\partial\hat{K}))$								
8	5.8551e-06	3.98	6.0106e-04	3.01	9.7812e-06	3.98	9.6587e-04	2.99
16	3.6734e-07	3.99	7.4966e-05	3.00	6.1484e-07	3.99	1.2020e-04	3.01
32	2.2981e-08	4.00	9.3650e-06	3.00	3.8507e-08	4.00	1.4972e-05	3.01
64	1.4367e-09	4.00	1.1704e-06	3.00	2.4086e-09	4.00	1.8684e-06	3.00
$(S_3^+(\hat{K}), E_2(\partial\hat{K}))$								
8	1.8198e-05	4.16	1.3107e-03	3.97	4.8452e-04	2.87	1.0400e-01	1.81
16	1.1056e-06	4.04	1.0364e-04	3.66	6.4011e-05	2.92	2.6936e-02	1.95
32	6.8612e-08	4.01	1.0370e-05	3.32	8.7026e-06	2.88	6.8010e-03	1.99
64	4.2807e-09	4.00	1.2030e-06	3.11	1.3567e-06	2.68	1.7061e-03	2.00

4.2. Second test problem

We now check the convergence rates in a test problem defined on the V-shaped domain shown in Figure 2, with anisotropic and heterogeneous tensor $\mathcal{K}$ given by

$$\mathcal{K}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ if } x < 0 \quad \text{and} \quad \mathcal{K}_2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \text{ if } x > 0,$$

and source function

$$f_1 = (2\sin y + \cos y)x + \sin y, \text{ if } x < 0 \quad \text{and} \quad f_2 = -2\mathrm{e}^x \cos y, \text{ if } x > 0.$$

This problem, originally proposed for a square domain in [28], has analytic solution

$$u = [2\sin y + \cos y]x + \sin y, \text{ if } x < 0 \quad \text{and} \quad u = \mathrm{e}^x \sin y, \text{ if } x > 0.$$

We may notice that the x component of the gradient of the solution has a discontinuity at $x = 0$. In the computations, we considered a mixed boundary condition. On the vertical left and right boundaries, we imposed a non-homogeneous Dirichlet condition on u, and on the rest of the boundary we imposed a non-homogeneous Neumann condition.

Again, two sequences of meshes based on parallelograms and trapezoids were adopted, with elements aligned with the discontinuity of the tensor $\mathcal{K}$ at $x = 0$ as shown in Figure 3. Despite the fact of the domain not being convex, the results presented in Table 2 show great agreement with the ones of the first test problem. They indicate convergence $\mathcal{O}(h^{m+2})$ for $\|u - u_h\|_{0,\Omega}$ and $\mathcal{O}(h^{m+1})$ for $\|\lambda - \lambda_h\|_{M_h}$ on squares, and optimal convergence rates for both $\|u - u_h\|_{0,\Omega}$ and $\|\lambda - \lambda_h\|_{M_h}$ when the spaces $(Q^+_{m+1}(\hat{K}), E_m(\partial\hat{K}))$, $m = 1, 2$ are adopted on trapezoidal meshes.

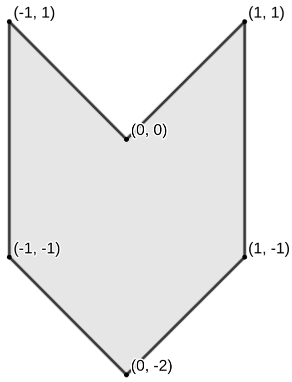

Figure 2: Non-convex domain of the second test problem.

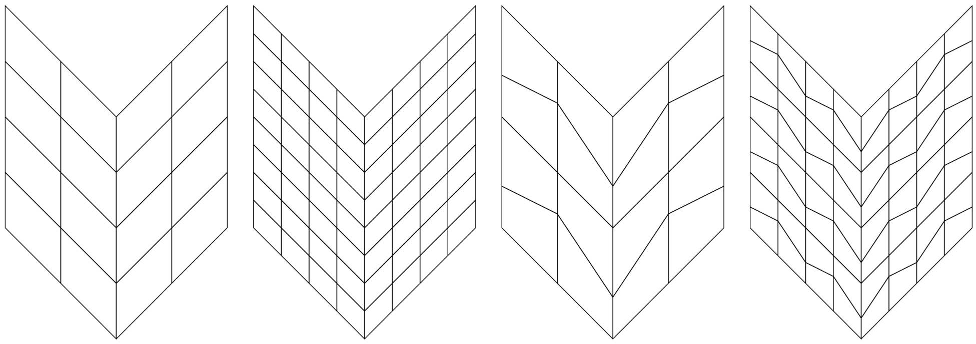

Figure 3: Second test problem; sequences of parallelogram and trapezoidal meshes, with $n = 4$ and 8.

Table 2: Second test problem; errors and rates of convergence for different spaces.

	Square meshes				Trapezoidal meshes			
	$\|u-u_h\|_{0,\Omega}$		$\|\lambda-\lambda_h\|_{M_h}$		$\|u-u_h\|_{0,\Omega}$		$\|\lambda-\lambda_h\|_{M_h}$	
n	err.	rate	err.	rate	err.	rate	err.	rate
				$(Q_2^+(\hat{K}), E_1(\partial\hat{K}))$				
8	7.6196e-04	2.96	9.7644e-02	1.74	8.8122e-04	2.95	9.6049e-02	1.66
16	9.7022e-05	2.97	2.5111e-02	1.96	1.1154e-04	2.98	2.5788e-02	1.90
32	1.2263e-05	2.98	6.2214e-03	2.01	1.4023e-05	2.99	6.5862e-03	1.97
64	1.5423e-06	2.99	1.5366e-03	2.02	1.7578e-06	3.00	1.6577e-03	1.99
				$(S_2^+(\hat{K}), E_1(\partial\hat{K}))$				
8	7.6200e-04	2.96	9.7644e-02	1.74	8.8762e-04	2.95	9.6410e-02	1.65
16	9.7017e-05	2.97	2.5111e-02	1.96	1.1361e-04	2.97	2.5900e-02	1.90
32	1.2262e-05	2.98	6.2214e-03	2.01	1.4904e-05	2.93	6.6201e-03	1.97
64	1.5423e-06	2.99	1.5366e-03	2.02	2.1487e-06	2.79	1.6705e-03	1.99
				$(Q_3^+(\hat{K}), E_2(\partial\hat{K}))$				
8	1.1394e-05	3.99	1.1379e-03	3.08	1.9470e-05	3.97	1.9972e-03	2.94
16	7.1560e-07	3.99	1.4051e-04	3.02	1.2369e-06	3.98	2.3274e-04	3.10
32	4.4845e-08	4.00	1.7534e-05	3.00	7.7981e-08	3.99	2.7221e-05	3.10
64	2.8068e-09	4.00	2.1924e-06	3.00	4.8952e-09	3.99	3.2678e-06	3.06
				$(S_3^+(\hat{K}), E_2(\partial\hat{K}))$				
8	5.6489e-05	4.20	1.1407e-02	3.46	6.7489e-04	2.58	1.1869e-01	1.86
16	3.1862e-06	4.15	1.0149e-03	3.49	9.8778e-05	2.77	3.1599e-02	1.91
32	1.8801e-07	4.08	9.2797e-05	3.45	1.4098e-05	2.81	8.2361e-03	1.94
64	1.1405e-08	4.04	8.6926e-06	3.42	2.2338e-06	2.66	2.1809e-03	1.92

Remark 4.1. In this paper, we have assumed homogeneous Dirichlet boundary condition. The treatment of more general boundary conditions, such as the mixed one adopted in the second test problem, requires some modifications on the variational formulation (1.3) (see, e.g. [1]). For the conditions of this second test, we consider disjoint Dirichlet and Neumann boundaries Γ_D and Γ_N, respectively, with $\Gamma_D \neq \varnothing$ and $\overline{\Gamma_D \cup \Gamma_N} = \partial\Omega$, and prescribed values $u|_{\Gamma_D} = u_D$ and $-\mathcal{K}\nabla u \cdot \boldsymbol{n}|_{\Gamma_N} = \bar{\lambda}_N$. In order to include these boundary conditions in the formulation, we redefine the space $M(\mathcal{T}_\mathrm{h})$ as

$$M(\mathcal{T}_\mathrm{h}) = \left\{ \mu \in \prod_{K\in\mathcal{T}_\mathrm{h}} H^{-\frac{1}{2}}(\partial K) \,:\, \exists\, \boldsymbol{q} \in H_{0,\Gamma_N}(\mathrm{div},\Omega) \text{ s.t. } \boldsymbol{q}\cdot\boldsymbol{n}^{\partial K} = \mu \text{ on } \partial K\,, \forall K \in \mathcal{T}_\mathrm{h} \right\},$$

where

$$H_{0,\Gamma_N}(\mathrm{div},\Omega) = \left\{ \boldsymbol{v} \in H(\mathrm{div},\Omega) \,:\, \langle \boldsymbol{v}\cdot\boldsymbol{n}, v\rangle = 0,\ \forall\, v \in H^1_{0,\Gamma_D}(\Omega) \right\},$$

with $H^1_{0,\Gamma_D}(\Omega) = \left\{ v \in H^1(\Omega) \,:\, v|_{\Gamma_D} = 0 \right\}$. Then, the primal hybrid formulation is stated as: find the pair $(u,\lambda) \in X \times M$ such that

$$\sum_{K\in\mathcal{T}_\mathrm{h}} \int_K (\mathcal{K}\nabla u)\cdot\nabla v \,\mathrm{dx} + \sum_{K\in\mathcal{T}_\mathrm{h}} \langle\lambda, v\rangle_{\partial K} = \int_\Omega f v \,\mathrm{dx} - \langle\bar{\lambda}_N, v\rangle_{\Gamma_N} \quad \forall\, v \in X, \tag{4.1a}$$

$$\sum_{K\in\mathcal{T}_\mathrm{h}} \langle\mu, u\rangle_{\partial K} = \langle\mu, u_D\rangle_{\Gamma_D} \quad \forall\, \mu \in M. \tag{4.1b}$$

In this case, it is important to highlight that the interpretation of the Lagrange multiplier stated in (1.6) remains valid except on Γ_N. Indeed, in (4.1) the Lagrange multiplier vanishes on Γ_N and the non-homogeneous flux is included through the term $\langle \bar{\lambda}_N, v\rangle_{\Gamma_N}$. Nonetheless, according to the results presented in [1], all the analysis (at continuous and discrete levels) can be carried over, provided that the source terms are bounded. For example, it suffices that $f \in L^2(\Omega)$, $\bar{\lambda}_N \in L^2(\Gamma_N)$ and $u_D \in H^{1/2}(\Gamma_D)$.

5. Conclusions

We developed the convergence analysis of the Primal Hybrid Finite Element Method on regular meshes of convex quadrilaterals generated by bilinear isomorphisms. The theoretical development made use of previous results presented by [25] for general approximations by quadrilateral finite elements, and Theorems 3.4 and 3.6 can be seen as a complement to the results of [4]. We also presented numerical results for test problems with homogeneous and heterogeneous coefficients and different boundary conditions, that are in agreement with the convergence theory.

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