Instantons on G_2−manifolds (with Addendum)

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by

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I certify that this thesis, and the research to which it refers, are the product of my own work, and that any ideas or quotations from the work of other people, published or otherwise, are fully acknowledged in accordance with the standard referencing practices of the discipline.

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ABSTRACT

This thesis sets the bases of a research line in the theory of G_2-instantons. Its immediate motive is to study a nonlinear parabolic flow of metrics on a holomorphic bundle $\mathcal{E} \to W$ over a certain asymptotically cylindrical Calabi-Yau $3-$fold, assuming $\mathcal{E}$ is stable 'at infinity'. Crucially, the solution's infinite time limit, if it exists, is a Hermitian Yang-Mills (HYM) metric on $\mathcal{E}$.

After a prelude on the group G_2 and G_2-manifolds $\left(M^7, \varphi\right)$, I develop the analogous versions of such aspects of lower dimensional gauge theory as the Chern-Simons functional, the notion of self-duality and topological energy bounds. I also explain how HYM metrics over a CY^3 give rise to G_2-instantons over the product $CY^3 \times S^1$, hence why our global analysis problem advances the cause of obtaining such instantons. I then move on to the general deformation theory of irreducible G_2-instantons, culminating at the local model for the moduli space. An illustration of some of these ideas appears as an exercise on 7-tori.

Secondly, I sketch A. Kovalev's construction of compact manifolds with holonomy G_2 and derive simple results, such as a Poincaré-Lelong-type equation of currents.

The core of the thesis is the study of the nonlinear 'heat flow'. Following the methods of S. Donaldson, C. Simpson et al., I establish the existence of a smooth solution $\{H_t\}$ defined for all time and having good asymptotic behaviour. Furthermore, I reduce the question of convergence as $t \to \infty$ to a conjectured lower energy bound over a cylindrical segment, down the tubular end, of measure roughly proportional to $\|H_t\|_{C^0}$.

Finally, I propose one example of a bundle satisfying the stability assumptions of our evolution problem, derived from the null-correlation bundle over $\mathbb{C}P^3$.

Addendum: A sufficient restatement with proof of the conjectured lower energy bound has been added *a posteriori* as *Appendix B.* This was not part of the original thesis.

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INTRODUCTION

The present work lays the foundation of a long-term research project in the theory of G_2-instantons. It fits in the wider context of gauge theory in higher dimensions, following the seminal works of S. Donaldson, R. Thomas, G. Tian and others. The common thread to such generalisations is the presence of an additional, closed structure on the base manifold M which induces an analogous notion of anti-self-dual connections, or *instantons*, on bundles over M. In the case at hand, G_2-manifolds are $7-$dimensional Riemannian manifolds with holonomy in the exceptional Lie group G_2, which translates exactly into the presence of such a closed structure. This allows one to make sense of G_2-*instantons* as the energy-minimising gauge classes of connections, solutions to the corresponding Yang-Mills equation.

Although the development of similar theories, notably in dimensions four [D-K] and six [Tho], has led to a very significant understanding of the invariants associated to moduli spaces of instantons, little is currently known about the $7-$dimensional case – indeed, no G_2-instanton has yet been constructed, much less any moduli space invariants studied. This is due not least to the success and attractiveness of the previous theories themselves, but partly also to the relative scarcity of working examples of G_2-manifolds [Bry][B-S][Joy].

In 2003, A. Kovalev provided an original construction of compact manifolds M with holonomy G_2. These are obtained by gluing two smooth asymptotically cylindrical Calabi-Yau 3-folds W_1 and W_2, truncated sufficiently far along the noncompact end, via an additional "twisted" circle component S^1. This opened a clear, three-step path in the theory of G_2-instantons: (1) to obtain a Hermitian Yang-Mills (HYM) connection over each W_i, which pulls back to a G_2-instanton A_i over the product $W_i \times S^1$, (2) to glue the A_i compatibly with the twisted connected sum $\left(W_1 \times S^1\right) \widetilde{\#} \left(W_2 \times S^1\right)$ in order to obtain a G_2-instanton $A = A_1 \widetilde{\#} A_2$ over M [Don_1][Tau], and (3) to study the moduli space of such instantons and eventually compute invariants in particular cases of interest.

The present work has a three-fold purpose in this context. First, to formalise in generality the basic elements of a theory of G_2-instantons and their moduli spaces. Second, to concretely frame the problem of obtaining G_2-instantons over Kovalev's manifolds. Finally, and most importantly, to make significant progress towards completing part (1) of the above strategy.

I begin *Chapter 1* recalling basic facts about the group G_2, in order to motivate the appearance of the G_2-structure φ (*Section 1.1*). This survey then proceeds to define G_2-manifolds and mention some of their key properties (*Section 1.2*). After this prelude I move on to the gauge theory set up, discussing broadly the Yang-Mills functional and the principle of generalisation to dimension $n \geq 4$ under the presence of a closed $(n-4)-$form [Tia][D-T]. I then start in earnest the adaptation of gauge-theoretical concepts and tools into the setting of G_2-manifolds, such as the Chern-Simons functional and its associated $1-$form (*Subsection 1.3.2*), the notion of self-duality of $2-$forms in the presence of φ, topological energy bounds for the Yang-Mills functional and the relationship between HYM connections over a Calabi-Yau $3-$fold W and G_2-instantons over $W \times S^1$ (*Section 1.4*).

The next fundamental adaptation is the deformation theory of G_2-instantons, via Fredholm differential topology; the discussion culminates at the finite-dimensional local model for the moduli space near irreducible G_2-instantons (*Section 1.5*). I illustrate some of these ideas with an exercise on T^3-fibred 7-tori (*Section 1.6*).

Chapter 2 is a quick introduction to Kovalev's manifolds [Kov_1][Kov_2], featuring the statement of his noncompact Calabi-Yau-Tian theorem and a discussion of its essential ingredients (*Section 2.1*). The main purpose is to set the scene for the analysis to follow, which takes place over an asymptotically cylindrical $SU(3)-$manifold W as given by that theorem. In the process we establish quite a bit of notation and prove a few basic results for the sequel, as well as a Poincaré-Lelong equation of currents which is just a curiosity (*Subsection 2.1.3*). Finally, I include a brief outline of the gluing process and some broader strategic comments (*Subsection 2.2.2*).

The core of the thesis is *Chapter 3*, concerning the HYM problem on a holomorphic bundle $\mathcal{E} \to W$ satisfying a certain asymptotic stability assumption along the noncompact end of W and carrying a suitable reference metric H_0 (*Section 3.1*). This amounts to studying a parabolic equation on the space of Hermitian metrics [Don_2][Don_4] over an asymptotically cylindrical base manifold, and it follows a standard pattern [Sim][Guo][But]. One begins by solving the associated Dirichlet problem on an arbitrary finite truncation

W_S, first obtaining short-time existence of smooth solutions $H_S(t)$ (*Subsection 3.1.1*), then extending them for all time. Fixing arbitrary finite time, one obtains $H_S(t) \underset{S\to\infty}{\longrightarrow} H(t)$ on compact subsets of W (*Subsection 3.1.2*). Moreover, every metric in the $1-$parameter family $H(t)$ approaches exponentially the reference metric H_0, in a suitable sense, along the cylindrical end. Hence one has solved the original parabolic equation and its solution has convenient asymptotia.

From *Section 3.2* onwards I tackle the issue of controlling $\lim_{t\to\infty} H(t)$, which is the final milestone towards step (1) of the project. Adapting to our context the 'determinant line norm' functionals introduced by Donaldson [Don$_2$][Don$_3$], I conjecture a time-uniform lower bound on the 'energy density' $\hat{F}$ over a finite piece, down the tubular end, of size roughly proportional to $\|H(t)\|_{C^0(W)}$, in *Conjecture 3.71* (*Subsection 3.2.3*). That, in turn, is a sufficient condition for uniform C^0-convergence of $H(t)$ over the whole of W, in view of an upper energy bound derived by the Chern-Weil method (*Section 3.3*). Therefore the HYM problem has been effectively reduced to establishing the conjectured lower energy bound over such a 'large' cylinder, as stated in *Theorem 3.76*.

Finally, *Chapter 4* presents an illustrative example of a setting $\mathcal{E} \to W$ satisfying the assumptions of our analysis in *Chapter 3*, based on the null-correlation bundle over $\mathbb{C}P^3$ [O-S-S][Bar].

ABOUT THE ADDENDUM: Please note that *Appendix B*, containing the correct restatement and a proof of *Conjecture 3.71*, hence of the main existence result claimed in *Theorem 3.76*, has been added to this document in November 2009. Its contents have not been scrutinised by the examiners of the original thesis, nor do they have any connection with the doctoral requirements of Imperial College London, and the author alone bears responsibility for any mistakes therein.

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CHAPTER 1
GAUGE THEORY ON G_2−MANIFOLDS

1.1 The group G_2

The exceptional Lie group G_2 comes last in the classification of families of simple Lie groups according to their (connected) Dynkin graphs [Jac]:

$$A_l, B_l, C_l, D_l, \underbrace{E_6, E_7, E_8, F_4, G_2}_{\text{exceptional}}$$

where subscript counts simple roots in the Lie algebra. Families E_6, E_7, E_8, F_4 and G_2 are called exceptional because each has a single element, and the ordering of letters corresponds to the presence of graph edges with increasing multiplicity. Yet again G_2 is 'exceptional', being the only one to admit a triple edge, the greatest root valence in a Dynkin graph.

Etymologies aside, in practice we will be interested in an alternative definition of G_2 [Sal, p.155]; let $\{e^i\}_{i=1,\ldots,7}$ denote the canonical basis of $\left(\mathbb{R}^7\right)^*$, $e^{ij} \doteq e^i \wedge e^j$ etc.:

Definition 1.1 *The* group G_2 *is the subgroup of* $GL(7)$ *preserving the* 3−*form*

$$\varphi_0 = \left(e^{12} - e^{34}\right) \wedge e^5 + \left(e^{13} - e^{42}\right) \wedge e^6 + \left(e^{14} - e^{23}\right)^7 + e^{567} \tag{1.1}$$

under the standard (pull-back) action on $\Lambda^3 \left(\mathbb{R}^7\right)^*$*, i.e.,*

$$G_2 \doteq \{g \in GL(7) \mid g^* \varphi_0 = \varphi_0\}.$$

This arguably obscure definition encodes the geometrical fact that G_2 is the group of automorphisms preserving a generalised vector cross-product in $\mathbb{R}^7$. To see this, recall that the usual vector cross-product in $\mathbb{R}^3$ comes from the identification of this space with the imaginary quaternions $\mathrm{Im}\,(\mathbb{H})$; then $a \times b$ is just the commutator under quaternionic algebra

$$a \times b \doteq \frac{1}{2}\,[a, b]\,. \tag{1.2}$$

The same relation (1.2) defines an analogous cross-product in $\mathbb{R}^7$ if we borrow the commutator this time from the imaginary octonions $\operatorname{Im}\mathbb{O} \simeq \mathbb{R}^7$, with multiplication conventions:

$\cdot$	e^1	e^2	e^3	e^4	e^5	e^6	e^7
e^1	-1	e^5	e^6	e^7	$-e^2$	$-e^3$	$-e^4$
e^2	$-e^5$	-1	$-e^7$	e^6	e^1	$-e^4$	e^3
e^3	$-e^6$	e^7	-1	$-e^5$	e^4	e^1	$-e^2$
e^4	$-e^7$	$-e^6$	e^5	-1	$-e^3$	e^2	e^1
e^5	e^2	$-e^1$	$-e^4$	e^3	-1	e^7	$-e^6$
e^6	e^3	e^4	$-e^1$	$-e^2$	$-e^7$	-1	e^5
e^7	e^4	$-e^3$	e^2	$-e^1$	e^6	$-e^5$	-1

It is then straightforward to verify that φ_0 induces the cross-product, in the presence of the Euclidean metric, by considering the following bilinear map from $\mathbb{R}^7$ to its dual:

$$\begin{aligned} \mathbb{R}^7 \times \mathbb{R}^7 &\to \left(\mathbb{R}^7\right)^* \\ (a,b) \mapsto (a,b)\lrcorner\varphi_0 &\doteq \varphi_0\left(a,b,.\right) = \left(a \times b\right)^* . \end{aligned} \tag{1.3}$$

For example, $\varphi_0\left(v_1,v_2,\cdot\right) = e^5\left(\cdot\right) = \left(v_5\right)^* = \left(v_1 \times v_2\right)^*$, where $e^i(v_j) \doteq \delta^i_j$. Note that the the restriction to $\operatorname{Im}\mathbb{O}$ of the natural inner product on $\mathbb{O}$,

$$\langle a,b\rangle \doteq \tfrac{1}{2}\left(\overline{a}.b + \overline{b}.a\right), \tag{1.4}$$

defined by polarisation (which means simply multiplication by -1 on $\operatorname{Im}\mathbb{O}$), coincides with the Euclidean inner product on $\mathbb{R}^7$:

$$\begin{aligned} \langle a,b\rangle &= \left\langle a^i v_i, b^j v_j\right\rangle = \tfrac{1}{2}\left(-a^i b^j\right)\underbrace{\left(v_i.v_j + v_j.v_i\right)}_{-2\delta_{ij}} \\ &= \textstyle\sum_i a^i b^i. \end{aligned} \tag{1.5}$$

Indeed, the Euclidean metric itself can be recovered from φ_0 using the relation

$$\langle a,b\rangle\, e^{1\ldots7} = \left(a\lrcorner\varphi_0\right) \wedge \left(b\lrcorner\varphi_0\right) \wedge \varphi_0 \tag{1.6}$$

(established e.g. by direct inspection on basis elements), and *Proposition 1.3* will show that G_2 preserves volume and orientation as on the left-hand side above. We conclude, in particular, that the induced cross-product (1.3) is obtained solely from the structure φ_0.

Actually, φ_0 encodes completely the octonions' algebra, as we can see from two elementary properties of the imaginary octonions:

Lemma 1.2 *For $a, b \in \mathrm{Im}\,\mathbb{O}$, we have:*

1. $a \times b = a.b + \langle a, b \rangle$;

2. $6 \langle a, b \rangle = -\mathrm{tr}\, T_{a,b}$,

where $T_{a,b}$ is the linear map given by the cross-product composition

$$\begin{aligned} T_{a,b} : \mathrm{Im}\,\mathbb{O} &\rightarrow \mathrm{Im}\,\mathbb{O} \\ v &\mapsto a \times (b \times v) \,. \end{aligned} \tag{1.7}$$

Proof

1. *We have $a \times b = \frac{1}{2}[a, b] = a.b - \frac{1}{2}(a.b + b.a)$; on the other hand, by definition, $2\langle a, b\rangle = \overline{a}.b + \overline{b}.a = -(a.b + b.a)$ since a and b are pure imaginary.*

2. *By linearity, it suffices to consider a, b among the canonical basis $\{v_1, ..., v_7\}$:*

$$\begin{aligned} \mathrm{tr}\, T_{v_i,v_j} &= \sum_{k=1}^{7} \left(T_{v_i,v_j}(v_k)\right)^k = \sum_{k=1}^{7} (v_i \times (v_j \times v_k))^k \overset{(1.)}{=} \sum_{j \neq k=1}^{7} (v_i \times \underbrace{(v_j.v_k)}_{v_l})^k \\ &= -\sum_{j \neq k=1}^{7} (v_l.v_i)^k = -\sum_{j \neq k=1}^{7} (\delta_{ij} v_k)^k = -6\delta_{ij} \\ &= -6 \langle v_i, v_j \rangle \,. \end{aligned}$$

■

Combining both parts of the above *Lemma 1.2* we have

$$a.b = a \times b + \frac{1}{6}\mathrm{tr}\, T_{a,b} \tag{1.8}$$

where the right-hand side is defined solely using the cross-product; hence the octonion product structure on $\mathrm{Im}\,\mathbb{O}$ is indeed recovered from the cross-product given by φ_0. Moreover, as claimed:

Proposition 1.3 $G_2 \subset SO(7)$.

Proof *We may think of the double cross-product (1.7) as a map*

$$T : \mathbb{R}^7 \times \mathbb{R}^7 \to \mathbb{R}^7 \otimes \left(\mathbb{R}^7\right)^*$$
$$(a, b) \mapsto T_{a.b}$$

and it is easy to see from the equivariance of the cross-product that T is G_2-covariant:

$$\begin{aligned} T_{ga,gb}(v) &= ga \times (gb \times v) \\ &= ga \times g\left(b \times g^{-1}v\right) \\ &= g\left(a \times \left(b \times g^{-1}v\right)\right) \\ &= g \circ T_{a,b} \circ g^{-1}(v). \end{aligned}$$

Hence $\operatorname{tr} T_{a,b}$ *is G_2-invariant for all a, b and so $G_2 \subset O(7)$ by* Lemma 1.2. *Regarding orientation, observe in the relationship (1.6) between φ_0 and the Euclidean inner product,*

$$\langle a, b\rangle e^{1\ldots 7} = (a \lrcorner \varphi_0) \wedge (b \lrcorner \varphi_0) \wedge \varphi_0,$$

that the r.h.s. is G_2-equivariant as a map $\mathbb{R}^7 \times \mathbb{R}^7 \to \Lambda^7\left(\mathbb{R}^7\right)^*$. Acting on the l.h.s. with any $g \in G_2$ and applying the transformation law for the volume element gives

$$\langle ga, gb\rangle = (\det g)^{-1}\langle a, b\rangle.$$

Consequently,

$$\begin{aligned} (\det g)^7 &= \det\left[(\det g)\,\delta_{ij}\right] = \det\left[(\det g)\langle v_i, v_j\rangle\right] \\ &= \det\left[\left\langle g^{-1}v_i, g^{-1}v_j\right\rangle\right] = \det\left[\langle v_i, v_j\rangle\right] \\ &= 1. \end{aligned}$$

■

Furthermore, in view of *Proposition 1.3*, part 1 of *Lemma 1.2* means that G_2 is exactly the group of algebra automorphisms of $\operatorname{Im}\mathbb{O}$. Extending the action trivially on $1 \in \mathbb{O}$, we realise in the same sense

$$G_2 = \operatorname{Aut}(\mathbb{O}).$$

The connection between this presentation of G_2 and the Lie algebraic setting in the beginning of this *Section* is the fact that $\mathfrak{g}_2 = \mathrm{Der}\,(\mathbb{O})$ has precisely the triple-edged diagram predicted in the Dynkin classification [Jac], but this will not be important for this text.

The following *Theorem* [Bry, 1.] summarises some relevant properties of G_2:

Theorem 1.4 *The subgroup $G_2 \subset SO(7) \subset GL(7)$ is compact, connected, simple, simply connected and* $\dim (G_2) = 14$. *Moreover, G_2 acts irreducibly on $\mathbb{R}^7$ and transitively on S^6.*

1.2 G_2-manifolds

Ultimately we want to consider Riemannian manifolds with holonomy group G_2, so the first thing to do is to spot suitable candidates. A famous result by Berger [*Theorem* A.80] classifies essentially all cases in which a subgroup of $SO\,(n)$ may occur as the holonomy group of an irreducible and nonsymmetric[1] simply-connected Riemannian manifold. Accordingly, our starting point will be the following instance of Berger's theorem:

Corollary 1.5 *If (M, g) is a simply-connected Riemannian manifold such that g is irreducible and nonsymmetric and* $\mathrm{Hol}\,(g) = G_2$, *then* $\dim M = 7$.

Let M be an oriented simply-connected smooth 7−manifold. We will see in this *Section* that if M carries a *torsion-free G_2−structure*, then it also admits a Riemannian metric with holonomy at least *contained* in G_2.

Definition 1.6 *A G_2−*structure *on the 7−manifold M is a 3−form $\varphi \in \Omega^3(M)$ such that, at every point $p \in M$, $\varphi_p = f_p^*(\varphi_0)$ for some frame $f_p : T_pM \to \mathbb{R}^7$.*

Given such φ, the set of all global frames f satisfying $\varphi = f^*\varphi_0$ is indeed a principal subbundle of the frame bundle $F \to M$, with fibre G_2, because the right-action of G_2 on F fixes φ_0 by definition. Moreover, as $G_2 \subset SO\,(7)$ [*Proposition 1.3*], φ fixes the orientation given by some (and consequently any) such frame f and also the metric $g = g\,(\varphi)$ given pointwise by *Lemma 1.2*. We may refer to (φ, g) as the G_2−structure whenever doing so is more economical than recalling explicitly that g is the associated metric.

The *torsion of* φ is the covariant derivative $\nabla\varphi$ given by the Levi-Civita connection of the induced metric g, thus we say that φ is *torsion-free* if $\nabla\varphi = 0$.

[1] for a discussion of Lie groups arising as holonomy of Riemannian symmetric spaces, see [Joy, pp. 50-53].

Definition 1.7 *A* G_2-manifold *is a pair* (M, φ) *where* M *is a* $7-$*manifold and* φ *is a torsion-free* G_2-*structure on* M*.*

The following theorem [Joy, 10.1.3, p.244][Fe-G] indicates the convenience of G_2-manifolds in the construction of examples of manifolds with holonomy group G_2.

Theorem 1.8 *Let* M *be a* $7-$*manifold with* G_2-*structure* φ *and associated metric* g*; then the following are equivalent:*

1. $\nabla\varphi = 0$, *i.e.,* M *is a* G_2-*manifold;*
2. $\mathrm{Hol}\,(g) \subset G_2$*;*
3. *let* $*_\varphi$ *denote the Hodge star operator induced by* g*; then*

$$d\varphi = d *_\varphi \varphi = 0.$$

NB.: Equations $\nabla\varphi = 0$ *and* $d *_\varphi \varphi = 0$ *are* nonlinear *in* φ*, since the metric* g *itself (hence also the Hodge star and the Levi-Civita connection) depends on* φ*.*

Finally, if we restrict our attention to *compact* G_2-manifolds, the condition to have holonomy *exactly* G_2 is given by the next theorem, which is a direct consequence of Cheeger-Gromoll decomposition [Joy, 10.2.2, p.245].

Theorem 1.9 *Let* (M, φ, g) *be a compact* G_2-*manifold; then*

$$\mathrm{Hol}\,(g) = G_2 \iff \pi_1(M) \text{ is finite.}$$

The purpose of the 'twisted' gluing in A. Kovalev's construction of asymptotically cylindrical G_2-manifolds [cf. *Section 2.2*] is precisely to secure this topological condition, hence strict holonomy G_2.

1.3 Gauge theory in higher dimensions

Gauge theory can be described as the study of bundles $E \to X$ with connections satisfying some gauge-invariant condition on the curvature [Jar, Ch. 1]. For example, the flatness condition

$$F_A = 0 \tag{1.9}$$

is clearly gauge-invariant. From a variational point of view, flat connections solve trivially the *Yang-Mills equation*

$$d_A^* F_A = 0, \tag{1.10}$$

which is the Euler-Lagrange equation of the *Yang-Mills functional*

$$YM(A) \doteq \|F_A\|^2 = \int_X |F_A|^2 .$$

When $\dim X = 4$, connections whose curvatures lie in the subbundle of $\pm 1-$eigenspaces of the Hodge star in $\Omega^2(\mathfrak{g}_E)$ (respectively *self-dual* and *anti-self-dual*) also solve (1.10), by the Bianchi identity:

$$d_A^* F_A = \pm d_A F_A = 0.$$

So both self-dual (SD) and anti-self-dual (ASD) connections are critical points[2] of the Yang-Mills functional. Indeed, it is well-known from Chern-Weil theory that, whenever a SD or ASD connection exists, it is an *absolute minimum*[3] of YM.

In *Subsection 1.3.1* I will sketch some of G. Tian's ideas [Tia] on the generalisation of these concepts to higher dimensions $n = \dim X \geq 4$, in the presence of a closed $(n-4)-$form on X.

In *Subsection 1.3.2* I will briefly describe the Chern-Simons picture, in which flat connections emerge analytically as critical points of a certain functional, over a base of real dimension three. We will then apply the same principle to detect G_2-instantons in dimension seven.

[2]Usually $\left(\text{i.e., when } c_2(E) \neq \left(1 - \frac{1}{\mathrm{rk}\,(E)}\right) c_1(E)\right)$, E admits at most one type: *either* SD *or* ASD; and possibly none.

[3]We shall carry an analogous discussion for G_2-manifolds in *Subsection 1.4.2*.

1.3.1 The Yang-Mills functional

Let $E \to X$ be a unitary bundle of rank r with structure group[4] G over a Riemannian manifold X of dimension $n \geq 4$. Given a connection A on E, we define the Yang-Mills functional as usual:

$$YM(A) \doteq \|F_A\|^2 = \int_X \langle F_A \wedge *F_A \rangle_{\mathfrak{g}},$$

where $\mathfrak{g} = \mathrm{Lie}(G)$, thus having its Euler-Lagrange equation:

$$d_A^* F_A = 0. \tag{1.11}$$

Following [Tia], the presence of a closed $(n-4)-$form Θ on X gives a criterion for finding classes of solutions to (1.11), which suggests a generalisation for the concept of instanton:

Lemma 1.10 *Let A be a unitary connection on $E \to X$ such that $\mathrm{tr}\,(F_A) \in \Omega^2(X)$ is harmonic and*

$$\Theta \wedge \left(F_A - \frac{1}{r}\mathrm{tr}\,(F_A) \otimes Id \right) = - * \left(F_A - \frac{1}{r}\mathrm{tr}\,(F_A) \otimes Id \right); \tag{1.12}$$

then A is a solution of (1.11). Moreover, if X is compact and without boundary, A satisfies

$$YM(A) = \left(2C_2(E) - \frac{r-1}{r} C_1(E)^2 \right) \cdot [\Theta] + \frac{1}{4\pi^2 r} \int_X |\mathrm{tr}\,(F_A)|^2 \, d\mu$$

with $[\Theta] \in H^{n-4}(X, \mathbb{R})$.

Remark 1.11 *Concerning the above* Lemma*:*

1. *The curvature splits invariantly into $\mathrm{tr}\, F_A \otimes Id$ and its traceless part F_A^0:*

$$F_A = \underbrace{\left(F_A - \frac{1}{r}\mathrm{tr}\,(F_A) \otimes Id \right)}_{F_A^0} + \frac{1}{r}\mathrm{tr}\,(F_A) \otimes Id$$

and indeed $\|\mathrm{tr}\,(F_A)\|_{L^2}$ is minimised by the harmonic representative of the de Rham cohomology class $[\mathrm{tr}\,(F_A)]$. So, when looking for energy-minimising connections, it suffices to minimise the L^2-norm of F_A^0 [Don$_2$, p. 4].

[4] We will soon specialise, for simplicity, to the case $G \subseteq SU(n)$.

2. *Assume $G \subseteq SU(n)$; at a minimising connection, the energy $YM(A)$ depends only on $E \to X$ and on the cohomology class of Θ. We define indeed* generalised instantons *as solutions of*

$$F_A \wedge \Theta = - * F_A.$$

3. *Quite often the condition (1.12) is overdetermined and the criterion is void, but in our present case of interest it is precisely the G_2-instanton equation [cf. (1.21)].*

In *Section 1.4* we will study the anti-self-dual property (of generalised instantons) in the case where $X = M$ is a G_2-manifold and $\Theta = *\varphi$ is the Hodge dual of its G_2-structure (see also [Rey]).

1.3.2 Chern-Simons theory

Consider a bundle P over a compact $3-$manifold Y [Don_1, §2.5]. The Chern-Simons functional is a multi-valued real function on the quotient $\mathcal{B} = \mathcal{A}/\mathcal{G}$, of the space $\mathcal{A}$ of connections on P by the gauge group, taking integer periods. It has the remarkable property that its critical points are precisely the flat connections on P modulo gauge.

Recall that $\mathcal{A}$ is an affine space modelled on $\Omega^1(\mathfrak{g}_P)$ so, fixing a reference $A_0 \in \mathcal{A}$,

$$\mathcal{A} = A_0 + \Omega^1(\mathfrak{g}_P).$$

In particular, $T\mathcal{A} \simeq \mathcal{A} \times \Omega^1(\mathfrak{g}_P)$, so we think of vectors on $\mathcal{A}$ as elements $a \in \Omega^1(\mathfrak{g}_P)$ and define a $1-$form on $\mathcal{A}$ by

$$\rho(a)_A \doteq \int_Y \operatorname{tr}(F_A \wedge a). \tag{1.13}$$

Let us first check that ρ is closed: since $F_{A+b} = F_A + d_A b + b \wedge b$, we have the first order difference

$$\rho(a)_{A+b} - \rho(a)_A = \int_Y \operatorname{tr}(d_A b \wedge a) + O\left(|a| . |b|^2\right). \tag{1.14}$$

Notice that the leading term in (1.14) is symmetric in a and b by Stokes's theorem:

$$\int_Y \operatorname{tr}(d_A b \wedge a - b \wedge d_A a) = \int_Y d(\operatorname{tr}(b \wedge a)) = 0. \tag{1.15}$$

This shows that ρ is closed when we compare the reciprocal Lie derivatives on parallel vector fields a, b around a point A:

$$
\begin{aligned}
d\rho\left(a,b\right)_A &= \left(\mathcal{L}_b\rho\left(a\right)\right)_A - \left(\mathcal{L}_a\rho\left(b\right)\right)_A \\
&= \lim_{h\to 0}\frac{1}{h}\left\{\left(\rho\left(a\right)_{A+hb}-\rho\left(a\right)_A\right)-\left(\rho\left(b\right)_{A+ha}-\rho\left(b\right)_A\right)\right\} \\
&= \lim_{h\to 0}\frac{1}{h^2}\underbrace{\left\{\left(\rho\left(ha\right)_{A+hb}-\rho\left(ha\right)_A\right)-\left(\rho\left(hb\right)_{A+ha}-\rho\left(hb\right)_A\right)\right\}}_{O\left(|h|^3\right)} \\
&= 0.
\end{aligned}
$$

Since $\mathcal{A}$ is contractible, the Poincaré Lemma tells us that ρ is the derivative of some function ϑ, say. Moreover, again by Stokes's Theorem, ρ vanishes on $\operatorname{img}\left(d_A\right)\simeq T_A\left\{\mathcal{G}.A\right\}$, which are precisely the directions tangent to the $\mathcal{G}$−orbits [cf. *Subsection 1.5.1* below]. Hence ρ descends to the quotient $\mathcal{B}$ and so does ϑ, at least locally. In fact, it can be shown that ϑ is given, up to an integer period, by the *Chern Simons functional*:

$$
\begin{aligned}
\vartheta \ :\ \mathcal{B} &\to \mathbb{R}/\mathbb{Z} \\
[A] &\mapsto \tfrac{1}{2}\int_Y \operatorname{tr}\left(d_{A_0}a\wedge a+\frac{2}{3}a\wedge a\wedge a\right)
\end{aligned}
$$

where $A=A_0+a$ and we assume for simplicity $F_{A_0}=0$. The first-order variation of ϑ is

$$
\begin{aligned}
\vartheta\left(A+b\right)-\vartheta\left(A\right) &= \tfrac{1}{2}\int_Y \operatorname{tr}\left(d_{A_0}a\wedge b+a\wedge d_{A_0}b+2a\wedge a\wedge b\right)+O\left(|b|^2\right) \\
&= \tfrac{1}{2}\int_Y \operatorname{tr} 2\underbrace{\left(d_{A_0}a+a\wedge a\right)}_{F_A}\wedge b+O\left(|b|^2\right) \\
&= \rho\left(b\right)_A+O\left(|b|^2\right),
\end{aligned}
$$

so one finds precisely

$$d\vartheta=\rho.$$

Comparing with (1.13), we see that the flat connections on $P\to Y$ are indeed the critical points of the Chern-Simons function in dimension 3.

Now, in the spirit of *Subsection 1.3.1*, a similar theory can be formulated in higher dimensions under the presence of a suitable closed $(n-3)$−form [D-T][Tho]. Our case of interest is $n=7$, in which the 4−form $*\varphi$, Hodge-dual to the G_2−structure on M, allows for the definition of a functional of Chern-Simons type[5] on the set of connections $\mathcal{A}$:

$$\vartheta(A) = \tfrac{1}{2}\int_M \operatorname{tr}\left(d_{A_0}a\wedge a + \frac{2}{3}a\wedge a\wedge a\right)\wedge *\varphi,$$

where we fix for simplicity $A_0 \in \mathcal{A}$ such that $F_{A_0}\wedge *\varphi = 0$. This function is obtained by integration of the analogous $1-$form

$$\rho(a)_A = \int_M \operatorname{tr}\left(F_A\wedge a\right)\wedge *\varphi. \tag{1.16}$$

We find ϑ explicitly by integrating ρ over paths $A(t) = A_0 + ta$, from A_0 to any $A = A_0 + a$:

$$\begin{aligned}\vartheta(A) - \vartheta(A_0) &= \int_0^1 \rho_{A(t)}\left(\dot{A}(t)\right)dt \\ &= \int_0^1\int_M \operatorname{tr}\left(\left(F_{A_0} + t d_{A_0}a + t^2 a\wedge a\right)\wedge a\right)\wedge *\varphi \\ &= \tfrac{1}{2}\int_M \operatorname{tr}\left(d_{A_0}a\wedge a + \frac{2}{3}a\wedge a\wedge a\right)\wedge *\varphi.\end{aligned}$$

It remains to check that (1.16) is closed, so that the above procedure doesn't depend on the path $A(t)$. This is true because of the property $d*\varphi = 0$ of the G_2-structure, which ensures that the leading term in the expansion of ρ,

$$\rho(a)_{A+b} - \rho(a)_A = \int_M \operatorname{tr}\left(d_A b\wedge a\right)\wedge *\varphi + O\left(|b|^2\right),$$

is still symmetric [cf. (1.14) and (1.15)] by Stokes' Theorem:

$$\int_M \operatorname{tr}\left(d_A b\wedge a - b\wedge d_A a\right)\wedge *\varphi = \int_M d\left(\operatorname{tr}\left(b\wedge a\right)\wedge *\varphi\right) = 0.$$

Hence

$$\rho(a)_{A+b} - \rho(a)_A = \rho(b)_{A+a} - \rho(b)_A + O\left(|b|^2\right)$$

and the previous $3-$dimension argument holds *ipsis litteris* to show that ρ is closed. At least locally, then, the functional ϑ descends to the orbit space $\mathcal{B}$.

To obtain the periods of ϑ we have to examine how its value is affected by the gauge action, so choose $g \in \mathcal{G}$ and consider some path $\{A(t)\}_{t\in[0,1]} \subset \mathcal{A}$ connecting A to $g.A$. The

[5]indeed, since only the condition $d*\varphi = 0$ is required, the discussion extends to cases in which the G_2-structure φ is not necessarily torsion-free.

natural projection $p_1 : M \times [0,1] \to M$ induces a bundle

$$\begin{array}{ccc} \mathbf{E}_g & \xrightarrow{\widetilde{p_1}} & E \\ \downarrow & & \downarrow \\ M \times [0,1] & \xrightarrow{p_1} & M \end{array}$$

and, using g to identify the fibres $(\mathbf{E}_g)_0 \overset{g}{\simeq} (\mathbf{E}_g)_1$, we may think of $\mathbf{E}_g$ as a bundle over $M \times S^1$. Moreover, in some local trivialisation, the path $A(t) = A_i(t)\, dx^i$ gives a connection $\mathbf{A} = \mathbf{A}_0 dt + \mathbf{A}_i dx^i$ on $\mathbf{E}_g$:

$$\begin{aligned} (\mathbf{A}_0)_{(t,p)} &= 0 \\ (\mathbf{A}_i)_{(t,p)} &= A_i(t)_p \, . \end{aligned}$$

The corresponding curvature 2−form is $F_{\mathbf{A}} = (F_{\mathbf{A}})_{0i}\, dt \wedge dx^i + (F_{\mathbf{A}})_{jk}\, dx^j \wedge dx^k$, where

$$\begin{aligned} (F_{\mathbf{A}})_{0i} &= \dot{A}_i(t) \\ (F_{\mathbf{A}})_{jk} &= (F_A)_{jk} \, . \end{aligned}$$

The periods of ϑ are then of the form

$$\begin{aligned} \vartheta(g.A) - \vartheta(A) &= \int_0^1 \rho_{A(t)}\left(\dot{A}(t)\right) dt \\ &= \int_{M \times [0,1]} \operatorname{tr}\left(F_{A(t)} \wedge \dot{A}_i(t)\, dx^i\right) \wedge dt \wedge *\varphi \\ &= \int_{M \times S^1} \operatorname{tr} F_{\mathbf{A}} \wedge F_{\mathbf{A}} \wedge *\varphi \\ &= \left\langle c_2(\mathbf{E}_g) \smile [*\varphi], M \times S^1 \right\rangle . \end{aligned}$$

The Künneth formula for the cohomology of $M \times S^1$ gives

$$H^4\left(M \times S^1\right) = H^4(M) \oplus H^3(M) \otimes \underbrace{H^1\left(S^1\right)}_{\mathbb{Z}}$$

and obviously $H^4(M) \smile [*\varphi] = 0$ so, denoting $c_2'(\mathbf{E}_g)$ the component lying in $H^3(M)$ and

$S_g = [c_2' (\mathbf{E}_g)]^{PD}$ its Poincaré dual, we are left with

$$\vartheta (g.A) - \vartheta (A) = \langle [*\varphi] , S_g \rangle .$$

Consequently, the periods of ϑ lie in the set

$$\left\{ \int_{S_g} *\varphi \mid S_g \in H_4 (M, \mathbb{R}) \right\}.$$

That may seem odd because this set is *dense* (as $*\varphi$ is not, in general, an integral class) but, as long as our interest remains in the study of the moduli space $\mathcal{M}$ of G_2-instantons, as the critical set of the Chern-Simons $1-$form ρ, there is not much to worry, for the gradient $\rho = d\vartheta$ is unambiguously defined on $\mathcal{B}$.

1.4 Yang-Mills theory on G_2-manifolds

The G_2-structure allows for a $7-$dimensional analogue of conventional Yang-Mills theory. The crucial fact is that one can use φ_0 to establish a notion of (anti-)self-duality for $2-$forms, by a convenient split of $\Lambda^2 = \Lambda^2 \left(\mathbb{R}^7\right)^*$ in terms of irreducible representations of G_2.

1.4.1 Self-duality in dimension 7

Since $G_2 \subset SO(7)$, we have $\mathfrak{g}_2 \subset \mathfrak{so}(7) \simeq \Lambda^2$, under the standard identification of $2-$forms with antisymmetric matrices. Denote $\Lambda^2_- \doteq \mathfrak{g}_2$ and Λ^2_+ its orthogonal complement in Λ^2:

$$\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_- . \tag{1.17}$$

Then $\dim \Lambda^2_+ = \dim \Lambda^2 - \dim \Lambda^2_- = 7$, and we identify $\Lambda^2_+ \simeq \mathbb{R}^7$ as the linear span of the contractions $\alpha_i \doteq v_i \lrcorner \varphi_0$. Indeed the (pull-back) G_2-action on Λ^2_+ corresponds to the (standard) action on $\mathbb{R}^7$:

$$\begin{aligned}(g.\alpha_i)(u_1, u_2) &= \alpha_i (g.u_1, g.u_2) = \varphi_0 (v_i, g.u_1, g.u_2) \\ &= \varphi_0 \left(g^{-1}.v_i, u_1, u_2\right) = \left(\left(g^{-1}.v_i\right) \lrcorner \varphi_0\right)(u_1, u_2).\end{aligned}$$

It is straightforward to verify that $\alpha_i \in (\mathfrak{g}_2)^{\perp} \subset \mathfrak{so}(7)$; e.g., letting $g_1 = \mathbf{e}^{\alpha_1} \in SO(7)$ one has $(g_1^* \varphi_0)(v_2, v_5, \cdot) = -e^1 \neq e^1 = \varphi_0 (v_2, v_{5,}.)$, so $g_1 \notin G_2$. Moreover [Bry, p. 541]:

Claim 1.12 *The space $\Lambda^2_\pm$ has the following properties:*

1. *$\Lambda^2_\pm$ is an irreducible representation of G_2;*
2. *$\Lambda^2_\pm$ is the $\substack{+2\\-1}$−eigenspace of the G_2−equivariant linear map*

$$\begin{aligned} T \,:\, \Lambda^2 &\to \Lambda^2 \\ \eta &\mapsto T\eta = *\left(\eta \wedge \varphi_0\right). \end{aligned}$$

By analogy with the 4−dimensional case, we will call Λ^2_+ (resp. Λ^2_-) the space of *self-dual* or *SD* (resp. *anti-self-dual* or *ASD*) forms. Still in the light of *Claim 1.12*, let us establish a convenient characterisation of the 'positive' projection in (1.17). The dual 4−form to the G_2−structure (1.1) in our convention is

$$*\,\varphi_0 = \left(e^{34} - e^{12}\right) \wedge e^{67} + \left(e^{42} - e^{13}\right) \wedge e^{75} + \left(e^{23} - e^{14}\right) \wedge e^{56} + e^{1234} \tag{1.18}$$

and we consider the G_2−equivariant linear map (between representations of G_2):

$$\begin{aligned} L_{*\varphi_0} \,:\, \Lambda^2 &\to \Lambda^6 \\ \eta &\mapsto \eta \wedge *\varphi_0. \end{aligned}$$

As $\Lambda^2_\pm$ and Λ^6 are irreducible representations and $\dim \Lambda^2_+ = \dim \Lambda^6$, Schur's Lemma gives:

Proposition 1.13 *The above map restricts as $L_{*\varphi_0}|_{\Lambda^2_+} : \Lambda^2_+ \xrightarrow{\sim} \Lambda^6$ and $L_{*\varphi_0}|_{\Lambda^2_-} = 0$.*

Proof *It only remains to check that the restriction $L_{*\varphi_0}|_{\Lambda^2_+}$ is nonzero. Using (1.1) and (1.18) we find, for instance,*

$$\begin{aligned} L_{*\varphi_0}\alpha_1 &= \left(v_1 \lrcorner \varphi_0\right) \wedge *\varphi_0 \\ &= \left(e^{25} + e^{36} + e^{47}\right) \wedge (\dots) \\ &= e^{234567}. \end{aligned}$$

Not only does this prove the statement, but it also suggests carrying out the full inspection of the $L_{*\varphi_0}\alpha_i$, which yields

$$\boxed{L_{*\varphi_0}\alpha_i = e^{1\ldots\hat{\imath}\ldots7}.}$$

as a somewhat aesthetical fact. ■

Hence we may think of the orthogonal projection of 2−forms into $\Lambda^2_+ \simeq \Lambda^6$ as the operation 'wedge product with $*\varphi_0$'.

1.4.2 Energy bound from Chern-Weil theory

Given a unitary bundle $E \to M$ over a compact G_2−manifold (M, φ) without boundary, the curvature $F_A \in \Omega^2(\mathfrak{g}_E)$ of some connection A conforms to the split (1.17):

$$F_A = F_A^+ \oplus F_A^-,$$

with $F_A^\pm \in \Omega^2_\pm(\mathfrak{g}_E)$. The L^2−norm of F_A gives the Yang-Mills functional, or 'energy',

$$YM(A) \doteq \|F_A\|^2 = \left\|F_A^+\right\|^2 + \left\|F_A^-\right\|^2. \tag{1.19}$$

The values of $YM(A)$ can be related to a certain characteristic class of the bundle E,

$$\kappa(E) = \int_M \operatorname{tr}\left(F_A^2\right) \wedge \varphi.$$

Using the property $d\varphi = 0$, a standard argument of Chern-Weil theory [M-S] shows that the de Rham cohomology class $\left[\operatorname{tr}\left(F_A^2\right) \wedge \varphi\right]$ is independent of A, thus the integral is indeed a topological invariant. Furthermore, by *Claim 1.12*,

$$\begin{aligned}\kappa(E) &= -\int_M \langle F_A \wedge F_A\rangle_{\mathfrak{g}} \wedge \varphi = -\int_M \langle F_A \wedge (F_A \wedge \varphi)\rangle_{\mathfrak{g}} \\ &= -\left(F_A, 2F_A^+ - F_A^-\right) \\ &= \left\|F_A^-\right\|^2 - 2\left\|F_A^+\right\|^2.\end{aligned} \tag{1.20}$$

Comparing with (1.19) we get:

$$YM(A) = 3\left\|F_A^+\right\|^2 + \kappa(E) = \frac{1}{2}\left(3\left\|F_A^-\right\|^2 - \kappa(E)\right)$$

It is then clear that $YM(A)$ attains its minimum at a connection whose curvature is either SD or ASD. Moreover, since $YM \geq 0$, the sign of $\kappa(E)$ obstructs the existence of one type or the other. Fixing $\kappa(E) \geq 0$, these facts motivate our interest in the G_2−*instanton equation*:

$$\boxed{F_A^+ = \left(L_{*\varphi}|_{\Omega^2_+}\right)^{-1}(F_A \wedge *\varphi) = 0.} \tag{1.21}$$

1.4.3 Relation with Hermitian Yang-Mills in complex dimension 3

Given a holomorphic bundle $\mathcal{E} \to W$ over a Kähler manifold (W, ω), to every Hermitian metric H on $\mathcal{E}$ there corresponds a unique unitary (Chern) connection $A = A_H$ compatible with the holomorphic structure [D-K, §2.1.5], and its curvature satisfies $F_A \in \Omega^{1,1}(\mathfrak{g})$. Such a metric is called *Hermitian Yang-Mills (HYM)* if its curvature has vanishing $\omega-$trace:

$$\hat{F}_A \doteq (F_A, \omega) = 0. \tag{1.22}$$

Let now W be a Calabi-Yau $3-$fold. We will see that the Cartesian product $M = W \times S^1$ is naturally a real $7-$dimensional G_2-manifold [Kov_1, eq. (2.1)], then verify that Hermitian Yang-Mills connections on some $\mathcal{E} \to W$ pull back to G_2-instantons over M.

Starting with the Kähler $2-$form $\omega \in \Omega^{1,1}(W)$ and holomorphic volume form Ω on W [G-H-J, p. 17], define

$$\begin{aligned} \varphi &= \omega \wedge d\theta + \operatorname{Im}\Omega, \\ *\varphi &= \tfrac{1}{2}\omega \wedge \omega - \operatorname{Re}\Omega \wedge d\theta. \end{aligned} \tag{1.23}$$

Here $d\theta$ is the coordinate $1-$ form on S^1, and the Hodge star on M is given by the product of the Kähler metric on W and the standard flat metric on S^1. Let us check explicitly that (1.23) is a G_2-structure; at any $p \in W$, we can choose coordinates $z^i = x^i + \mathbf{i}y^i$ so that

$$\begin{aligned} \omega_p &= \frac{\mathbf{i}}{2} \sum dz^i \wedge d\bar{z}^i \\ \Omega_p &= dz^1 \wedge dz^2 \wedge dz^3, \end{aligned}$$

hence

$$\begin{aligned} \omega_p \wedge d\theta &= \left(dx^1 \wedge dy^1 + dx^2 \wedge dy^2 + dx^3 \wedge dy^3\right) \wedge d\theta, \\ \operatorname{Im}\Omega_p &= dx^1 \wedge \left(dx^2 \wedge dy^3 + dy^2 \wedge dx^3\right) + dy^1 \wedge \left(dx^2 \wedge dx^3 + dy^3 \wedge dy^2\right). \end{aligned}$$

Setting

$$\begin{aligned} e^1 &= dx^2,\ e^2 = dx^3,\ e^3 = dy^2,\ e^4 = dy^3 \\ e^5 &= dy^1,\ e^6 = d\theta,\ e^7 = dx^1 \end{aligned}$$

one finds precisely the canonical form φ_0 in (1.1). The whole prescription varies smoothly with p so φ defines indeed a G_2-structure. Furthermore, inspection on tangent vectors using the inner product (1.6) shows that $g + d\theta \otimes d\theta \doteq \mathbf{g}(\varphi)$ is indeed the metric induced by φ on M. Since $d\omega = 0$ and $d\Omega = 0$, the pair (1.23) satisfies $d\varphi = 0$ and $d * \varphi = 0$ as in *Theorem 1.8*, so (M, φ) is a G_2-manifold. Alternatively, one may notice that the holonomy of $g(\varphi)$ preserves the '$d\theta$' direction, thus $\mathrm{Hol}\,(\mathbf{g}(\varphi)) = \mathrm{Hol}\,(g) \subset SU(3) \subset G_2$ (since W is Calabi-Yau) and we obtain the same conclusion, again from *Theorem 1.8*.

On the other hand, a connection A on $\mathcal{E} \to W$ pulls back to $p_1^*\mathcal{E} \to M$ via the canonical projection $p_1 : W \times S^1 \to W$, and similarly do the forms ω and Ω (for simplicity I keep the same notation for objects on W and their pull-backs to M). In particular, under the isomorphism $L_{*\varphi}|_{\Omega^2_+} : \Omega^2_+ \tilde{\to} \Omega^6$ [*Proposition 1.13*], the SD part of curvature maps to

$$L_{*\varphi}\left(F_A^+\right) = \; F_A \wedge *\varphi = \tfrac{1}{2} F_A \wedge (\omega \wedge \omega - 2\mathrm{Re}\,\Omega \wedge d\theta)\,. \tag{1.24}$$

Proposition 1.14 *A Hermitian Yang-Mills connection A on a holomorphic vector bundle $\mathcal{E} \to W$ over a Calabi-Yau $3-$fold W lifts to a G_2-instanton on the pull-back bundle $p_1^*\mathcal{E} \to M = W \times S^1$ induced by the canonical projection.*

Proof *Recall that A satisfies $F_A \in \Omega^{1,1}(W)$ and $\hat{F}_A = 0$. Since $\Omega \in \Omega^{3,0}(W)$, the former implies $F_A \wedge \Omega = F_A \wedge \overline{\Omega} = 0$, hence $F_A \wedge 2\mathrm{Re}\,\Omega = F_A \wedge \left(\Omega + \overline{\Omega}\right) = 0$. Replacing this in (1.24), we check that F_A^+ must be zero, as it maps isomorphically to the origin:*

$$\begin{aligned} F_A^+ &\cong \tfrac{1}{2} F_A \wedge \omega \wedge \omega \qquad \in \quad \Omega^{3,3}(W) \\ &= (cst.)\, \hat{F}_A . d\mathrm{Vol}\,(W) \\ &= 0 \end{aligned}$$

by the HYM condition $\hat{F}_A = 0$, using the fact that $\omega \wedge \omega = \frac{(cst.)}{\|\omega\|^2} * \omega$. ■

Thus, when M is of the form $CY^3 \times S^1$, we may obtain G_2-instantons by solving the HYM equation over CY^3. This is the essential motivation for the analysis in *Chapter 3*.

1.5 Local model for the moduli space of G_2−instantons

In this *Section* I obtain a local characterisation of the moduli space of solutions to equation (1.21), away from reducible connections, following [D-K, Chapter 4].

Let $E \to M$ be a vector bundle over a compact G_2−manifold M, with gauge group[6] $\mathcal{G}$ and Lie $(\mathcal{G}) = \Gamma(\mathfrak{g}_E)$. Denote $\mathcal{A}$ the set of connections on E and $\mathcal{B} = \mathcal{A}/\mathcal{G}$ the quotient space by the gauge action of $\mathcal{G}$. Foreseeing the main result in *Proposition 1.23*, we restrict attention to *irreducible* connections and assume, for simplicity, that E is an $SU(n)$ −bundle.

1.5.1 Topology of the orbit space $\mathcal{B}$

The first thing to do is to describe $\mathcal{B}$ locally, which means endowing $\mathcal{A}$ with slices transverse to the $\mathcal{G}$−orbits. The $\mathcal{G}$−action on $\mathcal{A}$ is given by

$$\begin{array}{rcl} \mathcal{G} \times \mathcal{A} & \to & \mathcal{A} \\ (g, A) & \mapsto & g.A = A - (d_A g)\, g^{-1}. \end{array}$$

In terms of generators: let $f \in \Omega^0(\mathfrak{g}_E)$ such that $g = \exp f$; then $g.A = A - d_A f$, so the derivative of the action with respect to $\mathcal{G}$ at a point $A \in \mathcal{A}$ is the map

$$-\,d_A : \Omega^0(\mathfrak{g}_E) \to \Omega^1(\mathfrak{g}_E)\,. \tag{1.25}$$

A natural transverse slice to the $\mathcal{G}$−orbit at A is the L^2−orthogonal complement of $\operatorname{img} d_A$ in $\Omega^1(\mathfrak{g}_E) = \operatorname{img} d_A \oplus \ker d_A^*$ [D-K, p.131], so we model neighbourhoods in $\mathcal{A}$ by

$$T_\varepsilon(A) = \left\{ a \in \Omega^1(\mathfrak{g}_E) \mid d_A^* a = 0, \|a\| < \varepsilon \right\}. \tag{1.26}$$

Remark 1.15 *Here $\|.\|$ is the L^2_k−norm induced by (φ, g) for some $k \geq 4$:*

$$\|a\| \doteq \|a\|_{L^2_k} = \left(\int_M \sum_{l=0}^{k} \left| \nabla^{i_1} ... \nabla^{i_l} a \right|^2 d\mu \right)^{\frac{1}{2}}.$$

*This choice has in view the use of Sobolev's embedding (*Lemma A.89*) in* Subsection 1.5.4.

[6] $\mathcal{G}$ is a *Hilbert Lie group*, i.e., an infinite dimensional Lie group modelled on a Hilbert space [Fr-U, p.53]. The bundle of Lie algebras $\mathfrak{g}_E \subset \operatorname{End} E$ is given pointwise by the adjoint representation of the structure group [D-K, p. 33].

In order to obtain a local structure for the orbit space $\mathcal{B}$ from the transverse slices $T_\varepsilon(A)/\mathcal{G}$, we need to consider the role of stabilisers. The isotropy group

$$\Gamma_A = \{g \in \mathcal{G} \mid g.A = A\}$$

of a connection A under gauge action is a Lie group, as a closed subgroup of $G = SU(n)$, since it is pointwise isomorphic to the centraliser in G of the holonomy group of A [D-K, Lemma 4.2.8]. It is clear from (1.25) that $\mathrm{Lie}(\Gamma_A) = \ker d_A \subset \Omega^0(\mathfrak{g}_E)$; in particular, A is irreducible if and only if Γ_A is discrete, hence finite since $SU(n)$ is compact. The local model for $\mathcal{B}$ is then given by [D-K, Prop. 4.2.9, p.132]:

Proposition 1.16 *For small ε the projection from $\mathcal{A}$ to $\mathcal{B}$ induces a homeomorphism h from the quotient $T_\varepsilon(A)/\Gamma_A$ to a neighbourhood of $[A]$ in $\mathcal{B}$. For $a \in T_\varepsilon(A)$, the isotropy group of a in Γ_A is isomorphic to that of $h(a)$ in $\mathcal{G}$.*

We may now apply the above principles to our case of interest:

Definition 1.17 *The* moduli space of (irreducible) G_2−instantons *on $E \to M$ is the set of gauge-classes of connections on E satisfying the G_2−instanton equation* (1.21)*:*

$$\mathcal{M}_E \doteq \left\{[A] \in \mathcal{B} \mid F_A^+ \doteq p_+(F_A) = 0\right\}.$$

NB.: Recall from Proposition 1.13 *that $F_A^+ = 0 \Leftrightarrow L_{*\varphi}(F_A) \doteq F_A \wedge *\varphi = 0$.*

For the local description of $\mathcal{M}_E$, define around a solution A of (1.21) the map

$$\begin{array}{rcl} \psi \,:\, T_\varepsilon(A) \subset \mathcal{A} & \to & \Omega^2_+(\mathfrak{g}_E) \\ a & \mapsto & \psi(a) \doteq p_+(F_{A+a}) = p_+(d_A a + a \wedge a) \end{array} \tag{1.27}$$

and write $Z(\psi) \subset T_\varepsilon(A)$ for its zero set. As in *Proposition 1.16*, h induces a homeomorphism from $Z(\psi)/\Gamma_A$ to a neighbourhood of $[A]$ in $\mathcal{M}_E$, which we proceed to examine.

1.5.2 Fredholm theory

We have just seen that the moduli space $\mathcal{M}_E$ of G_2−instantons on $E \to M^7$ is locally described as the zero set of a map ψ [cf. (1.27)] between the Banach spaces $T_\varepsilon(A)/\Gamma_A \subset \mathcal{A}$ and $\Omega^2_+(\mathfrak{g}_E)$. This suggests applying Fredholm theory [D-K, 4.2.5] to study the local model:

Proposition 1.18 *A Fredholm map Ξ from a neighbourhood of* 0 *is locally right-equivalent*[7] *to a map of the form*

$$\begin{array}{rcl} \widetilde{\Xi}: \quad U\times F & \to & V\times G \\ \widetilde{\Xi}\left(\xi,\eta\right) & = & \left(L\left(\xi\right),\sigma\left(\xi,\eta\right)\right) \end{array}$$

where $L=\left(D\Xi\right)_0 : U \tilde{\to} V$ *is a linear isomorphism,* $F=\ker L$ *and* $G=\operatorname{coker} L$ *are finite-dimensional and* $\left(D\sigma\right)_0=0$*.*

Corollary 1.19 *A neighbourhood of* 0 *in* $Z\left(\Xi\right)$ *is diffeomorphic to* $Z\left(\chi\right)$*, where*

$$\begin{array}{rcl} \chi\ :\ F & \to & G \\ \chi\left(\eta\right) & \doteq & \sigma\left(0,\eta\right). \end{array}$$

So, if we show that our map ψ is Fredholm on $Z\left(\psi\right)$, *Corollary 1.19* will provide a local model for a neighbourhood of $[A]$ in $\mathcal{M}_E$ on the finite-dimensional set $\chi^{-1}\left(0\right)/\Gamma_A$. Now, ψ is just the self-dual part of the curvature, so $\psi\left(a\right)-\psi\left(0\right)=\left(p_+\circ d_A\right)a+O\left(\left|a\right|^2\right)$ and

$$\left(D\psi\right)_0=p_+\circ d_A.$$

Moreover, by the 'slicing' condition (1.26) across orbits, we consider in fact the restriction

$$\begin{array}{cccc} p_+\circ d_A & : & \ker d_A^* & \longrightarrow \quad \Omega_+^2\left(\mathfrak{g}_E\right). \\ & & \cap & \\ & & \Omega^1\left(\mathfrak{g}_E\right) & \end{array} \tag{1.28}$$

In the analogous 4−dimensional case, one shows that (1.28) is Fredholm via the map

$$\mathbb{D}_A=d_A^+\oplus d_A^*:\Omega^1\left(\mathfrak{g}_E\right)\to\Omega_+^2\left(\mathfrak{g}_E\right)\oplus\Omega^0\left(\mathfrak{g}_E\right), \tag{1.29}$$

(we denote, *for the rest of this* Subsection *only,* $d_A^+\doteq p_+\circ d_A$) associated to the complex

$$\Omega^0\left(\mathfrak{g}_E\right)\overset{d_A}{\to}\Omega^1\left(\mathfrak{g}_E\right)\overset{d_A^+}{\to}\Omega_+^2\left(\mathfrak{g}_E\right). \tag{1.30}$$

[7] i.e., there exists a local diffeomorphism g, say, such that $\widetilde{\Xi}=\Xi\circ g$ [D-K, p.136].

Whenever A is anti-self-dual, (1.30) is an elliptic complex with cohomology[8]

$$H_A^0 = \ker d_A = \mathrm{Lie}\,(\Gamma_A), \quad H_A^1 = \frac{\ker d_A^+}{\mathrm{img}\, d_A} \simeq \ker \mathbb{D}_A, \quad H_A^2 = \mathrm{coker}\, d_A^+. \tag{1.31}$$

Consequently $\mathbb{D}_A$ is elliptic, hence Fredholm, so (1.28) is Fredholm: a neighbourhood of $[A]$ in $\mathcal{M}_E$ is modelled diffeomorphically on $Z(\chi)/\Gamma_A$, where

$$\chi : H_A^1 \to H_A^2$$

from *Corollary 1.19* is Γ_A−equivariant. Notice that H_A^0 does not appear in the image of χ, again by the 'slicing' restriction to $\ker d_A^*$ [cf. (1.28)].

Remark 1.20 *In dimension* 4*, taking the self-dual part of curvature gives a map*

$$F^+ : \mathcal{A} \to \Omega_+^2\,(\mathfrak{g}_E)$$

equivariant under the gauge group $\mathcal{G}$*, which acts linearly (by pull-back) on* $\Omega_+^2\,(\mathfrak{g}_E)$*. Therefore the map* ψ *translates into a section* Ψ *of the bundle of Banach spaces*

$$\mathcal{E} = \mathcal{A} \times_{\mathcal{G}} \Omega_+^2\,(\mathfrak{g}_E) \to \mathcal{B}$$

and $Z\,(\Psi)$ *represents the moduli space* $\mathcal{M}_E$ *of (*$\mathcal{G}$*-classes of irreducible) ASD connections.*

1.5.3 The extended elliptic complex

Back to dimension 7, an operador $\mathbb{D}_A$ as in (1.29) would certainly not be elliptic [e.g. compare bundle ranks $(1 \to 7 \to 7)$ in (1.30)], so our analysis requires a further subterfuge. Since the map $L_{*\varphi} =$ '$\,*\,\varphi \wedge .$' now plays the role of 'SD projection' [cf. *Proposition 1.13*], we denote henceforth

$$d_A^+ = L_{*\varphi} \circ d_A : \Omega^1\,(\mathfrak{g}_E) \to \Omega^6\,(\mathfrak{g}_E) \tag{1.32}$$

and consider instead the extended deformation complex

$$\Omega^0\,(\mathfrak{g}_E) \underset{d_A^*}{\overset{d_A}{\rightleftarrows}} \Omega^1\,(\mathfrak{g}_E) \overbrace{\xrightarrow{d_A} \Omega^2\,(\mathfrak{g}_E) \xrightarrow{*\varphi\wedge\cdot}}^{d_A^+} \Omega^6\,(\mathfrak{g}_E) \underset{d_A^*}{\overset{d_A}{\rightleftarrows}} \Omega^7\,(\mathfrak{g}_E)\,. \tag{1.33}$$

[8] One chooses $\mathrm{coker}\, d_A^+ \simeq \ker d_A \cap \Omega_+^2\,(\mathfrak{g}_E)$ as a Γ_A−equivariant complement.

Using $d * \varphi = 0$ [*Theorem 1.8*],

$$[L_{*\varphi}, d_A] = 0, \tag{1.34}$$

so, when A is anti-self-dual, (1.33) is indeed a complex and the identification of the self-dual $2-$forms with the $6-$forms is consistent with the relevant differential operators[9]. Moreover:

Lemma 1.21 *The operator d_A^+ defined by (1.32) has formal adjoint*

$$\left(d_A^+\right)^* = *d_A^+* : \Omega^6(\mathfrak{g}_E) \to \Omega^1(\mathfrak{g}_E).$$

Proof *For $a \in \Omega^1(\mathfrak{g}_E)$ and $\eta \in \Omega^6(\mathfrak{g}_E)$,*

$$\begin{aligned}
\left\langle d_A^+ a, \eta\right\rangle (*1) &= (*\varphi \wedge d_A a) \wedge *\eta = (d_A a) \wedge *(*(*\varphi \wedge *\eta)) \\
&= \langle d_A a, *(L_{*\varphi} * \eta)\rangle = \langle a, d_A^*(*L_{*\varphi} * \eta)\rangle (*1) \\
&= \langle a, *(d_A L_{*\varphi}) * \eta\rangle (*1) \overset{(1.34)}{=} \langle a, *(L_{*\varphi} d_A) * \eta\rangle (*1) \\
&= \left\langle a, \left(*d_A^+*\right)\eta\right\rangle (*1)
\end{aligned}$$

■

This is all one needs in order to establish ellipticity of the extended complex for G_2-instantons:

Proposition 1.22 *When A is anti-self-dual, the complex (1.33) is elliptic.*

Proof *First of all, since $\left(d_A^+\right)^* = *d_A^+*$ [Lemma 1.21] and $d_A^* = *d_A*$, notice that our complex is self-dual with respect to the Hodge star:*

$$\begin{array}{ccccccc}
\Omega^0 & \xrightarrow{d_A} & \Omega^1 & \xrightarrow{d_A^+} & \Omega^6 & \xrightarrow{d_A} & \Omega^7 \\
\shortparallel & & \shortparallel & & \shortparallel & & \shortparallel \\
*\Omega^7 & \xleftarrow[d_A^*]{} & *\Omega^6 & \xleftarrow[\left(d_A^+\right)^*]{} & *\Omega^1 & \xleftarrow[d_A^*]{} & *\Omega^0
\end{array}.$$

By Corollary A.86*, it suffices to show ellipticity at $\Omega^1(\mathfrak{g}_E)$, as that is equivalent to the ellipticity of the dual $*\Omega^7 \xleftarrow{d_A^*} *\Omega^6 \xleftarrow{\left(d_A^+\right)^*} *\Omega^1$, which is just $\Omega^1 \xrightarrow{d_A^+} \Omega^6 \xrightarrow{d_A} \Omega^7$. Fixing a section ξ of $T^{'}M$ (the cotangent bundle minus its zero section), we have symbol maps*

$$0 \to \pi^*\left(\Omega^0(\mathfrak{g}_E)\right)_\xi \xrightarrow{\xi.(\cdot)} \pi^*\left(\Omega^1(\mathfrak{g}_E)\right)_\xi \xrightarrow{*\varphi\wedge\xi\wedge(\cdot)} \pi^*\left(\Omega^6(\mathfrak{g}_E)\right)_\xi \longrightarrow \dots$$

[9] For more on elliptic complexes under the condition $d * \varphi = 0$, see [Fe-U].

*For $\alpha \in \Omega^1(\mathfrak{g}_E)$ such that $*\varphi \wedge \xi \wedge \alpha = 0$, exactness means α has to lie in $\xi.\Omega^0(\mathfrak{g}_E)$. Since G_2 acts transitively on S^6 (*Theorem 1.4*), take $g \in G_2$ such that $g^*\xi = \|\xi\| .e^1$ and denote $\widetilde{\alpha} = g^*\alpha$, so that*

$$*\varphi \wedge e^1 \wedge \widetilde{\alpha} = 0.$$

That is just the statement that $e^1 \wedge \widetilde{\alpha}$ is anti-self-dual, but this cannot occur unless $e^1 \wedge \widetilde{\alpha} = 0$, as

$$\left(e^1 \wedge \widetilde{\alpha}\right) \wedge \varphi = \widetilde{\alpha} \wedge \left(e^{1567} - e^{1345} - e^{1426} + e^{1237}\right)$$

has non-vanishing components involving e^1 and $*\left(e^1 \wedge \widetilde{\alpha}\right)$ obviously has not. Therefore $\widetilde{\alpha} = f.e^1$ for some $f \in \Omega^0(\mathfrak{g}_E)$, and

$$\alpha = (g^*)^{-1}\left(f.e^1\right) = \frac{f}{\|\xi\|}.\xi \in \xi.\Omega^0(\mathfrak{g}_E).$$

■

1.5.4 Reduction to a subbundle of kernels

In the light of *Remark 1.20* and the isomorphism $L_{*\varphi}|_{\Omega^2_+} : \Omega^2_+ \tilde{\to} \Omega^6$, the moduli space of G_2–instantons [*Definition 1.17*] is the zero set of the section $\Psi([A]) = F_A \wedge *\varphi$ of

$$\mathcal{E} = \mathcal{A} \times_{\mathcal{G}} \Omega^6(\mathfrak{g}_E) \to \mathcal{B}.$$

As an immediate consequence of (1.34) and the Bianchi identity, we have

$$\Psi([A]) \in \ker d_A \subset \Omega^6(\mathfrak{g}_E),$$

so, intuitively, the image of Ψ lies in the 'subbundle $\widetilde{\mathcal{E}}$ of kernels of d_A' inside $\mathcal{E}$:

$$\begin{array}{ccccccccc} \mathcal{E} & \supset & \widetilde{\mathcal{E}} & = & (\mathcal{A} & \times_{\mathcal{G}} & \ker d_A) & \to & \mathcal{B}. \\ & & & & & & \cap & & \\ & & & & & & \Omega^6(\mathfrak{g}_E) & & \end{array} \tag{1.35}$$

Let us make this idea rigorous in the case where E is an $SU(n)$–bundle, using the L^2–orthogonal projections $p_a : \ker d_A \to \ker d_{A_0}$ to trivialise the fibres into $\ker d_{A_0}$ over each neighbourhood $T_\varepsilon([A_0]) \subset \mathcal{B}$, where ε is a small global constant:

Proposition 1.23 *Let $E \to M$ be an $SU(n)-$bundle over a compact G_2-manifold (M, φ) and A_0 an irreducible connection; then there exists $\varepsilon > 0$ such that the orthogonal projection*

$$p_a : \ker d_A \to \ker d_{A_0}$$

in $\Omega^6(\mathfrak{g}_E)$ is an isomorphism for all $A = A_0 + a$, $a \in T_\varepsilon(A_0)$.

Proof *The proof consists in showing that the linear map p_a is both surjective and injective. We consider, throughout, the covariant derivatives and respective formal adjoints [cf. (1.33)]*

$$\Omega^6(\mathfrak{g}_E) \underset{d^*_{A_0}, d^*_A}{\overset{d_{A_0}, d_A}{\rightleftarrows}} \Omega^7(\mathfrak{g}_E).$$

Moreover, writing $\rho = d^*_{A_0} f$ for some $f \in \Omega^7(\mathfrak{g}_E)$, we denote an element of $\Omega^6(\mathfrak{g}_E)$ by

$$\eta = (\eta_0 \oplus \rho) \in \left(\ker d_{A_0} \oplus \operatorname{img} d^*_{A_0}\right) = \Omega^6(\mathfrak{g}_E).$$

Surjectivity *Given $\eta_0 \in \ker d_{A_0}$, write $g_0 \doteq -a \wedge \eta_0$; surjectivity of p_a means finding $\rho \in \operatorname{img} d^*_{A_0} \subset \Omega^6(\mathfrak{g}_E)$ such that $\eta = \eta_0 \oplus \rho \in \ker d_A$, i.e., solving for ρ the equation*

$$d_A \rho = g_0. \tag{1.36}$$

Notice first that the orthogonal complement of $\operatorname{img} d_{A_0}$ in $\Omega^7(\mathfrak{g}_E)$ is the origin, since $(\operatorname{img} d_{A_0})^\perp = \ker d^*_{A_0} = *\left(\ker d_{A_0} \subset \Omega^0(\mathfrak{g}_E)\right) = \{0\}$ (the vanishing is just Theorem A.87*, as A_0 is irreducible), so*

$$\operatorname{img} d_{A_0} = \Omega^7(\mathfrak{g}_E).$$

This means that we can think of the restriction of d_A to $\operatorname{img} d^*_{A_0}$ as a map

$$d_A : \operatorname{img} d^*_{A_0} \to \operatorname{img} d_{A_0}. \tag{1.37}$$

*Bijectivity of linear maps between Banach spaces is an open condition [*Lemma A.88*], so one can show that (1.37) is invertible by checking that, for suitably small a, this map is arbitrarily close to the isomorphism $d_{A_0} : \operatorname{img} d^*_{A_0} \tilde{\to} \operatorname{img} d_{A_0}$. Indeed,* *writing $L_a : \eta \mapsto a \wedge \eta$, we have the estimate*

$$\|d_A - d_{A_0}\| = \|L_a\| \leq \|a\| < \varepsilon.$$

Here we have used Lemma A.89*, since $a \in L^2_k$, $k \geq 4$, satisfies the hypothesis of Sobolev's embedding theorem [*Remark 1.15*]. So* (1.37) *is also an isomorphism for ε small enough, consequently we can find a unique $\rho \in \ker d^*_{A_0}$ solving* (1.36)*.*

Injectivity *Let $\eta \in \ker d_A \subset \Omega^6(\mathfrak{g}_E)$; then*

$$\begin{aligned} p_a(\eta) = 0 \quad &\Leftrightarrow \quad \rho = \eta \in \ker d_A \\ &\Leftrightarrow \quad d_A\rho = 0 \\ &\Leftrightarrow \quad \rho = 0 \end{aligned}$$

since $\rho \in \operatorname{img} d^*_{A_0}$ and we have just seen that $d_A : \operatorname{img} d^*_{A_0} \tilde{\to} \operatorname{img} d_{A_0}$ is an isomorphism (for suitably small a); so $\eta = \rho = 0$.

■

Now, the intrinsic derivative[10] of Ψ at $[A]$ is

$$\begin{aligned} (D\Psi)_{[A]} : \ker d^*_A &\to \ker d_A \subset \Omega^6(\mathfrak{g}_E) \\ a &\mapsto d^+_A a. \end{aligned}$$

To see that $(D\Psi)_{[A]}$ is Fredholm, consider the extended operator

$$\begin{aligned} \widetilde{\mathbb{D}_A} : \Omega^1(\mathfrak{g}_E) \oplus \Omega^7(\mathfrak{g}_E) &\to \Omega^0(\mathfrak{g}_E) \oplus \Omega^6(\mathfrak{g}_E) \\ (a, f) &\mapsto \left(d^*_A a,\, d^+_A a + d^*_A f\right). \end{aligned}$$

Claim 1.24 *The operator $\widetilde{\mathbb{D}_A}$ is elliptic when $[A] \in Z(\Psi)$.*

Proof *By* Proposition 1.22*, $\widetilde{\mathbb{D}_A} = d^*_A \oplus \left(d^+_A \oplus d^*_A\right)$ maps even to odd terms in the elliptic complex*

$$\Omega^0(\mathfrak{g}_E) \xrightarrow{d_A} \Omega^1(\mathfrak{g}_E) \xrightarrow{d^+_A} \Omega^6(\mathfrak{g}_E) \xrightarrow{d_A} \Omega^7(\mathfrak{g}_E). \tag{1.38}$$

■

[10] i.e. the component of the total derivative tangent to the gauge-fixing slices in $\Omega^1(\mathfrak{g}_E)$.

Consequently, $\widetilde{\mathbb{D}_A}$ is Fredholm, i.e., it has finite-dimensional kernel and cokernel, for all $[A] \in Z(\Psi)$. This shows that $(D\Psi)_{[A]}$ is also Fredholm:

$$\begin{array}{ccccccc} \ker (D\Psi)_{[A]} & \hookrightarrow & \ker \widetilde{\mathbb{D}_A} & & & & \\ \operatorname{coker} (D\Psi)_{[A]} & = & \operatorname{coker} d_A^+ & \cap & \operatorname{coker} \left(d_A^* |_{\Omega^7(\mathfrak{g}_E)} \right) & \hookrightarrow & \operatorname{coker} \widetilde{\mathbb{D}_A} \\ & & \cap & & \| & & \\ & & \ker d_A & = & \ker d_A & & \end{array}$$

using that $\Omega^6(\mathfrak{g}_E) = \ker d_A \oplus \operatorname{img} d_A^*$, by Hodge theory. Hence, by *Corollary 1.19*:

Proposition 1.25 *If $A \in \mathcal{A}$ is ASD, then an ε−neighbourhood of $[A] \in \mathcal{M}_E$ is modelled on $Z(\chi)/\Gamma_A$, where χ is an invertible map between finite-dimensional spaces defined by*

$$\chi : \underbrace{\ker \left(d_A^* \oplus d_A^+ \right)}_{\substack{\cap \\ \Omega^1(\mathfrak{g}_E)}} \longrightarrow \underbrace{\operatorname{coker} d_A^+ \cap \ker d_A}_{\substack{\cap \\ \Omega^6(\mathfrak{g}_E)}}$$

$$\chi(a) = \sigma(0, a)$$

and σ is the non-linear part of the local Fredholm decomposition of ψ.

1.6 An exercise on 7−tori

A 7−torus $T^7 = \mathbb{R}^7/\Lambda$ naturally inherits the G_2−structure φ from $\mathbb{R}^7$, setting the canvas for some less-than-trivial but still illustrative applications of the material in this *Chapter*. Recall from *Subsection 1.3.2* that a connection A on some bundle over T^7 is a G_2−instanton if and only if it is a zero of the 1−form (1.16):

$$\rho(b)_A = \int_{T^7} \operatorname{tr} (F_A \wedge b_A) \wedge *\varphi. \tag{1.39}$$

The exercise consists in studying, via Chern-Simons formalism, the behaviour of the moduli space of G_2−instantons under perturbations $\varphi \to \varphi + \phi$ of the G_2−structure. More precisely, given suitable assumptions, one asks whether $(\varphi + \phi)$ −instantons (if any at all) are 'nowhere near' the initial moduli space, once we deform the lattice. As a working example, the following class of T^3−fibred 7−tori has all the properties we need:

Definition 1.26 *A G_2−*fibred torus *is a triplet (η, L, α) in which:*

- *η is a metric on the vector space $\mathbb{R}^4$;*
- *L is a lattice in the subspace $\Lambda^2_+(\mathbb{R}^4, \eta)$ of 2−forms self-dual with respect to η;*
- *$\alpha : \mathbb{R}^4 \to \Lambda^2_+(\mathbb{R}^4, \eta)$ is a linear map.*

Given the above data, set $V \doteq \mathbb{R}^4 \oplus \Lambda^2_+$ and form the torus $\mathbb{T} = V/\tilde{L}$, with the lattice

$$\tilde{L} \doteq \left\{ (\mu, \nu + \alpha\mu) \mid \mu \in \mathbb{Z}^4,\ \nu \in L \right\} \subset V.$$

Then $\mathbb{T}$ inherits from V the G_2−structure φ_0 which makes $\tilde{L}$ orthonormal. Although $\mathbb{T}$ fibres over $T^4 \doteq \mathbb{R}^4/\mathbb{Z}^4$, the associated metric $g(\varphi_0)$ is *not*, in general, a Riemannian product.

1.6.1 Lifting instantons from T^4

Let $\mathbb{T}$ be a G_2−fibred torus and $\tilde{E} \to \mathbb{T}$ the pull-back from some bundle $E \to T^4$. We obtain trivial solutions to the instanton equation simply by lifting the moduli space $\mathcal{M}^+ \subset \mathcal{B}^4$ of self-dual connections on E:

Proposition 1.27 *Let $\mathbb{T}$ be a G_2−fibred torus; if $[A] \in \mathcal{M}^+$ is a self-dual connection on $E \to T^4$, then its lift $\left[\tilde{A}\right]$ by the fibration map $f : \mathbb{T} \to T^4$ is a G_2−instanton on $\tilde{E}$.*

Proof *At every point $p \in \mathbb{T}$, we have covectors $e^5, e^6, e^7 \in T^*_p(\mathbb{T})$ to complete a coframe $\left\langle e^1, ..., e^4 \right\rangle \in T^*_{f(p)}\left(T^4\right)$ into a coframe in $T^*_p(\mathbb{T})$, so that the natural 4−form associated to the G_2−structure reads, as usual,*

$$*\varphi = \left(e^{34} - e^{12}\right) \wedge e^{67} + \left(e^{42} - e^{13}\right) \wedge e^{75} + \left(e^{23} - e^{14}\right) \wedge e^{56} + e^{1234}.$$

Now, the projection into the self-dual subspace $\Omega^2_+(\mathbb{T})$ is identified with the map

$$\begin{aligned} L_{*\varphi} : \Omega^2(\mathbb{T}) &\to \Omega^6(\mathbb{T}) \simeq \Omega^2_+(\mathbb{T}) \\ \eta &\mapsto \eta \wedge *\varphi \end{aligned}$$

and, if A is SD we have:

$$\left(F_{\tilde{A}}\right)_{12} = \left(F_{\tilde{A}}\right)_{34}, \quad \left(F_{\tilde{A}}\right)_{13} = \left(F_{\tilde{A}}\right)_{42}, \quad \left(F_{\tilde{A}}\right)_{14} = \left(F_{\tilde{A}}\right)_{23}, \quad \left(F_{\tilde{A}}\right)_{ij} = 0, \quad i, j = 5, 6, 7$$

which clearly implies $L_{*\varphi} F_{\tilde{A}} = 0$ ■

Remark 1.28 *About the previous* Proposition*:*

1. *There is nothing special about tori here; an analogous statement holds indeed for any Riemannian submersion with associative fibres over a (compact) 4−manifold.*
2. *The converse raises a natural problem: to describe, if any, the instantons on such $\tilde{E} \to \mathbb{T}$ which are* not *pull-backs from self-dual connections over T^4. However, such an investigation is beyond the scope of the present work.*

For future reference, I denote the set of such φ−instantons obtained by lifts from $\mathcal{M}^+$ by

$$\widetilde{\mathcal{M}^+} \doteq \left\{ \left[\tilde{A}\right] \in \mathcal{B}^7 \mid [A] \in \mathcal{M}^+ \subset \mathcal{B}^4 \right\}. \tag{1.40}$$

We know from 4−dimensional gauge theory that there are $SU(2)$−bundles $E \to T^4$, say, on which SD connections exist. In such cases $\mathcal{M}^+$ is not empty, and we have examples of G_2−instantons on pull-back bundles over the fibred torus.

1.6.2 Persistence of instantons under deformations of $\mathbb{T}$

Working, for simplicity, on an $SU(n)$−bundle $\tilde{E} \to \mathbb{T}$ over a fixed G_2−fibred torus, let us ponder in generality about the behaviour of instantons under a deformation of G_2−structure:

$$\varphi \to \varphi + \phi, \qquad *\varphi \to *\varphi + \xi_\phi, \qquad \xi_\phi \doteq *_\varphi \phi \in \Omega^4(\mathbb{T}).$$

Remark 1.29 *Notice that an arbitrary ϕ does not preserve the fibred structure of $\mathbb{T}$. Indeed, $\xi_\phi \in \Omega^4(\mathbb{T})$ has four orthogonal components, with the following significance:*

$$\Lambda^4(\mathbb{R}^4 \oplus \Lambda^2_+) = \underbrace{\Lambda^4(\mathbb{R}^4)}_{(I)} \oplus \underbrace{\Lambda^3(\mathbb{R}^4) \otimes \Lambda^1(\Lambda^2_+)}_{(II)} \oplus \underbrace{\Lambda^2(\mathbb{R}^4) \otimes \Lambda^2(\Lambda^2_+)}_{(III)} \oplus \underbrace{\Lambda^1(\mathbb{R}^4) \otimes \Lambda^3(\Lambda^2_+)}_{(IV)}$$

(I) *corresponds to a rescaling of the metric η on $\mathbb{R}^4$;*

(II) *redefines the map α;*

(III) *splits as $\left(\Lambda^2_+ \otimes \Lambda^2_+\right) \oplus \left(\Lambda^2_- \otimes \Lambda^2_+\right)$, where the first factor modifies the lattice L and the second one affects the conformal class of η;*

(IV) *parametrises deformations transverse to the fibred structures.*

We must examine what happens to the zeroes of (1.39) under the corresponding perturbation of the Chern-Simons 1−form:

$$\rho \to \rho_\phi = \rho + r_\phi, \qquad r_\phi(b)_A = \int_{T^7} \mathrm{tr}\,(F_A \wedge b_A) \wedge \xi_\phi.$$

A trivial but crucial observation here is that a φ−instanton A is also a $(\varphi+\phi)$−instanton if and only if $(r_\phi)_A \equiv 0$. In general, however, there is no reason to expect such a coincidence; for example, the topology of the bundle may constrain the persistence of instantons under certain deformations:

Proposition 1.30 *Let $\tilde{E} \to (\mathbb{T},\varphi)$ be the pull-back of an $SU(n)$−bundle $E \to T^4$. If $c_2(E) \neq 0$, then φ−instantons over T^4 do not lift to $(\varphi+\phi)$−instantons on $\tilde{E}$, for any perturbation ϕ away from a fibred structure [i.e. of type (IV) in* Remark 1.29*].*

In order to prove this fact, and for later reference, let us briefly digress into the translation action of some vector $v \in \mathbb{T}$ on a connection A. The first order variation is given by the bundle-valued 1−form

$$(b_v)_A = v \lrcorner F_A \tag{1.41}$$

which we interpret as a tangent vector in $T_A\mathcal{A}^7$. Evaluating r_ϕ on this vector gives

$$\begin{aligned} r_\phi(b_v)_A &= \int_{\mathbb{T}} \mathrm{tr}\,(F_A \wedge (b_v)_A) \wedge \xi_\phi \\ &= -\tfrac{1}{2}\int_{\mathbb{T}} \mathrm{tr}\,(F_A \wedge F_A) \wedge (v \lrcorner \xi_\phi) \\ &= \left\langle c_2\left(\tilde{E}\right), S_\phi(v)\right\rangle, \end{aligned}$$

where $S_\phi(v) \doteq -\frac{1}{2}[v \lrcorner \xi_\phi]^{PD}$, and this depends only on the topology of $\tilde{E}$, not on the base point A. Hence we may interpret ϕ as defining a linear functional

$$\begin{aligned} N_\phi : \mathbb{R}^7 &\to \mathbb{R} \\ v &\mapsto \left\langle c_2\left(\tilde{E}\right), S_\phi(v)\right\rangle \end{aligned}$$

such that $N_\phi \neq 0$ implies A is not a $(\varphi+\phi)$−instanton, $\forall A \in Z(\rho)$. In other words, N_ϕ is an obstruction to the existence of $(\varphi+\phi)$−instantons inherited from φ. This is however a rather weak negative criterion, since the map $\phi \mapsto N_\phi$ has kernel of dimension at least 28 and thus, in principle, leaves plenty of possibilities for instantons of perturbed G_2−structures.

Let us focus back on the case when $\tilde{E}$ is a pull-back via $f : \mathbb{T} \to T^4$. At each point $p \in \mathbb{T}$, ϕ doesn't contribute to the integral in (1.39) unless $(\xi_\phi)_p \in \Lambda^1(\mathbb{R}^4) \otimes \Lambda^3(\Lambda^2_+) \subset \Lambda^4(T_p\mathbb{T})$, which means precisely that the perturbed $7-$torus is no longer fibred [cf. *Remark 1.29*]. Thus, denoting T^3 the typical fibre of f (and setting Vol $\left(T^3\right) = 1$), we may assume

$$\xi_\phi = -2\varepsilon \wedge d\mathrm{Vol}\left(T^3\right)$$

for some $\varepsilon \in \Omega^1\left(T^4\right)$. Similarly, we only consider $\left(b_v\right)_A = v \lrcorner F_A$ for some $v \in T^4$. Then

$$\begin{aligned} r_\phi\left(b_v\right)_A &= -2\int_{\mathbb{T}} \mathrm{tr}\left(F_A \wedge v \lrcorner F_A\right) \wedge \varepsilon \wedge d\mathrm{Vol}\left(T^3\right) \\ &= -2\int_{T^4} \mathrm{tr}\left(F_A \wedge v \lrcorner F_A\right) \wedge \varepsilon \\ &= \varepsilon\left(v\right).c_2\left(E\right) \end{aligned}$$

again doesn't depend on the base-point A. Writing $\widetilde{\mathcal{A}^4} \subset \mathcal{A}^7$ for the set of connections on $\tilde{E} \to \mathbb{T}$ that are lifts from $\mathcal{A}^4$, we see right above that, if $c_2\left(E\right) \neq 0$, one can always choose $v \in T^4$ such that $r_\phi\left(b_v\right)_A \neq 0$, $\forall A \in \widetilde{\mathcal{A}^4}$. This proves *Proposition 1.30*.

1.6.3 Further perturbative investigation

So far we know from *Proposition 1.27* that the set $\mathcal{M}^+$ of self-dual connections (modulo gauge) over T^4 lifts to instantons [cf. (1.40)] of the original G_2-structure φ (i.e. zeroes of ρ). On the other hand, from *Proposition 1.30*, we know that there are in general no zeroes of the *perturbed* Chern-Simons $1-$form ρ_ϕ in that same set $\widetilde{\mathcal{M}^+}$. In such cases, it is a somewhat suggestive next step to ask whether we can find any zeroes of ρ_ϕ at least *in a neighbourhood* of a point in $\widetilde{\mathcal{M}^+} \subset \widetilde{\mathcal{A}^4} \cap Z\left(\rho\right)$. After establishing some more notation, I will show that the answer to that question is still negative in *Proposition 1.31*.

For a real constant $h > 0$ and a unit vector $a \in \Omega^1\left(\mathbb{T}, \mathfrak{g}_E\right)$, we have

$$F_{A+ha} = F_A + h.d_A a + h^2.a \wedge a \tag{1.42}$$

Since $\mathcal{A}^7$ is an affine space modelled on $\Omega^1\left(\mathbb{T}, \mathfrak{g}_E\right)$, for a general vector field b on $\mathcal{A}^7$ we have the first order expansion

$$b_{A+ha} = b_A + h.\left(Db\right)_A\left(a\right) + o\left(h^2\right) \tag{1.43}$$

where $(Db)_A(a)$ is defined by the above formula as the infinitesimal variation of b, in the direction a, at the point A. Replacing both variation formulas (1.42) and (1.43) in the Chern-Simons 1−form (1.39) we get

$$\rho(b)_{A+ha} = \rho(b)_A + h.\int_{\mathbb{T}} \operatorname{tr}\left[d_A a \wedge b_A + F_A \wedge (Db)_A(a)\right] \wedge \xi_\phi + o\left(h^2\right)$$

hence its infinitesimal variation is

$$D\left[\rho(b)\right]_A(a) = \int_{\mathbb{T}} \operatorname{tr}\left[d_A a \wedge b_A + F_A \wedge (Db)_A(a)\right] \wedge \xi_\phi \ .$$

Recall that the translation action of $v \in \mathbb{T} \curvearrowright \mathbb{T}$ induces a vector field $(b_v)_A = v \lrcorner F_A$ on $\mathcal{A}^7$. In particular, by (1.43), we have $(Db_v)_A(a) = v \lrcorner (d_A a)$; so, evaluating ρ on b_v gives a function on $\mathcal{A}^7$ with gradient

$$D\left[\rho(b_v)\right]_A(a) = \int_{\mathbb{T}} \operatorname{tr}\left[v \lrcorner (F_A \wedge d_A a)\right] \wedge *\varphi.$$

Proposition 1.31 *If $c_2(E) \neq 0$, then around every point of $\widetilde{\mathcal{M}^+}$ there is an open neighbourhood in $\mathcal{A}$ where ρ_ϕ is not zero (as a 1−form).*

Proof *Fix $A \in \widetilde{\mathcal{M}^+}$. Any connection in $\mathcal{A}^7$ can be written in the form $A + ha$ where $h > 0$ is a constant and $\|a\| = 1$. Taking a vector $v \in T^4$ as in* Proposition 1.30,

$$\rho_\phi(b_v)_{A+ha} \quad = \quad \underbrace{r_\phi(b_v)_A}_{>0} + h.\int \operatorname{tr}\left[v \lrcorner (F_A \wedge d_A a)\right] \wedge *_\varphi(\varphi + \phi) + O\left(h^2\right),$$

which proves the result for a small enough h ∎

Notice that h depends on A so the above *Proposition* only extends to a *neighbourhood of $\widetilde{\mathcal{M}^+}$* when this moduli space is compact. In other words, if $\mathcal{M}^+$ is compact, then $\widetilde{\mathcal{M}^+}$ is *disconnected* from the set $\left(\mathcal{M}^7\right)^+_\phi$ of $(\varphi + \phi)$ −instantons.

Remark 1.32 *A next natural question would be whether any $(\varphi + \phi)$ −instantons exist* at all, *or if, instead, $\rho_\phi(b_v)_{A+ha}$ can be shown to remain non-zero arbitrarily far from $\widetilde{\mathcal{M}^+}$. This could lead to a gauge-theoretic criterion as to whether a given 7−torus is a T^3−fibration in our sense.*

CHAPTER 2
KOVALEV MANIFOLDS

In this *Chapter* I will give an abbreviated account of the construction of compact Riemannian G_2-manifolds developed by A. Kovalev [Kov$_1$][Kov$_2$]. This is achieved by gluing together, in an ingenious way, a pair of non-compact asymptotically cylindrical $7-$manifolds of holonomy $SU(3)$ along their cylindrical ends. Such components are of the form $W \times S^1$, where (W, ω) is $3-$fold given by a non-compact version of the Calabi conjecture, so we can anticipate that they will carry G_2-structures as discussed in *Subsection 1.4.3*. The G_2-structures are superposed along the gluing region using cut-off functions to yield a global G_2-structure φ on the compact manifold M, which can be chosen to be torsion-free, by a 'stretch the neck' argument. In the process I will establish a few results of relevance to *Chapters 3* and *4* and, as a side exercise, a Lelong-Poincaré equation for the metric ω.

2.1 Asymptotically cylindrical Calabi-Yau manifolds

Let us describe the complete Calabi-Yau $3-$folds with cylindrical ends W_i that one intends to glue together. The next *Definition* summarises the basic ingredients.

Definition 2.33 *A* base manifold *for our purposes is a compact, simply-connected Kähler $3-$fold $(\bar{W}, \bar{\omega})$ satisfying the following conditions:*

- *there is a $K3-$surface*[1] $D \in |-K_{\bar{W}}|$ *with holomorphically trivial normal bundle* $\mathcal{N}_{D/\bar{W}}$*;*
- *The complement $W = \bar{W} \setminus D$ has finite fundamental group $\pi_1(W)$.*

One wants to think of W as a compact manifold W_0 with boundary $D \times S^1$ and a cylindrical end attached there:

$$\begin{aligned} W &= W_0 \cup W_\infty \\ W_\infty &\simeq \left(D \times S^1 \times \mathbb{R}_+\right). \end{aligned} \tag{2.1}$$

[1] i.e., a simply-connected compact complex surface with $c_1(D) = 0$.

2.1.1 Elementary properties

Let $s_0 \in H^0\left(\bar{W}, K_{\bar{W}}^{-1}\right)$ be the defining section of the *divisor at infinity* D; then, s_0 defines a holomorphic coordinate z on a neighbourhood $U \subset (W_\infty \cup D)$ of D. Since $\mathcal{N}_{D/\bar{W}}$ is trivial, we may assume U is a *tubular neighbourhood of infinity*, i.e.,

$$U \simeq D \times \{|z| < 1\} \tag{2.2}$$

as real manifolds. Denoting $s \in \mathbb{R}_+$ and $\alpha \in S^1$, we pass to the asymptotically cylindrical picture (2.1) via $z = e^{-s-\mathbf{i}\alpha}$. For later reference, let us establish a straightforward result on the decay of coordinates induced by z on differential forms:

Lemma 2.34 *With respect to the model cylindrical metric,*

$$|dz|, |d\bar{z}| = O\left(\left|e^{-s}\right|\right).$$

Proof *In holomorphic coordinates* $\left(z, \xi^1, \xi^2\right)$ *on* $D \subset U \subset \bar{W}$ *such that* $z = e^{-s-\mathbf{i}\alpha}$ *and* $D = \{z = 0\} \subset U$, *we have*

$$\begin{aligned} dz &= -z\,(ds - \mathbf{i}d\alpha) \\ d\bar{z} &= -\bar{z}\,(ds + \mathbf{i}d\alpha) \end{aligned}$$

and $|z| = |\bar{z}| = e^{-s}$. ■

Furthermore, the holomorphic coordinate z on U is the same as a local holomorphic function τ, say. The assumptions that $\bar{W}$ is simply connected, compact and Kähler imply the vanishing of Dolbeault cohomology [Huy, Cor. 3.2.12, p.129]

$$H^{0,1}\left(\bar{W}\right) \oplus \overline{H^{0,1}\left(\bar{W}\right)} = H^1(\bar{W}, \mathbb{C}) = 0,$$

in which case one can solve Mittag-Leffler's problem for $\frac{1}{\tau}$ on $\bar{W}$ [G-H, pp.34-35] and τ extends to a global fibration

$$\tau : \bar{W} \xrightarrow{D} \mathbb{C}P^1 \tag{2.3}$$

by $K3-$surfaces diffeomorphic (though not in general biholomorphic) to D. In fact, this holomorphic coordinate can be seen as pulled-back from $\mathbb{C}P^1$, i.e., $K_{\bar{W}}^{-1}$ is the pull-back of a degree-one line bundle $L \to \mathbb{C}P^1$ and $z = \tau^* s_0$ for some $s_0 \in H^0(L)$ [Kov_2, §3].

2.1.2 Kovalev's noncompact Calabi-Yau theorem

The $K3$ divisor D is a compact complex surface, with complex structure I inherited from $\bar{W}$; by Yau's theorem, it admits a unique Ricci-flat Kähler metric in the Kähler class $\bar{\omega}|_D$:

$$\kappa_I \doteq \bar{\omega}|_D + \frac{1}{2\pi} dd^c u_0 \tag{2.4}$$

where $u_0 \in C^\infty(D, \mathbb{R})$. Ricci-flat metrics on a surface are *hyper-Kähler* [G-H-J, §4.6], which means D admits additional complex structures J and $K = IJ$ satisfying the quaternionic relations, and the metric is also Kähler with respect to any combination $aI + bJ + cK$ with $(a, b, c) \in S^2$. Let us denote their Kähler forms by κ_J and κ_K. In those terms we may state Kovalev's noncompact version of the Calabi conjecture [Kov_1, Theorem 2.2]:

Theorem 2.35 *For $W = \bar{W} \setminus D$ as above, the following hold:*

1. *W admits a complete Ricci-flat Kähler structure ω;*
2. *along the cylindrical end $D \times S^1_\alpha \times (\mathbb{R}_+)_s$, the Kähler form ω and the corresponding holomorphic volume form Ω are exponentially asymptotic*[2] *to those of the product Ricci-flat Kähler metric on D:*

$$\begin{aligned} \omega_\infty &= \kappa_I + ds \wedge d\alpha \\ \Omega_\infty &= (ds + \mathbf{i} d\alpha) \wedge (\kappa_J + \mathbf{i}\kappa_K)\,; \end{aligned} \tag{2.5}$$

3. $\mathrm{Hol}(\omega) = SU(3)$*, i.e., W is Calabi-Yau.*

NB.: The details in the rest of this *Section* are not essential for *Chapter 3*.

More precisely, the Ricci-flat structure ω has the form[3]

$$\omega = \bar{\omega} + \mu\tau^*\omega_1 + dd^c f_0, \tag{2.6}$$

[2] This means they can be written along the tubular end as

$$\omega\,|_{W_\infty} = \omega_\infty + d\psi$$

$$\Omega\,|_{W_\infty} = \Omega_\infty + d\Psi$$

where the $1-$form ψ and the $2-$form Ψ are smooth and all derivatives are $O\left(e^{-\lambda t}\right)$ with respect to ω_∞, for any $\lambda < \min\left\{1, \sqrt{\lambda_1(D)}\right\}$, and $\lambda_1(D)$ is the first eigenvalue of the Laplacian on D with the metric κ_I.

[3] Set $d^c = \frac{\mathbf{i}}{2}\left(\bar{\partial} - \partial\right)$, so that $dd^c = \mathbf{i}\partial\bar{\partial}$.

where:

- $\bar{\omega}$ is the initial metric on $\bar{W}$;
- ω_1 is a Kähler form on $\mathbb{C}P^1$ (which can be assumed to be the Fubini-Study metric) and $\mu > 0$ is a constant such that $\mu\tau^*\omega_1$ is semi-positive on $\bar{W}$ and positive transversely to the fibres of τ;
- Given a certain bundle metric $|.|_a$ pulled-back from $\mathbb{C}P^1$ via τ [cf. (2.3)],

$$f_0 = \frac{1}{4}\left(\log|s_0|_a^2\right)^2 + \tilde{u}, \tag{2.7}$$

  where $\tilde{u} = \dfrac{u+u_0}{2\pi}$, for real functions $u \in C^\infty(W)$ and $u_0 \in C^\infty\left(\bar{W}\right)$ with the following properties:

  - $u|_{U\smallsetminus D} \underset{s\to\infty}{\longrightarrow} 0$ uniformly on $D \times S^1$ and its derivatives are bounded (w.r.t. ω_∞);
  - u_0 is an extension to U of a solution of Yau's theorem on D [cf. (2.4)], so that Supp $(u_0) \subset U$ and

$$\left(\bar{\omega} + \tfrac{1}{2\pi}dd^c u_0\right)|_D = \kappa_I.$$

In order to understand ω in further detail, we need to take a deeper look into $\bar{W}$.

2.1.3 An exercise on density currents

The interesting structure of divisors arising in the construction of (W,ω) and their interplay with the Kähler metric invite us into a brief exploration from the perspective of the theory of density currents. Although most of it will have no bearing on any other argument in the text, this little effort is rewarded by a deeper understanding of Kovalev's construction, as well as the somewhat aesthetically pleasing result (2.16).

We produce $\left(\bar{W},\bar{\omega}\right)$ starting from a compact, simply-connected $3-$fold V that is assumed *Fano*, i.e., $c_1(V) > 0$. A Kähler structure is then given by some representative $(1,1)-$form $\omega_V \in c_1(V)$. A generic divisor $D \in |-K_{\bar{W}}|$ is always a $K3-$surface [Sho] but the Fano condition means $D.D \doteq C \neq 0$. Our desired $3-$fold carrying an anti-canonical divisor with trivial normal bundle emerges as the blow-up along this self-intersection:

$$\bar{W} \doteq \mathrm{Bl}_C V. \tag{2.8}$$

Example 2.36 *Let $V = \mathbb{C}P^3$. In this case, $K_V^{-1} = \mathcal{O}(4)$ and we may take D to be a quartic surface given by $s_0 \in H^0(\mathcal{O}(4))$, hence it is a $K3-$surface [cf.* (2.10) *below]. The self-intersection class of D, represented by a curve $C = D.D$, is Poincaré dual to $c_1(\mathbb{C}P^3)\big|_D = 4\,[H]|_D \neq 0$, so we blow up along C to get a trivial normal bundle $\mathcal{N}_{D/\bar{W}}$:*

$$\sigma : \bar{W} \doteq \mathrm{Bl}_C\left(\mathbb{C}P^3\right) \to \mathbb{C}P^3 \tag{2.9}$$

For future use in Chapter 4, *let us observe that the proper transform restricts to a biholomorphism of complex surfaces $\sigma : \tilde{D} \to D$ so, by the adjunction formula,*

$$K_{\tilde{D}} \simeq \sigma^* K_D = \sigma^*\left(K_{\mathbb{P}^3}|_D \otimes \mathcal{N}_{D/\mathbb{P}^3}\right) = \mathcal{O}_D \tag{2.10}$$

using $\mathcal{N}_{D/\mathbb{P}^3} = K_{\mathbb{P}^3}^{-1}\big|_D$. Furthermore, by a generic choice of s_0 [G-H, p.594], we may assume

$$\mathrm{Pic}\,(D) \simeq \mathbb{Z}. \tag{2.11}$$

Now, it is possible to find k sufficiently large so that the positive $(1,1)-$form

$$\bar{\omega} = \sigma^*\omega_V - \frac{1}{k}\omega_E \tag{2.12}$$

defines a Kähler structure on $\bar{W}$. Here σ denotes the blow-up map,

$$\begin{aligned} E = \sigma^{-1}(C) &\simeq \mathbb{P}\left(N_{C/V}\right) = \mathbb{P}\left(\mathcal{O}_C(D) \oplus \mathcal{O}_C(D)\right) \\ &\simeq C \times \mathbb{C}P^1 \end{aligned}$$

is the exceptional divisor and ω_E is a closed, semi-positive $(1,1)-$form representing $c_1(E)$ [G-H, p.186-187]. On the other hand, the first Chern class of the blow-up along a curve of complex codimension 2 is $c_1\left(K_{\bar{W}}^{-1}\right) = \sigma^* c_1(V) - c_1(E)$ [G-H, pp. 608-609], so a representative is given by

$$\tilde{\omega} \doteq \bar{\omega} + \frac{1-k}{k}\omega_E \in c_1\left(K_{\bar{W}}^{-1}\right). \tag{2.13}$$

The punctured neighbourhood $U^* \simeq D \times \{0 < |z| < 1\}$ is sliced by fibres $D_z = \tau^{-1}(z)$ in the same divisor class $|-K_{\bar{W}}|$; let $s_z \in H^0\left(K_{\bar{W}}^{-1}\right)$ denote their defining sections. The

Lelong-Poincaré equation of currents [Dem, (4.4)] for each D_z reads [4]

$$\tilde{\omega} = T_{D_z} - dd^c \log |s_z|^2_{\tilde{\omega}} \,. \tag{2.14}$$

Similarly, writing $E = s_E^{-1}(0)$ for $s_E \in H^0(\mathcal{O}(E))$ and taking some Hermitian bundle metric $|.|_E$,

$$\omega_E = T_E - dd^c \log |s_E|^2_E \,. \tag{2.15}$$

Substituting (2.13), (2.14) and (2.15), equation (2.6) becomes

$$\boxed{\omega = T_{D_z} + \tfrac{k-1}{k} T_E + \mu \tau^* \omega_1 + dd^c (f_0 - f_z)} \tag{2.16}$$

where

$$f_z \doteq \frac{1}{\pi} \left(\log |s_z|^2_{\tilde{\omega}} + \frac{k-1}{k} \log |s_E|^2_E \right) \in L^1_{\text{loc}} \left(\bar{W} \right) \tag{2.17}$$

is a meromorphic function on $\bar{W}$ with poles along $D_z \cup E$.

[4] [Op. cit. §2] On an oriented smooth manifold X^n, denote $\Omega^p_0(X) \subset \Omega^p(X)$ the compactly supported differential $p-$forms. A smooth oriented submanifold Y^m defines a *density current* $T_Y \in \Omega^{n-m}_0(X)^*$ by:

$$T_Y(\eta) \doteq \int_Y \eta.$$

2.2 A generalised connected sum

Although the gluing itself will not be further discussed in this text, it is instructive to describe it briefly, in order to get some perspective on future developments [see *Remark 2.39* below]. Consider an $SU(3)$ −manifold W' satisfying the assertions of *Theorem 2.35*. For some $S \geq 1$, we prune (W', ω') at $s' = S$ on the cylindrical end and cut-off[5] the Kähler form ω' and the holomorphic volume form Ω' to their asymptotic model (2.5) smoothly along the interval $s' \in [S-1, S]$, thus obtaining

$$W'_S \simeq W'_0 \cup \left(D \times S^1 \times [0, S]\right). \tag{2.18}$$

2.2.1 The gluing condition

The 7−dimensional product $W'_S \times S^1$ has boundary $D' \times S^1 \times S^1$. Comparing the asymptotic model (2.5) with the standard form of the G_2−structure on a product $CY \times S^1$ (1.23) we find that $W'_S \times S^1$ carries a G_2−structure on a collar neighbourhood of the boundary that is asymptotic to:

$$\varphi'_S = \kappa'_I \wedge d\alpha + \kappa'_J \wedge d\theta + \kappa'_K \wedge ds + d\alpha \wedge d\theta \wedge ds. \tag{2.19}$$

Since the set of all G_2−structures $\mathcal{P}^3\left(W' \times S^1\right) \subset \Omega^3\left(W' \times S^1\right)$ is open [Joy, p. 243], φ'_T is itself a G_2−structure on $W' \times S^1$ for large enough S.

Condition 2.37 *Two manifolds W' and W'' as above will be suitable for the gluing procedure if there is a hyper-Kähler isometry*

$$f : D'_J \to D''$$

[5] Given a cut-off function $\sigma : \mathbb{R} \to [0,1]$, $\sigma(t) = \begin{cases} 0, & t \leq 0 \\ 1, & t \geq 1 \end{cases}$, one writes

$$\begin{aligned} \omega'_T(t') &= \omega_{W'} - d\left(\sigma\left(t'+1-T\right)\psi'\right) \\ \Omega'_T(t') &= \Omega_{W'} - d\left(\sigma\left(t'+1-T\right)\Psi'\right) \end{aligned}$$

where ψ and Ψ are the exponentially decaying forms of *Theorem 2.35*. Since G_2−structures form an open set, the 3−form

$$\varphi'_T(t) \doteq \omega'_T(t) \wedge d\alpha' + \operatorname{Im}\Omega'_T(t)$$

is also a G_2−structure on W' as ψ' and Ψ' tend to zero. Note that $d\varphi'_T(t) = 0$ by definition.

between D'' and the hyper-Kähler rotation of D' with complex structure[6] J*. In this case the (pull-back) action on Kähler forms is*

$$f^* : \kappa''_I \mapsto \kappa'_J, \qquad \kappa''_J \mapsto \kappa'_I, \qquad \kappa''_K \mapsto -\kappa'_K. \tag{2.20}$$

Assuming this holds, define a map between collar neighbourhoods of the boundaries by

$$\begin{aligned} F_S : D' \times S^1 \times S^1 \times [S-1, S] &\rightarrow D'' \times S^1 \times S^1 \times [S-1, S] \\ (y, \alpha, \theta, s) &\mapsto (f(y), \theta, \alpha, 2S - 1 - s). \end{aligned}$$

This identification gives a compact oriented $7-$manifold

$$M_S = \left(W'_S \times S^1\right) \cup_{F_S} \left(W''_S \times S^1\right) \doteq W' \widetilde{\#}_S W''.$$

The matching of Kähler forms (2.20) guarantees that the respective G_2-structures (2.19) on $W'_S \times S^1$ and $W''_S \times S^1$ agree along the gluing region $[S-1, S]$:

$$\begin{aligned} F_S^* \varphi''_S &= F_S^*(\underbrace{\kappa''_I}_{\kappa'_J} \wedge \underbrace{d\alpha''}_{d\theta'} + \underbrace{\kappa''_J}_{\kappa'_I} \wedge \underbrace{d\theta''}_{d\alpha'} + \underbrace{\kappa''_K}_{-\kappa'_K} \wedge \underbrace{ds''}_{-ds'} + \underbrace{d\alpha''}_{d\theta'} \wedge \underbrace{d\theta''}_{d\alpha'} \wedge \underbrace{ds''}_{-ds'}) \\ &= \kappa'_I \wedge d\alpha' + \kappa'_J \wedge d\theta' + \kappa'_K \wedge ds' + d\alpha' \wedge d\theta' \wedge ds' \\ &= \varphi'_S. \end{aligned}$$

so we obtain a globally well-defined G_2-structure φ_S on M_S. Thus, for large enough S, there is a $1-$parameter family (M_S, φ_S) of compact oriented manifolds M_S equipped with G_2-structures φ_S.

However, even though it is possible to arrange $d\varphi_S = 0$ for any S [Kov_2, eq. (4.23)], a pair (M_S, φ_S) is not in principle a G_2-manifold, as one has yet to satisfy the second torsion-freedom condition (see *Theorem 1.8*):

$$d *_{\varphi_S} \varphi_S = 0.$$

[6]There is an S^1-ambiguity in the choice of J; in fact one could take any combination $bJ + cK$, with $b^2 + c^2 = 1$.

2.2.2 'Stretching the neck' for $\mathrm{Hol}(\varphi) = G_2$

The cut-off functions involved in the asymptotic approximations leading to (2.19) add error terms to $d *_{\varphi_S} \varphi_S$, but these are controlled by the estimate [Kov$_2$, *Lemma 4.25*]

$$\|d *_{\varphi_S} \varphi_S\|_{L^p_k} \leq C_{p,k} e^{-\lambda S},$$

with $0 < \lambda < 1$. This exponential decay suggests that, by "stretching the neck" up to a large enough S_0, one can make the error so small as to be totally compensated by a suitably small perturbation of φ_S in $\mathcal{P}^3(M_S)$, $S > S_0$ [Kov$_1$, Theorem 2.3]:

Theorem 2.38 *There exists $S_0 \in \mathbb{R}$ and for every $S > S_0$ a unique $\eta_S \in \Lambda^2(M)$ such that*

1. $\|\eta_S\|_{C^1} < (const.)\, e^{-\mu S}$, *for some $0 < \mu < 1$. In particular, $\varphi_S + d\eta_S \in \mathcal{P}^3(M_S)$;*
2. *The (closed) $3-$form $\varphi_S + d\eta_S$ satisfies*

$$d *_{\varphi_S + d\eta_S} (\varphi_S + d\eta_S) = 0,$$

so it defines a metric of holonomy G_2 on M_S.

Hence one has achieved a $1-$parameter family of compact oriented G_2-manifolds:

$$(M_S, \varphi_S + d\eta_S), \qquad S > S_0.$$

Remark 2.39 *The task of obtaining G_2-instantons on some bundle*

$$\mathcal{E} \to (M, \varphi) = (M_S, \varphi_S + d\eta_S)$$

can thus be structured in three broad steps, as discussed in the Introduction*:*

1. *To show the existence of HYM metrics H' and H'' respectively on bundles $\mathcal{E}' \to W'_S$ and $\mathcal{E}'' \to W''_S$, which in turn pull back to G_2-instantons over $W'_S \times S^1$ and $W''_S \times S^1$ [*Proposition 1.14*].*
2. *To make rigorous sense, under additional assumptions if necessary, of the bundle*

$$\mathcal{E} = \mathcal{E}' \widetilde{\#} \mathcal{E}'' \to M$$

and to show that any error term appearing in the G_2-*instanton equation [cf.* (1.21)*] as a result of the gluing can also be dealt with*[7]*, by a 'stretch-the-neck'-type argument, say.*

3. *In possession of* G_2-*instantons, to study their moduli space, starting from the principles of deformation theory set up in* Section 1.5 *and aiming at the foreseeable computation of an invariant.*

[7] similar problems are treated in [Don_1] and in [Tau]

CHAPTER 3
HERMITIAN YANG-MILLS PROBLEM

Let (W, ω) be an asymptotically cylindrical Calabi-Yau $3-$fold as given by *Theorem 2.35* and $\mathcal{E} \to W$ the restriction of a holomorphic vector bundle on $\bar{W}$ satisfying certain stability assumptions. The guiding thread of this *Chapter* is the perspective of obtaining a smooth Hermitian metric H on $\mathcal{E}$ satisfying the *Hermitian Yang-Mills (HYM)* condition

$$\hat{F}_H \doteq (F_H, \omega) = 0, \tag{3.1}$$

which would effectively complete *Step 1* of the strategy in *Remark 2.39*. Considerable progress will be made in that direction by studying the following analytical problem.

Let W_S be the compact manifold (with boundary) obtained by truncating W at length S down the tubular end [cf. (2.18)]. On each W_S we consider the nonlinear 'heat flow'

$$\left\{ \begin{array}{rcl} H^{-1}\dfrac{\partial H}{\partial t} & = & -2\mathbf{i}\hat{F}_H \\ H(0) & = & H_0 \end{array} \right. \quad \text{on} \quad W_S \times [0, T[\tag{3.2}$$

with smooth solution $H_S(t)$, defined for some (short) T, since (3.2) is parabolic. Here H_0 is a fixed Hermitian metric on $\mathcal{E} \to W$ with 'good' asymptotic behaviour, and one imposes the *Dirichlet* boundary condition

$$H|_{\partial W_S} = H_0|_{\partial W_S}. \tag{3.3}$$

Taking suitable $t \to T$ and $S \to \infty$ limits of solutions to (3.2) over compact subsets W_{S_0}, we obtain a solution $H(t)$ to the evolution equation defined over W, for arbitrary $T < \infty$, with two key properties:

- each metric $H(t)$ is exponentially asymptotic in all derivatives to the reference metric H_0 over typical finite cylinders along the tubular end;
- if $H(t)$ converges at all as $t \le T \to \infty$, then the limit is a HYM metric on $\mathcal{E}$.

Moreover, I will show that the infinite-time convergence of $H(t)$ over W, hence the existence of the smooth HYM metric, can be reduced to a conjectured lower 'energy bound' on $\hat{F}_{H(t)}$, over a domain down the tube roughly proportional to $\|H(t)\|_{C^0(W)}$.

3.1 The evolution equation on W

As outlined above, we start with a holomorphic vector bundle $\mathcal{E}$ over the original compact Kähler 3−fold $(\bar{W}, \bar{\omega})$ [cf. *Section 2.1*] and we ask that its restriction $\mathcal{E}|_D$ to the *divisor at infinity* D be (slope-)stable with respect to $[\bar{\omega}]$. As stability is an open condition, there exists $\delta > 0$ such that $\mathcal{E}|_{D_z}$ is also stable for all $|z| < \delta$. In view of this hypothesis, we will say colloquially that $\mathcal{E}|_{\bar{W}\setminus D}$, denoted simply $\mathcal{E} \to W$, is *stable at infinity* over the noncompact Calabi-Yau (W, ω) given by *Theorem 2.35*; this is consistent since $[\omega] = [\bar{\omega}|_W]$. In summary:

Definition 3.40 *A bundle $\mathcal{E} \to W$ will be called* stable at infinity *(or* asymptotically stable*) if it is the restriction of a holomorphic vector bundle $\mathcal{E} \to \bar{W}$ satisfying:*

- *$\mathcal{E}$ is irreducible;*
- *$\mathcal{E}|_D$ is stable, hence also $\mathcal{E}|_{D_z}$ for $|z| < \delta$.*

The last ingredient is a suitable metric for comparison. Fixing a smooth trivialisation of $\mathcal{E}|_U$ over the tubular 'neighbourhood of infinity' $U \simeq D \times \{|z| < 1\} \subset \bar{W}$, we have a C^∞ identification between each $\mathcal{E}|_{D_z}$ and $\mathcal{E}|_D$. Define $H_0|_D$ as the Hermitian Yang-Mills metric on $\mathcal{E}|_D$ and denote K its pull-back over U. Then, for each $0 \le |z| < \delta$, the stability assumption gives a self-adjoint element $h_z \in \operatorname{End}\left(\mathcal{E}|_{D_z}\right)$ such that $H_0|_{D_z} \doteq K.h_z$ is the HYM metric on $\mathcal{E}|_{D_z}$. I claim the family h_z varies smoothly with z across the fibres[1]. This can be read from the HYM condition:[2]

$$\widehat{F_{0|z}} = 0 \quad \Leftrightarrow \quad P_z(h_z) \doteq \Delta_K h_z + \mathbf{i}\left(\hat{F}_K.h_z + h_z.\hat{F}_K\right) + 2\mathbf{i}\Lambda_z\left(\bar{\partial}_K h_z.h_z^{-1}.\partial_K h_z\right) = 0$$

[1] supposing, for simplicity, that $\mathcal{E}$ is an $SL(n,\mathbb{C})$−bundle and fixing $\det h_z$.

[2] writing $F_{0|z}$ for the curvature of the Chern connection of $H_0|_{D_z}$ and Λ_z for the contraction with the Kähler form on D_z. Also, by definition, $\hat{F} \doteq \Lambda_z F$.

where $\{P_z\}$ is a family of nonlinear partial differential operators depending smoothly on z. Linearising [Don$_2$, p.14] and using the assumption $\widehat{F_{0|0}} = 0$ we find $(D_z P)|_{z=0} = \Delta_0$, which is invertible on metrics with fixed determinant. This proves the claim, by (the implicit function) *Theorem A.82*.

Extending H_0 in any smooth way over the compact end $\bar{W} \setminus U$, we obtain a smooth Hermitian bundle metric on $\mathcal{E}$. For technical reasons, I also require H_0 to have *finite energy*:

$$\left\| \hat{F}_{H_0} \right\|_{L^2(W,\omega)} < \infty.$$

Definition 3.41 *A* reference metric *H_0 on an asymptotically stable bundle $\mathcal{E} \to W$ is (the restriction of) a smooth Hermitian metric on $\mathcal{E} \to \bar{W}$ such that:*

- *$H_0|_{D_z}$ are the corresponding Hermitian Yang-Mills metrics on $\mathcal{E}|_{D_z}$, $0 \leq |z| < \delta$;*
- *H_0 has finite energy.*

Remark 3.42 *Denote A_0 the Chern connection*[3] *of H_0; then by assumption each $A_0|_{D_z}$ is ASD [D-K, Prop. 2.1.59, p.47]. In particular, $A_0|_D$ induces an elliptic deformation complex [cf.* Subsection 1.5.2*]*

$$\Omega^0(\mathfrak{g}) \overset{d_{A_0}}{\to} \Omega^1(\mathfrak{g}) \overset{d^+_{A_0}}{\to} \Omega^2_+(\mathfrak{g})$$

where $\mathfrak{g} = \mathrm{Lie}\left(\mathcal{G}|_D\right)$ generates the gauge group $\mathcal{G} = \mathrm{End}\,\mathcal{E}$ over D. Thus, the requirement that $\mathcal{E}|_D$ be irreducible imposes a constraint on the associated cohomology:

$$\mathbf{H}^0_{A_0|_D} = 0,$$

*as a non-zero horizontal section would otherwise would split $\mathcal{E}|_D$ [*Theorem A.87*].*

Furthermore, although this will not be essential here, it is worth observing that one might want to restrict attention to acyclic *connections [Don$_1$, Def. 2.4, p.25], i.e., whose gauge class $[A_0]$ is* isolated *in $\mathcal{M}_{\mathcal{E}|_D}$. Such requirement would, in other words, prohibit infinitesimal deformations of A_0 across gauge orbits, which translates simply into the vanishing of the next cohomology group:*

$$\mathbf{H}^1_{A_0|_D} = 0.$$

[3]throughout this text the holomorphic structure of $\mathcal{E}$ will be fixed.

The underlying heuristic in our definitions is the analogy between looking for solutions with finite energy to partial differential equations over a compact manifold, with fixed values over a hypersurface, say, and over a space with a topologically cylindrical end, imposing their exponential decay to a suitable boundary condition at infinity.

3.1.1 Short time existence of solutions and C^0-bounds

The short-time existence of a solution to our evolution equation is a standard result:

Proposition 3.43 *Equation (3.2) admits a smooth solution $H_S(t)$, $t \in [0,\varepsilon)$, for ε sufficiently small.*

Proof *Using the Kähler identities, (3.2) is equivalent to the parabolic equation*

$$\begin{aligned} \frac{\partial h}{\partial t} &= -\left\{\Delta_0 h + \mathbf{i}\left(\hat{F}_0.h + h.\hat{F}_0\right) + 2\mathbf{i}\Lambda\left(\bar{\partial}_0 h.h^{-1}.\partial_0 h\right)\right\} \\ h(0) &= I, \qquad h|_{\partial W_S} = I. \end{aligned}$$

for a positive self-adjoint endomorphism $h(t) = H_0^{-1}H_S(t)$ of a unitary bundle with Chern connection from H_0. The result is an instance of [Ham, Part IV, §11, p.122]. ■

The task of extending solutions to all time will be postponed to the next *Subsection.* For now let us introduce some preliminary results and tools that will set the tune of the whole investigation; we begin by recalling the parabolic maximum principle:

Lemma 3.44 (Maximum principle) *Let X be a compact Riemannian manifold with boundary and suppose $f \in C^\infty\left(\mathbb{R}_t^+ \times X\right)$ is a nonnegative function satisfying:*

$$\left(\frac{d}{dt} + \Delta\right) f_t(x) \le 0, \quad \forall (t,x) \in \mathbb{R}_t^+ \times X$$

and the Dirichlet condition:

$$f_t|_{\partial X} = 0.$$

Then either $\sup\limits_X f_t$ is a decreasing function of t or $f \equiv 0$.

The crucial role of the Kähler structure in this type of problem is that it often suffices to control $\sup\left|\hat{F}_H\right|$ in order to obtain uniform bounds on H and its derivatives, hence to take limits in one-parameter families of solutions. Let us then establish such a bound from the start; I denote generally $\hat{e} \doteq \left|\hat{F}_H\right|_H^2$ and, in the immediate sequel, $\hat{e}_t \doteq \left|\hat{F}_{H_S(t)}\right|_{H_S(t)}^2$.

Corollary 3.45 *Let* $\{H_S(t)\}_{0<T}$ *be a smooth solution to* (3.2) *on* W_S*. Then* $\sup_{W_S} \hat{e}_t$ *is non-increasing with* t*; in fact, there exists* $B > 0$*, independent of* S *and* T*, such that*

$$\sup_{W_S} \left| \hat{F}_{H_S(t)} \right|^2 \leq B. \tag{3.4}$$

Proof *Using the Weitzenböck formula [D-K, p. 221], one finds*

$$\left(\frac{d}{dt} + \Delta \right) \hat{e}_t = - \left| d^*_{H_S(t)} F_{H_S(t)} \right|^2 \leq 0. \tag{3.5}$$

At the boundary ∂W_S*, for* $t > 0$*, the Dirichlet condition* (3.3) *means precisely that* $H_S|_{\partial W_S}$ *is constant, hence the evolution equation gives* $\hat{e}_t|_{\partial W_S} = \left. \left| H_S^{-1} \dot{H}_S \right|^2 \right|_{\partial W_S} \equiv 0$*. Then* $B = \sup_{W_S} \hat{e}_0$*.* ■

In order to obtain C^0-bounds and state our first convergence result, let us digress briefly into two ways of measuring metrics, which will be convenient at different stages in this *Chapter*. First, given two metrics H and K of the same determinant[4] we write

$$H = K.e^{\xi}$$

where $\xi \in \Gamma(\operatorname{End} \mathcal{E})$ is traceless and self-adjoint with respect both to H and K, and define

$$\bar{\lambda} : \operatorname{Dom}(\xi) \subseteq W \to \mathbb{R}_{\geq 0} \tag{3.6}$$

as the highest eigenvalue of ξ [Don_3, §III]. Clearly

$$|H - K| \leq (cst). |K| . \left(e^{\bar{\lambda}} - 1 \right), \tag{3.7}$$

so it is enough to control $\sup \bar{\lambda}$ to get a bound on $\|H\|_{C^0}$ relatively to K. Except where otherwise stated, we will assume from now on $K = H_0$ to be the reference metric.

Remark 3.46 *The space of continuous (bounded) bundle metrics is* complete *with respect to the* C^0-*norm [Rud, Theorem 7.15], so convergence as metrics is equivalent to uniform convergence in the pointwise norm.*

[4] for the purpose of Yang-Mills theory we may assume $\det H = 1$, since the L^2-norm of $\operatorname{tr} F_H$ is minimised independently by the harmonic representative of $[c_1(\mathcal{E})]^{dR}$.

Second, there is an alternative notion of 'distance' [Don$_2$, Def. 12], which is more natural to our evolution equation, as will become clear in the next few results:

Definition 3.47 *Given two Hermitian metrics H, K on a complex vector bundle $\mathcal{E}$, let*

$$\begin{aligned}\tau(H,K) &= \operatorname{tr} H^{-1}K\\ \sigma(H,K) &= \tau(H,K)+\tau(K,H)-2\operatorname{rk}\mathcal{E}.\end{aligned}$$

The function σ is nonnegative (since $a+a^{-1}\geq 2$, $\forall a\geq 0$), it vanishes if and only if $H=K$ and its expression is symmetric in H and K.

Although σ is not strictly speaking a distance, it compares to $\bar{\lambda}$ by increasing functions:

$$\begin{aligned}\sigma &= \operatorname{tr}\left(e^{\xi}+e^{-\xi}-2.\mathrm{Id}\right)\\ &\geq e^{\bar{\lambda}}+e^{-\bar{\lambda}}-2=e^{-\bar{\lambda}}\left(e^{\bar{\lambda}}-1\right)^2.\end{aligned}$$

Remark 3.48 *A sequence $\{H_i\}$ converges to H in C^0 if and only if $\sup\sigma(H_i,H)\to 0$. The former because $\sup e^{\bar{\lambda}}\to 1$ and the latter because obviously $\lim H_i^{-1}H=\lim H^{-1}H_i=\mathrm{Id}$.*

From the previous inequality we deduce, in particular,

$$e^{\bar{\lambda}}\leq\sigma+2. \tag{3.8}$$

Conversely, it is easy to see that

$$\sigma\leq 2r.e^{(r-1)\bar{\lambda}}$$

where $r=\operatorname{rk}\mathcal{E}$. Furthermore, in the context of our evolution problem, σ lends itself to applications of the maximum principle [Don$_2$, Prop. 13]:

Lemma 3.49 *If $H_1(t)$ and $H_2(t)$ are solutions of evolution equation (3.2), then $\sigma(t)=\sigma(H_1(t),H_2(t))$ satisfies*

$$\left(\frac{d}{dt}+\Delta\right)\sigma\leq 0.$$

Corollary 3.50 *If a smooth solution $H_S(t)$ to (3.2) on W_S, with Dirichlet boundary conditions, is defined for $0 \leq t < T$, then $H_S(t) \overset{C^0}{\to} H_S(T)$ and $H_S(T)$ is continuous.*

Proof *The argument is analogous to [Don$_2$, Cor. 15]. By* Remarks 3.46 *and* 3.48*, it suffices to show that $\sup \sigma(H_S(t), H_S(t'))$ converges to zero as $t' > t \to T$. Clearly $f_t = \sup_{W_S} \sigma(H_S(t), H_S(t+\tau))$ satisfies the (boundary) conditions of* Lemma 3.44*, so it is decreasing and*

$$\sup_{W_S} \sigma(H_S(t), H_S(t+\tau)) < \sup_{W_S} \sigma(H_S(0), H_S(\tau))$$

for all $t, \tau, \delta > 0$ such that $0 < T - \delta < t < t + \tau < T$. Taking $\delta < \varepsilon$ in Proposition 3.43 *we ensure continuity of $H_S(t)$ at $t = 0$, so the right-hand side is arbitrarily small for all t sufficiently close to T. Hence $H_S(t)$ is a uniformly Cauchy sequence as $t \to T$.* ∎

Looking back at (3.2), we may interpret $\hat{F}_H$ intuitively as a velocity vector along $1-$parameter families $H(t)$ in the space of Hermitian metrics. In this case, *Corollary 3.45* suggests an absolute bound on the variation of H for finite (possibly small) time intervals $0 \leq t \leq T$ where solutions exist. A straightforward calculation yields

$$\begin{aligned} \dot{\sigma} &= \frac{d}{dt}\sigma(H_S(t), H_0) \\ &\leq \operatorname{tr}\left[\left(H_S^{-1}\dot{H}_S\right).\left(e^{\xi} - e^{-\xi}\right)\right] \\ &\leq 2\left|\operatorname{tr}\left(\mathbf{i}\hat{F}_{H_S}\right).\left(e^{\xi} - e^{-\xi}\right)\right| \\ &\leq (cst.)\left|\hat{F}_{H_S}\right|.e^{\bar{\lambda}} \\ &\leq \tilde{B}.e^{\bar{\lambda}} \end{aligned}$$

using the evolution equation and *Corollary 3.45*. Combining with (3.8) and integrating,

$$\boxed{e^{\bar{\lambda}} \leq \sigma + 2 \leq 2e^{\tilde{B}T} \doteq C_T, \quad \forall t \leq T.} \tag{3.9}$$

Consequently, for any fixed $S_0 > 0$, the restriction to W_{S_0} of $H_S(t)$ lies in a C^0-ball of radius $\log C_T$ about H_0 in the space of Hermitian metrics, for all $S \geq S_0$ and $t \leq T$. Since C_T doesn't depend on S, the next *Lemma* shows that the H_S converge uniformly on compact subsets $W_{S_0} \subset W$ for any fixed interval $[0, T]$ (possibly trivial) where solutions exist for all $S > S_0$:

Lemma 3.51 *If there exist $C_T > 0$ and $S_0 > 0$ such that, for all $S' > S \geq S_0$, the evolution equation*

$$\begin{cases} H^{-1}\dfrac{\partial H}{\partial t} = -2\mathbf{i}\hat{F}_H & \\ H(0) = H_0, \quad H|_{\partial W_S} = H_0|_{\partial W_S} \end{cases} \quad \text{on} \quad W_S \times [0,T]$$

admits a smooth solution H_S satisfying

$$\sigma(H_S, H_{S'})|_{W_S} \leq C_T,$$

then the H_S converge uniformly to a (continuous) family H defined on $W_{S_0} \times [0,T]$.

Proof *It is of course possible to find a function $\phi : W \to \mathbb{R}$ such that*

$$\begin{cases} \phi \equiv 0, \text{ on } W_0 \\ \phi(y,\alpha,s) = s, \text{ for } s \geq 1 \\ |\Delta\phi| \leq L \end{cases}$$

thus giving an exhaustion of W by our compact manifolds $W_S \simeq \{p \in W \mid \phi(p) \leq S\}$, $S \geq S_0$. Taking $S_0 < S < S'$, I claim

$$\sigma\left(H_S(t)|_{W_S}, H_{S'}(t)|_{W_S}\right)(p) \leq \frac{C_T}{S}(\phi(p) + Lt), \quad \forall (p,t) \in W_S \times [0,T],$$

which yields our statement, since its restriction to W_{S_0} gives

$$\sigma(H_S, H_{S'})|_{W_{S_0}} \leq \frac{C_T(S_0 + LT)}{S} \underset{S\to\infty}{\longrightarrow} 0.$$

*The inequality holds trivially at $t = 0$ and on ∂W_S by (3.9), hence on the whole of $W_S \times [0,T]$, by the maximum principle [*Lemma 3.44*]:*

$$\left(\frac{d}{dt} + \Delta\right)\left(\sigma(H_S, H_{S'}) - \frac{C_T}{S}(\phi + Lt)\right) \leq -\frac{C_T}{S}(\Delta\phi + L) \leq 0,$$

using $|\Delta\phi| \leq L$ and Lemma 3.49. ■

This defines a *continuous* Hermitian bundle metric over our non-compact W:

$$\boxed{H(t) \doteq \lim_{S\to\infty} H_S(t), \quad t \leq T.} \tag{3.10}$$

It remains of course to show that any H_S can be smoothly extended for all $t \in [0, \infty[$ and that the limit metric $H(t)$ is itself a smooth solution of the evolution equation on W, with satisfactory asymptotic properties along the tubular end.

3.1.2 Smooth solutions for all time

The bound (3.4) allows us to extend solutions H_S to $t = T$, hence past T, for all time [Sim, Corolary 6.5]. More precisely, we can exploit the features of our problem to control a Sobolev norm $\|\Delta H\|$ by $\sup\left|\hat{F}\right|$ and weaker norms of H.

Lemma 3.52 *Let H and K be smooth Hermitian metrics on a holomorphic bundle over a Kähler manifold with Kähler form ω; then, for any submultiplicative pointwise norm $\|.\|$,*

$$\boxed{\|\Delta_K H\| \leq (cst.)\left[\left(\left\|\hat{F}_H\right\| + 1\right)\|H\| + \|\nabla_K H\|^2 \left\|H^{-1}\right\|\right]} \tag{3.11}$$

where $\Delta_K \doteq 2\mathbf{i}\Lambda_\omega \bar{\partial}\partial_K$ is the Kähler Laplacian and $(cst.)$ depends on K and $\|.\|$ only.

Proof *Write $h = K^{-1}H$ and ∇_K for the Chern connection of K. Since $\nabla_K K = 0$,*

$$\Delta_K H = K.\Delta_K h.$$

On the other hand, the Laplacian satisfies [D-K, p.46][Don$_2$, p.15]

$$\Delta_K h = h\left(\hat{F}_H - \hat{F}_K\right) + \mathbf{i}\Lambda_\omega\left(\bar{\partial}h.h^{-1} \wedge \partial_K h\right)$$

so the triangular inequality and again $\nabla_K K = 0$ yield the result. ∎

That will be the key to the recurrence argument behind *Corollary 3.57*, establishing smoothness of H_S as $t \to T$. Let us first collect some preliminary results [Sim, Lemma 6.4].

Lemma 3.53 *Let $\{H_i\}_{0 \leq i < I}$ be a one-parameter family of Hermitian metrics on a bundle $\mathcal{E} \to X$ over a compact Kähler manifold-with-boundary such that*

1. *$H_i \xrightarrow{C^0} H_I$, where H_I is a continuous metric,*
2. *$\sup\limits_X \left|\hat{F}_{H_i}\right|$ is bounded uniformly in i,*
3. *$H_i|_{\partial X} = H_0$;*

then $\{H_i\}$ is bounded in $L^p_2(X)$ uniformly in i, for all $p < \infty$, so H_I is of class C^1.

Corollary 3.54 *If $\{H_S(t)\}_{0\leq t<T}$ is a solution of* (3.2) *with Dirichlet condition on ∂W_S, then the $H_S(t)$ are bounded in $L_2^p(W_S)$ uniformly in t, for all $1\leq p<\infty$, and $H_S(T)$ is of class C^1.*

Proof *By* Corollary 3.50 *and* (3.4), *$\{H_S(t)\}_{0\leq t<T}$ satisfies* Lemma 3.53. ■

The *Corollary* gives, in particular, a time-uniform bound on $\|F_{H_S}\|_{L^p(W_S)}$. This can actually be improved to a uniform bound on all derivatives of curvature:

Lemma 3.55 *F_{H_S} is bounded in $L_k^\infty(W_S)$, uniformly in $0\leq t<T<\infty$, for each $k\geq 0$.*

Proof *By induction in k:*

$\underline{k=1}$: following [Don$_2$, Lemma 18], we obtain a uniform bound on $e_S(t)\doteq\left|F_{H_S(t)}\right|^2$, using the fact that

$$\left(\frac{d}{dt}+\Delta\right)e_S\leq(cst.)\left((e_S)^{\frac{3}{2}}+e_S\right)$$

[Don$_2$, Prop.16, (ii)], and consequently

$$e_S(t)\leq(cst.)\left(1+\int_0^t\|K_{t-\tau}\|_{L^p(W_S)}\left\|(e_S)^{\frac{3}{2}}+e_S\right\|_{L^q(W_S)}\right),\tag{3.12}$$

*where K_t is the heat kernel associated to $\frac{d}{dt}+\Delta$ and $\frac{1}{p}+\frac{1}{q}=1$. On the complete [*Theorem 2.35*] $6-$dimensional Riemannian manifold W_S, K_t satisfies [E-S, §9] the diagonal condition*

$$K_t(x,x)\leq\frac{(cst.)}{t^3},\qquad\forall x\in W_S$$

of Theorem A.90, *which gives a 'Gaussian' bound on the heat kernel. So, fixing $C>4$ and denoting $r(.,.)$ the geodesic distance, we have*

$$K_t(x,y)\leq\frac{(cst.)}{t^3}\exp\left\{-\frac{r(x,y)^2}{Ct}\right\},\qquad\forall x,y\in W_S.$$

Hence, for each $x\in W_S$, we obtain the bound

$$\begin{aligned}\|K_t(x,.)\|_{L^p(W_S)} &\leq \frac{(cst.)}{t^3}\left(\int_{W_S}\exp\left\{-p\frac{r(x,y)^2}{Ct}\right\}dy\right)^{\frac{1}{p}}\\ &\leq \frac{(cst.)}{t^3}\left(\int_0^\infty\left(\tfrac{Ct}{p}\right)^3u^5e^{-u^2}du\right)^{\frac{1}{p}}\\ &\leq \tilde{c}_p\,t^{\frac{3}{p}(1-p)}.\end{aligned}$$

Now, $p < \frac{3}{2} \Leftrightarrow \frac{3}{p}(1-p) > -1$*, in which case*

$$\int_0^T \|K_t(x,.)\|_{L^p(W_S)}\, dt \le c_p(T).$$

Inequality (3.12) *gives the desired result provided* $(e_S)^{\frac{3}{2}} \in L^q(W_S)$ *for some* $q > 3$*; this means* $F_{H_S(t)} \in L^{\tilde{q}}(W_S)$ *for some* $\tilde{q} > 9$*, which is guaranteed by* Corollary 3.54*.*

$\underline{k \Rightarrow k+1}$*: The general recurrence step is identical to [Don$_2$, Cor. 17 (ii)], using the maximum principle [*Lemma 3.49*] with boundary conditions.* ■

We are now in shape to put into use the Kähler setting, combining the L_k^∞−bounds on F (hence on $\hat{F}$) with inequality (3.11), via elliptic regularity:

Lemma 3.56 *Let* $\{H_i\}_{0 \le i < I}$ *be a one-parameter family of Hermitian metrics on a holomorphic vector bundle* $\mathcal{E} \to X$ *over a compact Kähler manifold-with-boundary such that*

1. $H_i \xrightarrow[i \to I]{L_2^p} H$*, where* H *is a continuous metric,* $\forall (1 \le p < \infty)$*,*
2. $\left\|\hat{F}_{H_i}\right\|_{L_k^p(X)} \le C'_{p,k} \quad \forall k \in \mathbb{N}, \quad \forall (1 \le p < \infty)$*,*
3. $\|H_i\|_{L_k^p(\partial X)} \le C_{p,k} \quad \forall k \in \mathbb{N}, \quad \forall (1 \le p < \infty)$*;*

then $\{H_i\}$ *is* C^∞−*bounded and* H *is smooth.*

Proof *Fixing* $p \ge 1$*, I will prove the following statement by induction in* k*:*

$$\|H\|_{L_{k+2}^p} \text{ and } \left\|H^{-1}\right\|_{L_k^p} \text{ are bounded, } \forall k \ge 0.$$

The first hypothesis gives step $k = 0$*. Now, assuming the statement up to step* $k-1$*,*

$$\begin{aligned} \left\|H^{-1}\right\|_{L_k^p} &= \left\|H^{-1}\right\|_{L^p} + \left\|\nabla\left(H^{-1}\right)\right\|_{L_{k-1}^p} \\ &\le \left\|H^{-1}\right\|_{L^p} + \left\|H^{-1}\right\|_{L_{k-1}^p}^2 \|\nabla H\|_{L_{k-1}^p} \\ &\le \left\|H^{-1}\right\|_{L_{k-1}^p}\left(1 + \left\|H^{-1}\right\|_{L_{k-1}^p}\|H\|_{L_k^p}\right) \end{aligned}$$

so $\left\|H^{-1}\right\|_{L_k^p}$ *is bounded. On the other hand, elliptic regularity on manifolds-with-*

boundary and (3.11), *with* $K = H_0$, *give*

$$\begin{aligned}\|H\|^2_{L^p_{k+2}} &\leq (cst.)\left(\|\Delta H\|^2_{L^p_k} + \|H\|^2_{L^p_{k+1}} + \|H\|^2_{L^p_{k+\frac{3}{2}}(\partial X)}\right)\\ &\leq (cst.)\left[\|H\|^2_{L^p_{k+1}}\left(1+\left(1+\left\|\hat{F}_H\right\|_{L^p_k} + \|H\|_{L^p_{k+1}}\left\|H^{-1}\right\|_{L^p_k}\right)^2\right) + \|H\|^2_{L^p_{k+\frac{3}{2}}(\partial X)}\right]\end{aligned}$$

where $(cst.)$ *depends on* H_0 *and* X *only, and all those terms are bounded by assumption. Passing perhaps to a subsequence,* $\{H_i\}$ *is bounded in* C^∞ *and its limit* H *is smooth.* ■

Corollary 3.57 *Under the Dirichlet condition, the limit metric* $H_S(T)$ *is smooth.*

Proof Corollary 3.54 *gives hypothesis* 1., Lemma 3.55 *gives* 2. *and the Dirichlet condition on* ∂W_S *gives* 3., *as* H_0 *is smooth.* ■

Since $H_S(t) \xrightarrow[t\to T]{C^\infty} H_S(T)$, the solution can be smoothly extended beyond T, by short-time existence, hence for all time [Sim, Prop 6.6]:

Proposition 3.58 *Given any* $T > 0$, *the family of Hermitian metrics* $H(t)$ *on* $\mathcal{E} \to W$ *defined by* (3.10) *is the unique, smooth solution of the evolution equation*

$$\begin{cases} H^{-1}\dfrac{\partial H}{\partial t} = -2\mathbf{i}\hat{F}_H \\ H(0) = H_0 \end{cases} \quad \text{on } W\times[0,T]$$

with $\sup_W |H| < \infty$. *Furthermore,* $\sup_W \left|\hat{F}_{H(t)}\right| \leq B = \sup_W \left|\hat{F}_{H_0}\right|$.

Proof *On any compact subset* $W_{S_0}\times[0,T]$, H_S *are* $C^\infty-$*bounded so, by the evolution equation,* $\frac{\partial H_S}{\partial t} \xrightarrow[S\to\infty]{C^\infty} \frac{\partial H}{\partial t}$ *and* H *is a solution on* $W_{S_0}\times[0,T]$ *satisfying the same bounds. This is independent of the choice of* S_0. ■

3.1.3 Asymptotic behaviour of the solution

We have a solution $\{H(t)\}$ of the flow on W [*Proposition 3.58*], giving a Hermitian metric on $\mathcal{E} \to W$ for each $t \in [0,T]$. Let us study the asymptotic properties of $H(t)$ along the non-compact end. Set

$$\hat{e}_t = \left|\hat{F}_{H(t)}\right|^2 .$$

First of all, as a direct consequence of *Lemma 2.34*, I claim

$$\hat{e}_0 \leq B\epsilon, \qquad \epsilon \doteq \begin{cases} 1 & \text{on } W_0 \\ e^{-s} & \text{on } \partial W_s, \quad s \geq 0 \end{cases} \tag{3.13}$$

where $B = \sup_W \hat{e}_0$ [*Corollary 3.45*]. In a holomorphic trivialisation of $\mathcal{E}$ over the neighbourhood of infinity U, with coordinates $\left(z, \xi^1, \xi^2\right)$ such that $D = \{z = 0\} \subset \bar{W}$, the curvature F_{H_0} is a $(1,1)$ −form with values in $\operatorname{End}\mathcal{E}$:

$$F_{H_0}|_{U\smallsetminus D} = F_{zz} \underbrace{dz \wedge d\bar{z}}_{0\left(|z|^2\right)} + \sum_i (F_{zi} \underbrace{dz \wedge d\bar{\xi^i}}_{0(|z|)} + F_{iz} \underbrace{d\xi^i \wedge d\bar{z}}_{0(|z|)}) + \sum_{ij} F_{ij}\, d\xi^i \wedge d\bar{\xi^j}. \tag{3.14}$$

The terms involving dz or $d\bar{z}$ decay at least as $0(|z|)$ along the tubular end [*Lemma 2.34*], and all the coefficients of F_{H_0} are bounded, so $F_{H_0} \underset{|z|\to 0}{\longrightarrow} \sum F_{ij} d\xi^i \wedge d\xi^j$. Consequently,

$$\hat{F}_{H_0}\left(z, \xi^1, \xi^2\right) \underset{|z|\to 0}{\longrightarrow} \hat{F}_{H_0}\Big|_D \left(\xi^1, \xi^2\right) = 0,$$

i.e., $\hat{F}_{H_0}$ decays exponentially to zero as $s \to \infty$. From (3.13) we now obtain the exponential decay of each $\hat{e}_t$ along the cylindrical end:

Proposition 3.59 *Take B and ϵ as in (3.13); then*

$$\hat{e}_t \leq \left(Be^t\right)\epsilon \quad \text{on} \quad W.$$

Proof *The statement is obvious on W_0. For any $s_0, t_0 \geq 0$, take $T = S > \max\{s_0, t_0\}$, let $\Sigma_S \doteq W_S \smallsetminus W_0$ and consider on $\Sigma_S \times [0,T]$ the comparison function $g(t,s) \doteq Be^{t-s}$. Using the Weitzenböck formula one shows $\left(\frac{d}{dt} + \Delta\right)\hat{e}_S \leq 0$ [cf. (3.5)], where $\hat{e}_S = \left|\hat{F}_{H_S}\right|^2$ and H_S is a solution of our flow (with Dirichlet condition) on W_S as in* Lemma 3.51.

For $\psi \doteq \hat{e}_S - g$*, we have*[5] $\left(\frac{d}{dt} + \Delta\right)\psi \leq 0$ *and, by the maximum principle [*Lemma 3.44*],*

$$\psi \leq \max_{\partial([0,T]\times\Sigma_S)} \left\{\hat{e}_S - Be^{t-s}\right\} \leq 0.$$

To see that the r.h.s. is zero, there are four boundary terms to check:

$\underline{s = S}$*: the Dirichlet condition implies* $\hat{e}_S(t,S) = 0$ *for all* $t > 0$*, hence* $\psi(t,S) \leq 0$*;*

$\underline{s = 0}$*:* $\psi(t,0) \leq B\left(1 - e^t\right) \leq 0$*;*

$\underline{t = 0}$*:* (3.13) *gives* $\psi(0,s) \leq 0$*;*

$\underline{t = T}$*: again by* Corollary 3.45 *we have* $\psi(T,s) \leq B\left(1 - e^{T-s}\right) \leq 0$.

This shows that $\hat{e}_S(t,s) \leq Be^{t-s}$ *on* $\Sigma_S \times [0,T]$*. Take* $T = S \to \infty$*.* ■

As a result, we may replace pointwise $\left(Be^T\right)e^{-s}$ for $\tilde{B}$ in (3.9) to obtain exponential C^0-convergence of $H(t)$ along the cylindrical end:

$$\sigma(H(t), H_0)\,|_{\partial W_S} = O\left(e^{-S}\right). \tag{3.15}$$

The next *Proposition* will establish exponential decay of $H(t)$ in C^1. I will state it in rather general terms to highlight the fact that essentially all one needs to control is the Laplacian, hence $\hat{F}$ in view of (3.11). The proof emulates the argument in [Don$_4$, Prop.8].

Proposition 3.60 *Let* V *be an open set of a Riemannian manifold* X*,* $V' \subset V$ *an interior domain and* $Q \to X$ *some bundle with connection* ∇ *and a continuous fibrewise metric. There exist constants* $\varepsilon, A > 0$ *such that, if a smooth section* $\phi \in \Gamma(Q)$ *satisfies:*

1. $\|\phi\|_{C^0(V)} \leq \varepsilon$*;*
2. $|\Delta\phi| \leq f(|\nabla\phi|)$ *on* V*, for some non-decreasing function* $f : \mathbb{R}^+ \to \mathbb{R}^+$*;*

2'. *the above assumption remains valid under local rescalings by a constant, in the sense that, in every ball* $B_r \subset V$*, it still holds for some function* $\tilde{f}$ *after the radial rescaling* $\tilde{\phi}(\tilde{x}) \doteq \phi(mx)$*,* $m > 0$*;*

then

$$\boxed{\|\phi\|_{C^1(V')} \leq A\,\|\phi\|_{C^0(V)}\,.}$$

[5] recall that our sign convention for the Laplacian is $\Delta = -\sum_i \frac{\partial^2}{\partial x_i^2}$, in local coordinates.

Proof *I first contend that ϕ obeys an a priori bound*

$$\left|(\nabla\phi)_x\right| r(x) \leq 1, \quad \forall x \in V$$

where $r(x) : V \to \mathbb{R}$ is the distance to ∂V. Since the term on the left-hand side is zero on ∂V, its supremum is attained at some $\hat{x} \in V^0 = V$ (possibly not unique). Write

$$m \doteq \left|(\nabla\phi)_{\hat{x}}\right|, \quad R = r(\hat{x})$$

and suppose, by contradiction, that $R > \frac{1}{m}$. If that's the case then we rescale the ball $B_R(\hat{x})$ by the factor m, obtaining a rescaled local section $\tilde{\phi}$ defined in $\tilde{B}_{mR} \supset \tilde{B}_1$. In this picture, any point in $\tilde{B} \doteq \tilde{B}_{\frac{1}{2}}$ is further from ∂V than $\frac{R}{2}$, hence, by definition of $\hat{x}$, $\left\|\nabla\tilde{\phi}\right\|_{C^0(\tilde{B})} \leq 2$. By assumptions 2. and 2'. there exists $L > 0$ such that $\left\|\Delta\tilde{\phi}\right\|_{C^0(\tilde{B})} \leq L$, and elliptic regularity gives

$$\left\|\nabla\tilde{\phi}\right\|_{C^{0,\alpha}(\tilde{B})} \leq c_\alpha.(L+\varepsilon) \doteq \tilde{c}_\alpha$$

using assumption 1.. Now, the rescaled gradient at $\hat{x}$ has norm $\left|\left(\nabla\tilde{\phi}\right)_{\hat{x}}\right| = 1$ so, taking $\alpha = \frac{1}{2}$ (say) in a smaller ball of radius $\rho = (\frac{1}{2\tilde{c}_{\frac{1}{2}}})^2$,

$$\left|\left(\nabla\tilde{\phi}\right)_x\right| \geq 1 - \tilde{c}_{\frac{1}{2}}.\rho^{\frac{1}{2}} \geq \frac{1}{2}, \quad \forall x \in \tilde{B}_\rho.$$

This means $\left|\tilde{\phi}\right|$ varies by some definite $\delta > 0$ inside $\tilde{B}_\rho$ and we reach a contradiction choosing $\varepsilon < \delta$. So

$$\left|(\nabla\phi)_x\right| \leq \left(\inf_{\partial U} r\right)^{-1}, \quad \forall x \in U \subset V^0$$

for some open set U containing V'. To conclude the proof, it suffices to apply Moser's estimate [Mos, Theorem 3] and control the L^2-norm of $\nabla\phi$ on U:

$$\begin{aligned}(cst.)\left\|\nabla\phi\right\|^2_{C^0(V')} &\leq \left\|\nabla\phi\right\|^2_{L^2(U)} = \int_U (\nabla\phi, \nabla\phi) = \int_U (\phi, \nabla^*\nabla\phi) = \int_U (\phi, \Delta\phi) \\ &\leq \left\|\nabla\phi\right\|^2_{C^0(U)} \left\|\phi\right\|^2_{L^2(V')}\end{aligned}$$

and the last term is obviously bounded by $(cst.)\left\|\phi\right\|^2_{C^0(V)}$. ■

Now let $\operatorname{End}\mathcal{E} = Q$ in *Proposition 3.60*, with connection ∇_0.

Notation 3.61 *Given $S > r > 0$, write $\Sigma_r(S)$ for the interior of the cylinder $(W_{S+r} \setminus W_{S-r})$ of 'length' $2r$. We denote the C^k*−exponential tubular limit *of an element in $C^k(\Gamma(Q))$ by:*

$$\phi \xrightarrow[S\to\infty]{C^k} \phi_0 \quad \dot{\Leftrightarrow} \quad \|\phi - \phi_0\|_{C^k(\Sigma_1(S),\omega)} = O\left(e^{-S}\right).$$

For $S \geq 3$, let $V = \Sigma_3(S)$ and $V' = \Sigma_2(S)$ so that the distance of V' to ∂V is always 1. In view of (3.15), for whatever $\varepsilon > 0$ given by the statement, it is possible to choose $S \gg 0$ so that $\phi = (H(t) - H_0)|_{\Sigma_3(S)}$ satisfies the first condition (for arbitrary fixed t), hence also the second one by (3.11), with $f(x) = (cst.)\left[(B+1)\varepsilon + x^2\right]$ and $(cst.)$ depending only on H_0 and ε. We conclude, in particular, that $H(t)$ is C^1−exponentially asymptotic to H_0 in the tubular limit:

$$H(t) \xrightarrow[S\to\infty]{C^1} H_0. \tag{3.16}$$

Furthermore, in our case the bound on the Laplacian (3.11) holds for any L^p_k−norm, given our control over all derivatives of the curvature [*Lemma 3.55*], so the argument above lends itself to the obvious iteration over shrinking tubular segments $\Sigma_{1+\frac{1}{k}}(S)$:

Corollary 3.62 *Let $\{H(t) \mid t \in [0,T]\}$ be the solution to the evolution equation on $\mathcal{E} \to W$ given by* Proposition 3.58*; then*

$$\boxed{H(t) \xrightarrow[S\to\infty]{C^k} H_0, \quad \forall k \in \mathbb{N}.}$$

Combining existence and uniqueness of the solution for arbitrary time [*Proposition 3.58*] and C^∞−exponential decay [*Corollary 3.62*], one has the main statement:

Theorem 3.63 *Let $\mathcal{E} \to W$ be stable at infinity, with reference metric H_0, over an asymptotically cylindrical $SU(3)$−manifold W as given by the Calabi-Yau-Tian-Kovalev* Theorem 2.35*; then, for any $0 < T < \infty$, $\mathcal{E}$ admits a 1−parameter family $\{H_t\}$ of smooth Hermitian metrics solving*

$$\begin{cases} H^{-1}\dfrac{\partial H}{\partial t} = -2\mathbf{i}\hat{F}_H \\ H(0) = H_0 \end{cases} \quad on \quad W \times [0,T].$$

Moreover, each H_t approaches H_0 exponentially in all derivatives over tubular segments $\Sigma_1(S)$ along the non-compact end.

3.2 Another variational formulation

There is a standard way [Don$_2$, §1.2] to build a functional on the space of Hermitian bundle metrics over a compact Kähler manifold the critical points of which, if any, are precisely the Hermitian Yang-Mills metrics. This procedure is in fact completely analogous to the Chern-Simons construction discussed in *Subsection 1.3.2*, in that it amounts to integrating along paths a prescribed first-order variation, expressed by a closed $1-$form.

In this *Section* I will adapt this prescription to W, restricting attention to metrics with suitable asymptotic behaviour, and to the $K3$ divisors $D_z = \tau^{-1}(z)$ along the tubular end. On one hand, the resulting functional $\mathcal{N}_W$ will illustrate the fact that our evolution equation converges (if at all) to a HYM metric. On the other hand, crucially, the family $\mathcal{N}_{D_z}$ will mediate the role of stability in the time-uniform control of $\{H_t\}$ over W.

3.2.1 Definition of the functional

I will set up this analogous framework in some generality at first, defining an *a priori* path-dependent functional $\mathcal{N}_W$ on a suitable set of Hermitian metrics on $\mathcal{E}$. When restricted to the specific $1-$parameter family $\{H_t\}$ from our evolution equation, we will see that $\mathcal{N}_W(H_t)$ is in fact decreasing, from which the $t \to \infty$ limit, if it exists, must be HYM on $\mathcal{E}$. Let

$$\mathcal{I}_0 \doteq \left\{ h \in \operatorname{End}\mathcal{E} \mid h \text{ is Hermitian}, \quad h \xrightarrow[S\to\infty]{C^\infty} 0 \right\}$$

denote the space of fibrewise Hermitian matrices which decay exponentially along the tube.

Definition 3.64 *Let $\mathcal{H}_0$ be the set of smooth Hermitian metrics H on $\mathcal{E} \to W$ such that:*

$$\boxed{H \xrightarrow[S\to\infty]{C^\infty} H_0.}$$

Remark 3.65 *About the* Definition*:*

1. *The exponential decay* (3.13) *implies $H_0 \in \mathcal{H}_0$. Indeed, $\mathcal{H}_0$ is a* star domain *in the affine space $H_0 + \mathcal{I}_0$, in the sense that $H_0 + \ell(H - H_0) \in \mathcal{H}_0$, $\forall(\ell, H) \in [0,1] \times \mathcal{H}_0$, with $H - H_0 \in \mathcal{I}_0$. Thus $\mathcal{H}_0$ is contractible, hence* connected *and* simply connected*.*

2. *There is a well-defined notion of 'infinitesimal variation' of a metric H, as an object in the 'tangent space'*

$$T_H\mathcal{H}_0 \simeq \mathcal{I}_0.$$

3. *We know from* (3.13) *that* $\hat{F}_{H_0} \underset{S\to\infty}{\overset{C^0}{\longrightarrow}} 0$, *hence*

$$\left\|\hat{F}_H\right\|_{L^1(W,\omega)} < \infty, \quad \forall H \in \mathcal{H}_0.$$

4. *Any 'nearby'* $H \in \mathcal{H}_0$, *for which* $\xi = \log H_0^{-1}H$ *is well-defined,*[6] *is joined to* H_0 *by*

$$\begin{array}{c} \gamma : [0,1] \to \mathcal{H}_0 \\ \gamma(\ell) = H_0 e^{\ell\xi} \end{array}. \tag{3.17}$$

Clearly $\ell\xi \underset{S\to\infty}{\overset{C^\infty}{\longrightarrow}} 0$, *so* $\gamma(\ell) \in \mathcal{H}_0$, $\forall \ell \in [0,1]$.

5. *Given any* $T > 0$, *the solutions* $\{H_t\}_{t\in[0,T]}$ *of our flow form a path in* $\mathcal{H}_0$, *since* $\hat{F}_{H_t}$ *decays exponentially along the tube for any* t *[*Theorem 3.63*].*

Let $\theta \in \Omega^1\left(\mathcal{H}_0, \Omega^{1,1}(W)\right)$ be given by

$$\begin{array}{c} \theta_H : T_H\mathcal{H}_0 \to \Omega^{1,1}(W) \\ \theta_H(k) = 2\mathbf{i}\mathrm{tr}\left(H^{-1}.k.F_H\right) \end{array}. \tag{3.18}$$

Then we may, at first formally, write

$$(\rho_W)_H(k) = \int_W \theta_H(k) \wedge \omega^2, \tag{3.19}$$

which will define a smooth 1−form on any domain $H_0 \in \mathcal{U} \subset \mathcal{H}_0$ where the integral converges, for all $H \in \mathcal{U}$ and all $k \in T_H\mathcal{H}_0$. The crucial fact for us is that ρ is identically zero precisely at the HYM metrics:

$$\begin{aligned} (\rho_W)_H = 0 \quad &\Leftrightarrow \quad \int_W \mathrm{tr}\left(H^{-1}k.F_H\right) \wedge \omega^2 = 0, \quad \forall k \in T_H\mathcal{H}_0 \\ &\Leftrightarrow \quad \hat{F}_H = (F_H, \omega) = 0. \end{aligned}$$

Following the analogy with Chern-Simons formalism, this suggests integrating ρ_W over a path to obtain a function having the HYM metrics as critical points. Given $H \in \mathcal{H}_0$, let $\gamma(\ell) = H_\ell$ be a path in $\mathcal{H}_0$ connecting H to the reference metric H_0, and form the evaluation

[6] i.e., such that $\left\|H_0^{-1}H - I\right\|_{C^0(W,\omega)} < 1$.

of θ along γ:

$$\Phi^{\gamma}(\ell) \doteq [\theta_{\gamma}(\dot{\gamma})](\ell) = 2\mathbf{i}\mathrm{tr}\left(H_{\ell}^{-1}.\dot{H}_{\ell}.F_{H_{\ell}}\right) \in \Omega^{1,1}(W). \tag{3.20}$$

For instance, with γ as in (3.17), we have $H_{\ell}^{-1}.\dot{H}_{\ell} = \underbrace{H_{\ell}^{-1}.H_0}_{e^{-\ell\xi}}.\left(\frac{\partial}{\partial \ell}e^{\ell\xi}\right) = \xi$ and

$$\begin{aligned}(\rho_W)_{H_{\ell}}\left(\dot{H}_{\ell}\right) &= \int_W \Phi^{\gamma}(\ell) \wedge \omega^2 \\ &= 2\mathbf{i}\int_W \mathrm{tr}\ \xi.F_{H_{\ell}} \wedge \omega^2 \\ &= 2\mathbf{i}\int_W \mathrm{tr}\ \xi.\hat{F}_{H_{\ell}}\ d\mathrm{Vol}_{\omega}\end{aligned}$$

is well-defined near H_0, since $\xi = \log H_0^{-1}H$ is bounded and $\hat{F}_{H_{\ell}}$ is integrable [*Remark 3.65*]. Thus, in this setting at least, we have a rigorously defined integral:

$$\mathcal{N}_W^{\gamma}(H) \doteq \int_{\gamma} \rho_W. \tag{3.21}$$

Before we move on, let us establish a convenient relation between $\Phi^{\gamma}(\ell)$ and the rate of change of the 'topological' charge density $\mathrm{tr}\, F^2$ along γ, in the following *Corollary*.

Lemma 3.66 *Let H be a Hermitian bundle metric, let $h \in T_H\mathcal{H}_0$ be an infinitesimal variation of H and denote $\tau \doteq hH^{-1}$; then the curvature of the Chern connection of H varies, to first order, by*

$$F_{H+h} = F_H + \bar{\partial}\partial_H\tau + O\left(|\tau|^2\right).$$

Proof *Set $g = (H+h)H^{-1} = 1+\tau$, so that [D-K, p.46][Don$_2$, p.15]*

$$F_{H+h} = F_H + \bar{\partial}\left(g^{-1}\partial_H g\right).$$

Observing that $g^{-1} = 1 - \tau + O\left(|\tau|^2\right)$, we expand the variation of curvature to find:

$$\begin{aligned}\bar{\partial}\left(g^{-1}\partial_H g\right) &= -\left(g^{-1}.\bar{\partial}g.g^{-1}\right)\partial_H g + g^{-1}\bar{\partial}\partial_H g \\ &= -(1-\tau)\bar{\partial}\tau(1-\tau)\partial_H\tau + (1-\tau)\bar{\partial}\partial_H\tau + O\left(|\tau|^2\right) \\ &= \bar{\partial}\partial_H\tau + O\left(|\tau|^2\right).\end{aligned}$$

■

Corollary 3.67 *Let* $\{\gamma(\ell) = H_\ell\} \subset \mathcal{H}_0$ *be a* $1-$*parameter family of metrics on* $\mathcal{E}$*; then the evaluation* Φ^γ *from* (3.20) *satisfies*

$$-\mathbf{i}\bar{\partial}\partial\Phi^\gamma(\ell) = \frac{d}{d\ell}\mathrm{tr}\, F_{H_\ell}^2.$$

Proof *Using the first order variation of* F *[*Lemma 3.66*] and the Bianchi identity:*

$$\begin{aligned} \frac{d}{d\ell}\mathrm{tr}\, F_{H_\ell}^2 &= 2\mathrm{tr}\left(\frac{d}{d\ell}F_{H_\ell}\right)\wedge F_{H_\ell} \\ &= 2\mathrm{tr}\,\bar{\partial}\partial_{H_\ell}\left(H_\ell^{-1}.\dot{H}_\ell\right)\wedge F_{H_\ell} \\ &= -\mathbf{i}\bar{\partial}\partial\Phi^\gamma(\ell). \end{aligned}$$

■

By the same token, if we restrict attention to our family $\{\gamma(t) = H_t\} \subset \mathcal{H}_0$ satisfying the evolution equation

$$\begin{cases} H^{-1}\dfrac{\partial H}{\partial t} = -2\mathbf{i}\hat{F}_H \\ H(0) = H_0 \end{cases}, \tag{3.22}$$

set $\mathcal{N}_W(H_0) = 0$ and write for short $\Phi_t \doteq \Phi^\gamma(t)$, we obtain a real smooth function

$$\mathcal{N}_W(H_T) = \int_0^T (\rho_W)_{H_t}\left(\dot{H}_t\right)dt = \int_0^T\left(\int_W \Phi_t\wedge\omega^2\right)dt. \tag{3.23}$$

Proposition 3.68 *The function* $\mathcal{N}_W(H_t)$ *is well-defined,* $\forall t\in[0,\infty[$*, and*

$$\frac{d}{dt}\mathcal{N}_W(H_t) = -\frac{2}{3}\left\|\hat{F}_{H_t}\right\|^2_{L^2(W)}.$$

Proof *Using the evolution equation* (3.22)*:*

$$\begin{aligned} \frac{d}{dt}\mathcal{N}_W(H_t) &= (\rho_W)_{H_t}\left(\dot{H}_t\right) = \int_W \Phi_t\wedge\omega^2 \\ &= 2\int_W \mathrm{tr}\,\underbrace{\mathbf{i}H_t^{-1}.\dot{H}_t}_{2\hat{F}_{H_t}}.\underbrace{F_{H_t}\wedge\omega^2}_{\frac{1}{6}\hat{F}_{H_t}d\mathrm{Vol}_\omega} = \frac{2}{3}\int_W \mathrm{tr}\,\hat{F}_{H_t}^2.d\mathrm{Vol}_\omega \\ &= -\frac{2}{3}\left\|\hat{F}_{H_t}\right\|^2_{L^2(W)} \end{aligned}$$

and this is finite, as $\hat{F}_{H_t}$ *decays exponentially along the tube [*Proposition 3.59*].* ■

The above *Proposition* confirms that we are on the right track: if the $\{H_t\}$ converge to a smooth metric $H = H_\infty$ at all, then H must be HYM.

3.2.2 $\mathcal{N}_W$ is path-independent

Our definition of $\mathcal{N}_W$ by integration of ρ_W is *a priori* path dependent and we have briefly examined two examples [(3.21) and (3.23)] of such formulation, both of which will be relevant at different stages in the ensuing analysis. We will now establish that both settings are, in fact, completely equivalent.

Lemma 3.69 *Let $H \in \mathcal{H}_0$ and $h, k \in T_H\mathcal{H}_0 \cong \mathcal{I}_0$; in the terms of (3.18), the difference*

$$\eta_H(h,k) \doteq \frac{1}{2\mathbf{i}}\left(\theta_{H+h}(k) - \theta_H(k)\right)$$

is anti-symmetric to first order, modulo $\operatorname{img}\partial + \operatorname{img}\bar{\partial}$.

Proof *In the notation of* Lemma 3.66 *and setting $\sigma \doteq hK^{-1}$, the anti-symmetrisation of η_H is*

$$\begin{aligned}\xi_H(h,k) &\doteq \eta_H(h,k) - \eta_H(k,h) \\ &= \operatorname{tr}\left((\sigma.\tau - \tau.\sigma).F_H + \sigma.\bar{\partial}\partial_H\tau - \tau.\bar{\partial}\partial_H\sigma\right) \\ &\quad + O(|\sigma|.|\tau|^2) + O(|\tau|.|\sigma|^2).\end{aligned} \tag{3.24}$$

The curvature of the Chern connection of H obeys $F_H = \bar{\partial}\partial_H + \partial_H\bar{\partial}$, so

$$\begin{aligned}\sigma.\bar{\partial}\partial_H\tau &= \sigma.F_H.\tau - \sigma.\partial_H\bar{\partial}\tau \\ &= \sigma.F_H.\tau - \partial_H\left(\sigma.\bar{\partial}\tau\right) - \bar{\partial}\left(\partial_H\sigma.\tau\right) + \tau.\bar{\partial}\partial_H\sigma;\end{aligned}$$

mutatis mutandis,

$$\tau\bar{\partial}\partial_H\sigma = \tau.F_H.\sigma - \partial_H\left(\tau.\bar{\partial}\sigma\right) - \bar{\partial}\left(\partial_H\tau.\sigma\right) + \sigma.\bar{\partial}\partial_H\tau.$$

Substituting these and using the cyclic property of trace in (3.24) we find

$$\begin{aligned}\xi_H(h,k) &= \tfrac{1}{2}\operatorname{tr}\left(\partial_H\left(\tau.\bar{\partial}\sigma - \sigma.\bar{\partial}\tau\right) + \bar{\partial}\left(\partial_H\tau.\sigma - \partial_H\sigma.\tau\right)\right) + O(|\sigma|.|\tau|^2) + O(|\tau|.|\sigma|^2) \\ &\in \operatorname{img}\partial + \operatorname{img}\bar{\partial} \quad \textit{modulo} \quad O(|\sigma|.|\tau|^2) + O(|\tau|.|\sigma|^2).\end{aligned}$$

■

Corollary 3.70 *Let $\mathcal{U} \subset \mathcal{H}_0$ be a subset where the integral defining the 1−form ρ_W in (3.19) converges, for all $H \in \mathcal{U}$ and all $k \in T_H\mathcal{H}_0$; then $\rho_W|_{\mathcal{U}}$ is closed.*

Proof *Recall that a 1−form is closed precisely when its infinitesimal variation is symmetric to first order [p.20]. In view of the previous* Lemma, *it remains to check that $\xi_H(h,k) \wedge \omega^2$ integrates to zero modulo terms of higher order:*

$$\lim_{S\to\infty} \int_{W_S} \xi_H(h,k) \wedge \omega^2 = 0.$$

Taking account of bi-degree and de Rham's theorem, still modulo $O(|\sigma| \,.\, |\tau|^2) + O(|\tau| \,.\, |\sigma|^2)$,

$$\begin{aligned}
\int_{W_S} \xi_H(h,k) \wedge \omega^2 &= \tfrac{1}{2}\int_{\partial W_S} \mathrm{tr}\left[\left(\tau.\bar\partial\sigma - \sigma.\bar\partial\tau\right) + \left(\partial_H\tau.\sigma - \partial_H\sigma.\tau\right)\right] \wedge \omega^2 \\
&= \tfrac{1}{2}\int_{\partial W_S} \mathrm{tr}\left[2\tau.\bar\partial\sigma + 2\partial_H\tau.\sigma - \nabla_H(\sigma.\tau)\right] \wedge \omega^2 \\
&= \int_{\partial W_S} \mathrm{tr}\left(\tau.\bar\partial\sigma + \partial_H\tau.\sigma\right) \wedge \omega^2 \underset{S\to\infty}{\longrightarrow} 0
\end{aligned}$$

From (the Calabi-Yau-Tian-Kovalev) Theorem 2.35, *along the tube we have, as in* (2.5),

$$\omega^2|_{\partial W_S} = \kappa_I^2 + O\left(e^{-S}\right).$$

Consequently, as $S \to \infty$, the operation '$. \wedge \omega^2$' annihilates all components of the 1−form $\mathrm{tr}\left(\tau.\bar\partial\sigma + \partial_H\tau.\sigma\right)$ except those transversal to D_z ($|z| = e^{-S}$) in $\partial W_S \simeq D_z \times S^1$:

$$\begin{aligned}
\int_{\partial W_S} \mathrm{tr}\left(\tau.\bar\partial\sigma + \partial_H\tau.\sigma\right) \wedge \omega^2 &= \int_{\partial W_S} \mathrm{tr}\left(\tau.\frac{\partial\sigma}{\partial\bar z}d\bar z + \left((\partial_H\tau)_z.\sigma dz\right) \wedge \left(\kappa_I^2 + 0\left(e^{-S}\right)\right)\right) \\
&= \int_{|z|=e^{-S}} O(|z|) \wedge \left(\kappa_I^2 + 0(|z|)\right) \underset{|z|\to 0}{\longrightarrow} 0.
\end{aligned}$$

*Here we used that $|dz|, |d\bar z| = O\left(e^{-S}\right) = O(|z|)$ [*Lemma 2.34*], while τ and σ also decay exponentially in all derivatives [*Definition 3.64*].* ■

Since $\mathcal{H}_0$ is simply connected, we conclude that

$$\boxed{\mathcal{N}_W = \int_\gamma \rho_W} \tag{3.25}$$

doesn't depend on the choice of path γ.

3.2.3 A conjectured lower bound on 'energy density' via $\mathcal{N}_{D_z}$

In view of the analysis in the next *Section*, one would like to derive, for small enough $|z|$, a time-uniform *lower* bound on the 'energy density' given by the $\omega-$trace of the restriction of curvature $F_{H_t}|_{D_z}$:

$$\widehat{F_{t|z}} \doteq \widehat{F_{H_t}|_{D_z}} = \left(F_{H_t}|_{D_z}\,, \omega|_{D_z}\right).$$

Recalling that $\xi_t \in \Gamma\left(\mathrm{End}\,\mathcal{E}\right)$ is defined by $H_t = H_0 e^{\xi_t}$ (hence is self-adjoint with respect to both metrics), write $\bar{\lambda}_t$ for its highest eigenvalue as in (3.6) and set

$$L_t \doteq \sup_W \bar{\lambda}_t.$$

One would like to prove:

Conjecture 3.71 *For every $t \in \left]0, \infty\right[$, there exists an open set $A_t \subset \tau(W_\infty) \subset \mathbb{C}P^1$ of parameters along the tubular end [cf. (2.1) and (2.3)] such that:*

1. *$\forall z \in A_t$, $|z| < \delta$ as in* Definition 3.40, *i.e., $\mathcal{E}|_{D_z}$ is stable;*

2. *$\forall z \in A_t$,*

$$\boxed{\left\|\widehat{F_{t|z}}\right\|_{L^2(D_z)} \geq \frac{c}{2}\left(1 - \frac{c'}{L_t}\right)};$$

3. *in the measure μ_∞ induced on $\tau(W_\infty)$ by the asymptotically cylindrical metric ω,*

$$\mu_\infty(A_t) = c''\sqrt{L_t}$$

for some $c, c', c'' > 0$ independent of t and z.

Together with a uniform upper 'energy bound' on F_{H_t} over W, this should suffice to establish the time-uniform C^0-bound on $\bar{\lambda}_t$.

At last, the asymptotic stability assumption on $\mathcal{E}$ intervenes via a restriction of the 'norm' functional $\mathcal{N}_W\left(H_t\right)$ just discussed to transversal $K3-$divisors $D_z = \tau^{-1}\left(z\right)$ far enough along the tube. Setting each $\mathcal{N}_{D_z}\left(H_0\right) = 0$, we write [cf. (3.19) and (3.25)]

$$\mathcal{N}_{D_z} \doteq \int_\gamma (\rho)_{D_z} = \int_\gamma \int_{D_z} \theta \wedge \omega \tag{3.26}$$

where the objects involved are the obvious analogues over D_z of those defined over W; e.g., $\gamma(\ell) = H_\ell|_{D_z}$ is a curve of Hermitian metrics on $\mathcal{E}|_{D_z}$. Each D_z being a compact complex surface, the definition of $\mathcal{N}_{D_z}$ is standard [Don$_2$, pp.8-11], and it shares the analogous properties of $\mathcal{N}_W$ discussed in this *Section* (e.g. path-independence), dispensing with asymptotic considerations. Stability then enters the picture by an instance of [Don$_3$, Lemma 24]:

Lemma 3.72 *Suppose $\mathcal{E}|_{D_z}$ is stable with Hermitian Yang-Mills metric $H_0|_{D_z}$; then there exists a [positive] constant c_z such that*

$$\mathcal{N}_{D_z}(H_t) \geq c_z \left(\|\xi_t\|_{L^{\frac{4}{3}}(D_z)} - 1 \right).$$

Intuitively, the argument goes as follows: on a fixed D_z far enough along the tube, the quantity $\mathcal{N}_{D_z}(H_t)$ is controlled, in a certain sense, by the $\omega-$traced restriction of curvature $\widehat{F_{t|z}}$ [*Lemma 3.73* below]. On the other hand, the stability assumption implies that the same $\mathcal{N}_{D_z}(H_t)$ controls $\|\xi_t\|_{L^{\frac{4}{3}}(D_z)}$ [*Lemma 3.72*], so $\xi_t|_{D_z}$ arbitrarily 'big' would imply on $\widehat{F_{t|z}}$ being at least 'somewhat big'. Moreover, if this happens at some z_0 then it must still hold over a 'large' set $A_t \subset \mathbb{C}P^1$ of parameters z, roughly proportional to the supremum $L_t = \|\bar{\lambda}_t\|_{C^0(D_{z_0})}$ [*Conjecture 3.71*].

Adapting the archetypical Chern-Weil technique [cf. *Subsection 1.4.2*], I establish an absolute 'energy bound' on F_{H_t} over W [estimate (3.28) below], so that the set A_t, carrying a 'proportional amount of energy', cannot be too large in the measure μ_∞. Hence the supremum L_t, roughly of magnitude $\mu_\infty(A_t)$, can only grow up to a time-uniform value.

I will start by proving essentially 'half' of *Conjecture 3.71*:

Lemma 3.73 *There exists a constant $c_1 > 0$, independent of t and z, such that*

$$\mathcal{N}_{D_z}(H_t) \leq c_1 \, L_t \, \|\widehat{F_{t|z}}\|_{L^2(D_z)} \qquad \forall t \in \,]0, \infty[\,.$$

Proof *Fixing $t > 0$ and $|z| < \delta$, we simplify notation by $\xi = \xi_t$ and $\|.\| = \|.\|_{L^2(D_z)}$ and consider the curve*

$$\gamma : [0,1] \to \mathcal{H}_0|_{D_z}$$
$$\gamma(\ell) = H_0 e^{\ell \xi}$$

*with $\gamma(1) = H_t$ and $\gamma_\ell^{-1}.\dot{\gamma}_\ell = \xi$. Using the first variation of curvature [*Lemma 3.66*] we obtain $\frac{d}{d\ell} F_\ell = \bar{\partial}\partial_\ell \xi$, with $\partial_\ell \doteq \partial_{H_\ell}$ and $F_\ell \doteq F_{\gamma(\ell)}$.*

Form

$$m(\ell) \doteq \int_0^\ell \int_{D_z} \Phi^\gamma(\ell) \wedge \omega = 2\mathbf{i} \int_0^\ell \left(\int_{D_z} \xi . F_\ell \wedge \omega \right) d\ell$$

so that $m(1) = \mathcal{N}_{D_z}(H_t)$ *and* $m(0) = 0$*; differentiating along* γ *we have*

$$m'(\ell) = 2\mathbf{i} \int_{D_z} \xi . F_\ell \wedge \omega.$$

The function $m(\ell)$ *is in fact convex:*

$$m''(\ell) = 2\mathbf{i} \int_{D_z} \operatorname{tr} \xi . \left(\bar{\partial}_\ell \partial_\ell \xi \right) \wedge \omega = 2 \left\| \partial_\ell \xi \right\|^2 = \left\| \nabla_\ell \xi \right\|^2 \geq 0$$

since ξ *is real, and so* $|\partial_\ell \xi|^2 = \left| \bar{\partial}_\ell \xi \right|^2 = \frac{1}{2} |\nabla_\ell \xi|^2$*. Now, by the mean value theorem, there exists some* $\ell \in [0,1]$ *such that*

$$\begin{aligned} \mathcal{N}_{D_z}(H_t) &= m(1) = m(0) + m'(\ell) \leq m'(1) \leq 2 \int_{D_z} |\xi . F_1 \wedge \omega| \\ &\leq c_1 \; L_t \; \| \widehat{F_1|_{D_z}} \| \end{aligned}$$

using convexity and Cauchy-Schwarz, with $c_1 \doteq 2 \sup_{|z|<\delta} \sqrt{\operatorname{Vol}(D_z)}$ *and* $F_1 = F_{H_t}$*.* ■

Remark 3.74 *Convexity implies that* $\mathcal{N}_{D_z}(H_t) = m(1)$ *is* positive *for all* t*, because* $m'(0) = 0$ *gives an absolute minimum at* $\ell = 0$*, so* $m(1) \geq m(0) = 0$*.*

Finally, let us assume[7], for the sake of argument, that the following statement can be made rigorous:

> "In the terms of *Conjecture 3.71*, there exists a set A_t, 'proportional' to $c''\sqrt{L_t}$, such that
>
> $$\|\xi_t\|_{L^{\frac{4}{3}}(D_z)} \geq c_2 \; L_t \qquad \forall t \in]0, \infty[,$$
>
> where c_2 is independent of t and z."

Then, together with *Lemma 3.72* and *Lemma 3.73*, this would prove *Conjecture 3.71*, with

$$c = \frac{2c_2}{c_1} \left(\inf_{|z|<\delta} c_z \right) \quad \text{and} \quad c' = \frac{1}{c_2}.$$

[7] It can be shown that $\Delta\bar{\lambda}$ is (weakly) uniformly bounded [Don$_3$, p.246], hence the maximum principle suggests that $|\xi_t|$ cannot 'decrease faster' than a certain concave parabola along the cylindrical end.

3.3 Towards time-uniform convergence

Let $\{H_t\}$ be the family of smooth Hermitian metrics on $\mathcal{E} \to W$ given for arbitrary finite time by *Proposition 3.58*. In order to obtain a HYM metric as $H = \lim_{t\to\infty} H_t$ it would suffice to show that $\{H_t\}$ is C^0-bounded, for then it is actually $C^\infty-$bounded on any compact subset and the limit H is smooth [*Lemmas 3.51*, *3.53* and *3.56*]. Concretely, this would mean improving the constant C_T in (3.9) to a time-uniform bound C_∞ or, what is the same, controlling the sequence $\bar{\lambda}_t$ of highest eigenvalues of $\xi_t = \log H_0^{-1} H_t$ [cf. (3.6)]:

$$\boxed{\|\bar{\lambda}_t\|_{C^0(W)} \leq C_\infty.} \tag{3.27}$$

I will show that this task reduces essentially to *Conjecture 3.71*, as the problem (3.27) amounts in fact to controlling the size of the set A_t where the 'energy density' $\widehat{F_{t|z}}$ is bigger than a definite constant. I begin by stating the announced upper bound:

$$E(t) \doteq \int_W \left(|F_{H_t}|^2 - |F_{H_0}|^2\right) d\mathrm{Vol}_\omega \quad \leq \quad 0 \qquad \forall t \in]0, \infty[\,. \tag{3.28}$$

The curvature of a Chern connection splits orthogonally as $F = \hat{F}.\omega \oplus F^\perp$ in $\Omega^{1,1}(\mathrm{End}\,\mathcal{E})$, so $|F|^2 = \left|F^\perp\right|^2 + \left|\hat{F}\right|^2$ (setting $|\omega| = 1$). On the other hand, the Hodge-Riemann equation (*A.2*) reads

$$\mathrm{tr}\, F^2 \wedge \omega = \left(\left|F^\perp\right|^2 - \left|\hat{F}\right|^2\right) \omega^3.$$

Comparing we find $|F|^2 \omega^3 = \mathrm{tr}\, F^2 \wedge \omega + 2\left|\hat{F}\right|^2 \omega^3$ [cf. *Subsection 1.4.2*], so

$$\begin{aligned}
E(t) \quad &= \quad \int_W \left(\mathrm{tr}\, F_{H_t}^2 - \mathrm{tr}\, F_{H_0}^2\right) \wedge \omega + 2\int_W (\hat{e}_t - \hat{e}_0)\, \omega^3 \\
\overset{\frac{d}{dt}}{\Longrightarrow} \quad \dot{E}(t) \quad &\leq \quad \int_W \left(-\mathbf{i}\bar{\partial}\partial\Phi_t\right) \wedge \omega + 2\int_W (-\Delta \hat{e}_t)\, \omega^3 \\
&\leq \quad \lim_{S\to\infty} \int_{\partial W_S} 2\left[\partial \mathrm{tr}\left(\hat{F}_{H_t}.F_{H_t}\right) \wedge \omega + \frac{\partial \hat{e}_t}{\partial \nu}\, d\mathrm{Vol}_\omega|_{\partial W_S}\right] \\
&= \quad 0
\end{aligned}$$

using *Corollary 3.67* along with its *Proof* and $\left(\frac{d}{dt} + \Delta\right)\hat{e}_t \leq 0$ as in (3.5), then complex integration by parts [*Lemma A.81*] and the Gauss-Ostrogradsky theorem, and finally the exponential decay $\hat{F}_{H_t} \xrightarrow[S\to\infty]{C^\infty} 0$, a direct consequence from *Proposition 3.59* and *Corollary 3.62*. Since obviously $E(0) = 0$, this proves the upper bound (3.28).

From now on I will write, in cylindrical coordinates, D_s for D_z when $|z| = e^{-s}$. Reasoning as above we find, for the curvature of a Chern connection on $\mathcal{E}|_{D_s}$,

$$|F|^2 \omega^2 = \operatorname{tr} F^2 + 2\left|\hat{F}\right|^2 \omega^2. \tag{3.29}$$

Also recall, for immediate use, that

$$\begin{aligned} d\mathrm{Vol}_\omega = \tfrac{1}{6}\omega^3 &= \tfrac{1}{2} ds \wedge d\alpha \wedge (\kappa_I + d\psi)^2 + \widetilde{d\psi} \\ d\mathrm{Vol}_\omega|_{D_s} &= \tfrac{1}{2}\left(\omega|_{D_s}\right)^2 = \tfrac{1}{2}(\kappa_I + d\psi)^2\Big|_{D_s}. \end{aligned}$$

with $\widetilde{d\psi} = O\left(e^{-s}\right)$. In the terms of *Conjecture 3.71*, denote $\Sigma_t \doteq A_t \times D_s$ the finite cylinder $\tau^{-1}(A_t)$ along the tubular end W_∞. Then, isolating the component of curvature along D_s, the Hodge-Riemann property (3.29) gives a lower estimate on the curvature over Σ_t:

$$\begin{aligned} \int_{\Sigma_t} |F_t|^2 \left\{ d\mathrm{Vol}_\omega - \widetilde{d\psi} \right\} &\geq \int_{\Sigma_t} \left|F_{t|s}\right|^2 \left\{ d\mathrm{Vol}_\omega - \widetilde{d\psi} \right\} \\ &= \int_{A_t} \left\{ \int_{D_s} \left|F_{t|s}\right|^2 d\mathrm{Vol}_\omega|_{D_s} \right\} ds \wedge d\alpha \\ &\geq \int_{A_t} \left\{ \left\langle c_2\left(\mathcal{E}|_{D_s}\right), [D_s] \right\rangle + 2\int_{D_s} |\widehat{F_{t|s}}|^2 \, d\mathrm{Vol}_\omega|_{D_s} \right\} ds \wedge d\alpha. \end{aligned}$$

On the other hand, the asymptotia of F_0 (3.14) give $|F_0|^2 = \left|F_{0|s}\right|^2 + R_0$ over W_∞, where the remainder is obviously positive and satisfies $R_0 = O\left(e^{-s}\right)$, so

$$\begin{aligned} \int_{\Sigma_t} \left(|F_0|^2 - R_0\right) \left\{ d\mathrm{Vol}_\omega - \widetilde{d\psi} \right\} &= \int_{A_t} \left\{ \int_{D_s} \left|F_{0|s}\right|^2 d\mathrm{Vol}_\omega|_{D_s} \right\} ds \wedge d\alpha \\ &= \int_{A_t} \mathrm{YM}_{D_S}(H_0)\, ds \wedge d\alpha \end{aligned}$$

and, by *Definition 3.41*, $\mathrm{YM}_{D_S}(H_0) = \left\langle c_2\left(\mathcal{E}|_{D_s}\right), [D_s] \right\rangle$. Comparing and cancelling the topological terms:

$$\int_{\Sigma_t} \left(|F_t|^2 - |F_0|^2 + R_0\right) \left\{ d\mathrm{Vol}_\omega - \widetilde{d\psi} \right\} \geq 2 \int_{A_t} \|\widehat{F_{t|s}}\|^2_{L^2(D_s)} ds \wedge d\alpha. \tag{3.30}$$

This discussion culminates at the following result:

Proposition 3.75 *Under the hypotheses of* Conjecture 3.71*, there exists a constant* C_∞*, independent of* t *and* z*, such that*

$$L_t \le C_\infty, \quad \forall t \in \,]0,\infty[\,.$$

Proof *Since* $\mu_\infty(A_t) = c''\sqrt{L_t}$*, then either* L_t *is uniformly bounded in* t *and there is nothing to prove, or (a subsequence of) the sequence of sets* $\{A_t\}_{0<t<\infty}$ *gets arbitrarily* $\mu_\infty-$*large as* $t \to \infty$*. In the latter case, the factor of* c' *in* Conjecture 3.71 *becomes negligible, as does the* $\widetilde{d\psi}$ *term in* (3.30) *along the tube* W_∞*, so:*

$$\begin{aligned} c.\mu_\infty(A_t) &\le \int_{\Sigma_t}\left(|F_t|^2 - |F_0|^2 + R_0\right)\left\{d\mathrm{Vol}_\omega - \widetilde{d\psi}\right\} \\ &\overset{t\gg 0}{\simeq} \int_{\Sigma_t}\left(|F_t|^2 - |F_0|^2 + R_0\right) d\mathrm{Vol}_\omega. \end{aligned}$$

That integral, in turn, approaches the tail of $E(t) + \|R_0\|_{L^1(W)}$ *and we know from the negative energy condition* (3.28) *that this is bounded above, uniformly in* t*. Hence the tail cannot grow indefinitely and, by contradiction, there must exist* $C_\infty \simeq \left(\frac{\|R_0\|_{L^1(W)}}{c.c''}\right)^2$ *yielding the statement.* ∎

Finally, replacing the uniform bound for C_T in (3.9), this control cascades into the exponential C^0-decay in (3.15) and hence the $C^\infty-$decay in *Corollary 3.62*. We have thus proved the following reduction of the HYM problem on $\mathcal{E} \to W$:

Theorem 3.76 *Let* $\mathcal{E} \to W$ *be stable at infinity, equipped with a reference metric* H_0*, over an asymptotically cylindrical* $SU(3)-$*manifold* W *as given by the Calabi-Yau-Tian-Kovalev* Theorem 2.35*, and let* $\left\{H_t = H_0 e^{\xi_t}\right\}$ *be the* $1-$*parameter family of Hermitian metrics on* $\mathcal{E}$ *given by* Theorem 3.63*; then* Conjecture 3.71 *implies that the limit* $H = \lim_{t\to\infty} H_t$ *exists and is a smooth Hermitian Yang-Mills metric on* $\mathcal{E}$*, exponentially asymptotic in all derivatives*[8] *to* H_0 *along the tubular end of* W*:*

$$\boxed{\hat{F}_H = 0, \quad H \xrightarrow[S\to\infty]{C^\infty} H_0.}$$

[8] cf. *Notation 3.61.*

CHAPTER 4
CONSTRUCTION OF AN ASYMPTOTICALLY STABLE BUNDLE

It is fair to ask whether there are any holomorphic bundles at all satisfying the asymptotic stability conditions of *Definition 3.40*, thus providing concrete instances for the analysis in *Chapter 3*. Fixing the base manifold[1] $\bar{W} = \mathrm{Bl}_C \mathbb{P}^3$ [cf. *Example 2.36*], we will see that the pull-back of the null-correlation bundle on $\mathbb{P}^3$ [O-S-S][Bar] gives exactly such a case.

4.1 Null-correlation bundles

Let $n \in \mathbb{Z}$ be odd and take a holomorphic symplectic $(2,0)-$form ω relative to the standard complex structure on $\mathbb{C}^{n+1} \cong \mathbb{H}^{\frac{n+1}{2}}$. This defines a *null-correlation* [Bar, §7.1], i.e., an isomorphism

$$\begin{array}{ccc} \mathbb{P}^n & \to & (\mathbb{P}^n)^* \\ l & \mapsto & E_l \end{array}$$

such that $l \subset E_l$, $\forall l \in \mathbb{P}^n$, where $(\mathbb{P}^n)^*$ denotes the (projective) space of hyperplanes through the origin in $\mathbb{C}^{n+1}$. Specifically, such (non-degenerate) ω defines the $n-$planes

$$E_l = l^\omega \doteq \left\{ u \in \mathbb{C}^{n+1} \mid (l \lrcorner \omega)(u) = 0 \right\}$$

and, forming the linear quotients E_l/l, one obtains a $(n-1)-$plane bundle $N \to \mathbb{P}^n$.

4.1.1 The bundle N as kernel of a surjective map

Following [O-S-S, pp. 76-81], we will now see that the graph of the null-correlation in $\mathbb{P}^n \times (\mathbb{P}^n)^*$ determines precisely a non-vanishing, $\mathcal{O}(2)-$valued section of $\Omega^1_{\mathbb{P}^n} = (T_{\mathbb{P}^n})^*$. Equivalently, the corresponding (holomorphic) *null-correlation bundle* $N \to \mathbb{P}^n$ can be seen

[1] we denote henceforth $\mathbb{P}^n \doteq \mathbb{C}P^n$.

as the kernel of a surjective bundle map

$$T_{\mathbb{P}^n}(-1) \overset{s}{\twoheadrightarrow} \mathcal{O}(1). \tag{4.1}$$

For every point $l \in \mathbb{P}^n$, seen as a line in $\mathbb{C}^{n+1}$, contraction with ω defines an $n-$plane $E_l = l^\omega$. Clearly $l \subset l^\omega$, so l^ω cuts out a hyperplane in the tangent space $(T_{\mathbb{P}^n})_l$, which corresponds (as its null-space) to a line in $\left(\Omega^1_{\mathbb{P}^n}\right)_l$. Such data, for all $l \in \mathbb{P}^n$, determine a holomorphic line subbundle[2] of $\Omega^1_{\mathbb{P}^n}$. Since Pic $(\mathbb{P}^n) = \mathbb{Z}$, this must be of the form

$$\mathcal{O}(k) \overset{s}{\hookrightarrow} \Omega^1_{\mathbb{P}^n}$$

for some $k \in \mathbb{Z}$. To see that in fact $k = -2$, notice that the above inclusion defines a non-vanishing section $s \in H^0\left(\Omega^1_{\mathbb{P}^n}(-k)\right)$, therefore

$$c_n\left(\Omega^1_{\mathbb{P}^n}(-k)\right) = 0.$$

Applying Cartan's formula for the total Chern class of a bundle splitting to the $\mathcal{O}(1)-$twisted Euler sequence of $\mathbb{P}^n$, we find $c_i(\mathbb{P}^n) = \binom{n+1}{i}$, so:

$$\begin{aligned} 0 &= c_n\left(\Omega^1_{\mathbb{P}^n}(-k)\right) = -c_n\left(T_{\mathbb{P}^n}(k)\right) \\ &= -\sum_{i=0}^{n} \binom{n+1}{i} k^{n-i}. \end{aligned}$$

Certainly $k \neq 0$, for otherwise $\mathcal{O} \hookrightarrow \Omega^1_{\mathbb{P}^n}$ would give a non-vanishing section, contradicting $c_n\left(\Omega^1_{\mathbb{P}^n}\right) = -(n+1) \neq 0$; thus we can meaningfully multiply the above equation by k:

$$\begin{aligned} & (1+k)^{n+1} = 1 \\ \Leftrightarrow \quad & k = -2 \quad (\text{or } k = 0). \end{aligned}$$

Hence the non-vanishing section $s \in H^0\left(\Omega^1_{\mathbb{P}^n}(2)\right)$ defines a bundle map of the form (4.1).

[2] as we know from the Euler sequence, the fibre of $T_{\mathbb{P}^n} = \mathrm{Hom}\left(\mathcal{O}(-1), \mathcal{O}^{\oplus(n+1)}/\mathcal{O}(-1)\right)$ at l consists of 'linear deformations' of the form $v : l \to \mathcal{O}^{\oplus(n+1)}/l$, so a hyperplane in $(T_{\mathbb{P}^n})_l$ can be regarded as $E \doteq \mathrm{span}\left\langle v_1, \ldots, \hat{i}, \ldots, v_n\right\rangle \simeq \mathrm{span}\left\langle l, \mathrm{img}(v_1), \ldots, \hat{i}, \ldots, \mathrm{img}(v_n)\right\rangle \subset \mathbb{C}^{n+1}$ for some basis $\{v_i\}$. All such E form a bundle

$$\mathbb{P}\left(\Omega^1_{\mathbb{P}^n}\right) \simeq \{(l, E) \subset \mathbb{P}^n \times (\mathbb{P}^n)^* \mid l \subset E\} \to \mathbb{P}^n.$$

4.1.2 Properties and geometric interpretation

From the short exact sequence

$$0 \longrightarrow N \longrightarrow T_{\mathbb{P}^n}(-1) \longrightarrow \mathcal{O}(1) \longrightarrow 0 \tag{4.2}$$

we read, again by Cartan's formula, the Chern classes of N:

$$c_1(N) = 0, \qquad c_2(N) = 1. \tag{4.3}$$

Furthermore, N is *simple* [O-S-S, pp.77-78], hence irreducible. Indeed, from the long exact sequence associated to (4.2), one finds $h^0(N) = 0$ and so N is *stable* [O-S-S, p.180].

For some geometric intuition, let us restrict attention to the case $n = 3$. A point $l \in \mathbb{P}^3$ corresponds to a line through the origin in $\mathbb{C}^4$, so it determines a $3-$dimensional $\omega-$null space l^ω containing l as a subspace. Dualising (4.2), we get

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \Omega^1_{\mathbb{P}^n}(1) \longrightarrow N^* \longrightarrow 0$$

where we may think of the fibre of N^* over l as the (self-dual) $2-$plane given by the quotient

$$N_l^* \simeq \frac{l^\omega}{l} \simeq N_l \in \mathrm{Gr}_{\mathbb{C}}(2,4).$$

4.2 Asymptotic stability of $\tilde{N} \to \tilde{\mathbb{P}}^3$

Having seen that N is irreducible, our aim in this *Subsection* is to prove that $\tilde{N} = \sigma^* N \to \tilde{\mathbb{P}}^3$ satisfies the asymptotic stability condition of *Definition 3.40*:

Proposition 4.77 *The pull-back $\tilde{N} = \sigma^* N$ of the null-correlation bundle $N \to \mathbb{P}^3$ by the blow-up (2.9) is stable at infinity.*

Here our working definition of (Mumford-Takemoto or slope-) stability is [D-K, §6.1.1]:

Definition 4.78 *A holomorphic $SL(2,\mathbb{C})$-bundle $\mathcal{E}$ over a compact Kähler surface (D,ω) is* stable *if and only if any $\mathcal{F} \in \mathrm{Pic}(D)$ receiving a nontrivial holomorphic map $\mathcal{E} \to \mathcal{F}$ has strictly positive slope:*

$$\mu(\mathcal{F}) \doteq \langle c_1(\mathcal{F}) \smile [\omega], [D]\rangle > 0.$$

As $\sigma : \tilde{D} \to D$ is an isomorphism, it suffices to show stability for $N|_D$. In view of (4.3),

$$c_1\left(N|_D\right) = 0,$$

so $N|_D$ is topologically trivial; it is, in fact, holomorphically trivial, since $H^{0,1}(D) = 0$ implies the Jacobian is trivial, thus indeed $N|_D$ is an $SL(2,\mathbb{C})-$bundle. Moreover, as Pic $(D) = \mathbb{Z}$ [cf. (2.11)], line bundles on D are classified by their degree. Hence we are in position to apply a simple stability criterion derived from [O-S-S, p.165]:

Lemma 4.79 *A rank 2 vector bundle $\mathcal{E} \to X$ with $c_1(\mathcal{E}) = 0$ and* Pic $(X) = \mathbb{Z}$ *is stable if and only if it has no sections.*

Proving *Proposition 4.77* reduces therefore to showing that $h^0\left(N|_D\right) = 0$. Since D is cut-out by a section s of $K_{\mathbb{P}^3}^{-1} = \mathcal{O}(4)$, the restriction of N to D induces the exact sequence

$$0 \longrightarrow N(-4) \longrightarrow N \longrightarrow N|_D \longrightarrow 0.$$

Considering the associated long exact sequence in cohomology, I contend

$$h^0(N(-4)) = h^1(N(-4)) = 0, \tag{4.4}$$

which implies $h^0\left(N|_D\right) = h^0(N) = 0$, as desired. To check (4.4), we write the $\mathcal{O}(-4)-$twisted defining sequence of N [cf. (4.2)] over $\mathbb{P}^3$:

$$0 \longrightarrow N(-4) \longrightarrow T_{\mathbb{P}^3}(-5) \longrightarrow \mathcal{O}(-3) \longrightarrow 0.$$

On one hand, since $K_{\mathbb{P}^3} = \mathcal{O}(-4)$, Serre duality [Huy, Cor. 4.1.16, p.171] and Bott's formula [O-S-S, p.8] give:

$$\begin{aligned} H^0\left(T_{\mathbb{P}^3}(-5)\right) &= H^0\left(\Omega^1_{\mathbb{P}^3}(1)^* \otimes K_{\mathbb{P}^3}\right) \\ &\simeq H^2\left(\Omega^1_{\mathbb{P}^3}(1)\right)^* = \{0\} \end{aligned}$$

so $h^0(N(-4)) = 0$. Similarly,

$$H^1\left(T_{\mathbb{P}^3}(-5)\right) \simeq H^1\left(\Omega^1_{\mathbb{P}^3}(1)\right)^* = \{0\}$$

and, obviously, $H^0\left(\mathcal{O}(-3)\right) = \{0\}$. Thus $h^1(N(-4)) = 0$ and *Proposition 4.77* is proved.

APPENDIX A
BACKGROUND MATERIAL

A.1 Special holonomy

I cite *Berger's theorem* from [Joy, Th.3.4.1, p.55]; the original reference is [Ber, Th.3, p.318].

Theorem A.80 (Berger) *Suppose* (M, g) *is a simply-connected Riemannian* $n-$*manifold such that* g *is irreducible and nonsymmetric; then exactly one of the following holds:*

- $\mathrm{Hol}\,(g) = SO\,(n)$
- $n = 2m, m \geq 2$ $\quad$ $\mathrm{Hol}\,(g) = U\,(m) \subset SO\,(2m)$
- $n = 2m, m \geq 2$ $\quad$ $\mathrm{Hol}\,(g) = SU\,(m) \subset SO\,(2m)$
- $n = 4m, m \geq 2$ $\quad$ $\mathrm{Hol}\,(g) = Sp\,(m) \subset SO\,(4m)$
- $n = 4m, m \geq 2$ $\quad$ $\mathrm{Hol}\,(g) = Sp\,(m)\,Sp\,(1) \subset SO\,(4m)$
- $n = 7, m \geq 2$ $\quad$ $\mathrm{Hol}\,(g) = G_2 \subset SO\,(7)$
- $n = 8, m \geq 2$ $\quad$ $\mathrm{Hol}\,(g) = \mathrm{Spin}\;(7) \subset SO\,(8)\,.$

A.2 Bundle-valued differential forms

A.2.1 Inner product

Let G be a semi-simple Lie group, $E \to M$ a principal $G-$bundle over a smooth Riemannian $n-$manifold M; we denote $\mathfrak{g} = \mathrm{Lie}\;(G)$ and $\mathfrak{g}_E \simeq \mathrm{End}\,E$.

We use throughout the inner product of endomorphism-valued $p-$forms $\alpha, \beta \in \Omega^p\,(\mathfrak{g}_E)$

$$(\alpha, \beta) = \int_M \langle \alpha \wedge *\beta \rangle_{\mathfrak{g}}\,, \tag{A.1}$$

where $\langle .,. \rangle$ denotes the Killing metric on $\mathfrak{g}$ acting on the endomorphism part of the composition $\alpha \wedge *\beta \in \Omega^n\,(\mathfrak{g}_E)$ and the standard inner product (from the Riemannian metric on M) acting on the differential form part. In local coordinates, let $\alpha = \alpha_I \otimes dx^I$ and

$\beta = \beta_I \otimes dx^I$, with $\alpha_I, \beta_I \in \mathfrak{g}$; then

$$\langle \alpha \wedge *\beta \rangle_{\mathfrak{g}} =_I \langle \alpha_I, \beta_I \rangle_{\mathfrak{g}} \, d\mu$$

where $d\mu$ is the volume form.

In particular, the curvature of a connection A is formally written as $F_A = dA + A \wedge A$, which means locally:

$$F_A = \frac{1}{2} \left(dx^i \wedge dx^j \right) \otimes (F_A)_{ij} \in \Omega^2 \left(\mathfrak{g}_E \right)$$

with $(F_A)_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]$. In those terms we define, pointwise,

$$|F_A|^2 = \sum_{i,j}^{n} \left| (F_A)_{ij} \right|_{\mathfrak{g}}^2 = |F_A|_{\mathfrak{g}}^2 .$$

A.2.2 Complex integration by parts on manifolds with boundary

Lemma A.81 (Integration by parts) *Let $X^n \subseteq W$ be a compact complex (sub)manifold (possibly $n = 3$), Φ a $(1,1)$ −form, Ω a closed $(n-2, n-2)$ −form and f a meromorphic function on X; then*

$$\int_X \Phi \wedge dd^c f \wedge \Omega = \int_X f. \left(-\mathbf{i} \bar{\partial} \partial \Phi \right) \wedge \Omega + \mathbf{i} \int_{\partial X} \left(\Phi \wedge \bar{\partial} f + f. \partial \Phi \right) \wedge \Omega.$$

Proof *By the Leibniz rule and Stokes' theorem, using $d = \partial + \bar{\partial}$ and taking account of bi-degree, we have*

$$\begin{aligned} \int_X \Phi \wedge dd^c f \wedge \Omega &= \int_X \Phi \wedge \mathbf{i} \partial \bar{\partial} f \wedge \Omega \\ &= \mathbf{i} \int_{\partial X} \Phi \wedge \bar{\partial} f \wedge \Omega - \underbrace{\int_X \mathbf{i} \partial \Phi \wedge \bar{\partial} f \wedge \Omega}_{(*)} \end{aligned}$$

and again

$$(*) = \mathbf{i} \int_{\partial X} f. \partial \Phi \wedge \Omega + \int_X f. \left(-\mathbf{i} \bar{\partial} \partial \Phi \right) \wedge \Omega.$$

■

A.2.3 The Hodge-Riemann bilinear relation

The curvature on a Kähler $n-$fold splits as $F = \hat{F}.\omega \oplus F^{\perp} \in \Omega^{1,1}(\operatorname{End}\mathcal{E})$, so:

$$F^2 \wedge \omega^{n-2} = \hat{F}^2.\omega^n + \left(F^{\perp}\right)^2 \wedge \omega^{n-2}.$$

The Hodge-Riemann pairing $(\alpha, \beta) \longmapsto \alpha \wedge \beta \wedge \omega$ on $\Omega^{1,1}(W)$ is positive-definite along ω and negative-definite on the primitive forms in $\langle\omega\rangle^{\perp}$ [Huy, pp.39-40] (with respect to the reference Hermitian bundle metric); since the curvature F is real as a bundle-valued $2-$form, we have:

$$\operatorname{tr} F^2 \wedge \omega^{n-2} = \left(\left|F^{\perp}\right|^2 - \left|\hat{F}\right|^2\right)\omega^n \tag{A.2}$$

using that $\operatorname{tr}\xi^2 = -|\xi|^2$ on the Lie algebra part.

A.3 Banach analysis

A.3.1 The implicit function theorem

The following two theorems are taken from [D-K, Appendix I]. Recall that a map $f : U \subset E \to F$ between Banach spaces is *differentiable* at $p \in U$ if there is a bounded linear functional $(Df)_p : E \to F$ such that

$$\lim_{h\to 0} \frac{f(p+h) - f(p) - (Df)_p h}{\|h\|} = 0.$$

If f is differentiable for all p then $Df : E \to \operatorname{Hom}(E, F)$ is a map between Banach spaces and we can iterate the definition to C^r and C^∞ maps.

Theorem A.82 (Implicit function theorem in Banach spaces) *Let $E = E_1 \times E_2$ be a product of Banach spaces and $f : E \to F$ a smooth map such that $f\left(\widehat{\xi_1}, \widehat{\xi_2}\right) = 0$ for some fixed $\left(\widehat{\xi_1}, \widehat{\xi_2}\right)$; if $(D_2 f)_{\left(\widehat{\xi_1}, \widehat{\xi_2}\right)} : E_2 \to F$ is an isomorphism, then there exist open neighbourhoods $\widehat{\xi_1} \in U \subset E_1$ and $\widehat{\xi_2} \in V \subset E_2$ and a smooth $h : U \to V$ such that*

$$f(\xi_1, h(\xi_1)) = 0, \quad \forall \xi_1 \in U.$$

Moreover, for $(\xi_1, \xi_2) \in U \times V$, $f(\xi_1, \xi_2) = 0 \Rightarrow \xi_2 = h(\xi_1)$.

Theorem A.83 (Inverse function theorem) *Let $f : E \to F$ be a smooth map between Banach spaces; if $(Df)_{\widehat{\eta}}$ is surjective and admits a bounded right inverse, then there exist open neighbourhoods W and $\widetilde{W}$ of $\widehat{\eta}$ in E and a local homeomorphism $g : W \to \widetilde{W}$ such that*

$$f(\eta) = \left[(Df)_{\widehat{\eta}} \circ g\right](\eta), \quad \forall \eta \in W.$$

In particular, f is right-invertible on $\widetilde{W}$ and, for θ near $f(\eta)$, the equation $f(\eta) = \theta$ has a solution η near $\widehat{\eta}$.

We want to think about equations on E_2 parameterised by E_1 of the form

$$\Psi_{\xi_1}(\xi_2) \doteq f(\xi_1, \xi_2) = 0. \tag{A.3}$$

The following *Corollary* sets sufficient conditions to perturb ($A.3$) with respect to the parameter ξ_1 keeping solutions under control:

Corollary A.84 *If $f\left(\widehat{\xi}_1, \widehat{\xi}_2\right) = 0$ and $(D_2 f)_{\left(\widehat{\xi}_1, \widehat{\xi}_2\right)}$ is surjective and admits a bounded right inverse then, for every ξ_1 near $\widehat{\xi}_1$, there is a solution ξ_2 to the (perturbed) equation*

$$\Psi_{\xi_1}(\xi_2) = 0.$$

Moreover, ξ_2 is near $\widehat{\xi}_2$ for ξ_1 sufficiently near $\widehat{\xi}_1$.

Proof *For fixed ξ_1, the derivative at ξ_2 of $\Psi_{\xi_1} : E_2 \to F$ is:*

$$\begin{aligned} & (D\Psi_{\xi_1})_{\xi_2}(v) = (Df)_{(\xi_1,\xi_2)}(0, v) = (D_2 f)_{(\xi_1,\xi_2)}(v) \\ \overset{(\text{Theorem A.83})}{\Rightarrow} \quad & \exists \xi_2 \in (\Psi_{\xi_1})^{-1}(0). \end{aligned}$$

Moreover, $\xi_2 = h(\xi_1)$ [Theorem A.82], so ξ_2 varies smoothly with ξ_1. ■

A.3.2 Ellipticity of the dual complex

In this *Subsection* and the next one I collect a few results that are used mostly in the proofs of *Proposition 1.22* and *Proposition 1.23*.

Lemma A.85 *For a sequence* $A \overset{S}{\to} B \overset{T}{\to} C$ *of bounded linear operators on Hilbert spaces:*

$$\ker T = \operatorname{img} S \Leftrightarrow \ker S^* = \operatorname{img} T^*.$$

Proof *I claim* $B = \operatorname{img} S \oplus \ker S^*$*:*

$$\begin{aligned} b \in (\operatorname{img} S)^{\perp} \subset B \quad &\Leftrightarrow \quad \langle b, Sa \rangle = 0, \quad \forall a \in A \\ &\Leftrightarrow \quad \langle S^* b, a \rangle = 0, \quad \forall a \in A \\ &\Leftrightarrow \quad b \in \ker S^*. \end{aligned}$$

Since S^* *is continuous (as it is bounded),* $\ker S^* \subset B$ *is a closed subspace and so* $B = (\ker S^*)^{\perp} \oplus \ker S^*$*.*

Mutatis mutandis, $B = \ker T \oplus \operatorname{img} T^*$, *so:*

$$\begin{array}{ccccc} B & = & \operatorname{img} S & \oplus & \ker S^* \\ & & \| & \Leftrightarrow & \| \\ & = & \ker T & \oplus & \operatorname{img} T^* \end{array}$$

by the uniqueness of the orthogonal complement. ∎

Corollary A.86 *Let* $F \overset{L_1}{\to} G \overset{L_2}{\to} H$ *be a complex of differential operators between vector bundles over a compact manifold* M *with respective inner products on the fibres; if the associated symbols satisfy* $\sigma(L_i^*) = (\sigma(L_i))^*$*, then*

$$F \overset{L_1}{\to} G \overset{L_2}{\to} H \text{ is elliptic } \Leftrightarrow H \overset{L_2^*}{\to} G \overset{L_1^*}{\to} F \text{ is elliptic.}$$

Proof *Apply the* Lemma *to the associated sequence of symbol maps*

$$\pi^* F \overset{\sigma(L_1)}{\longrightarrow} \pi^* G \overset{\sigma(L_2)}{\longrightarrow} \pi^* H.$$

∎

A.3.3 Irreducible connections have no constant sections

The following is a slight improvement of [Fr-U, 3.1, p.54]:

Theorem A.87 *Let $E \to M$ be an $SU(n)-$bundle over a compact Riemannian manifold M and A a non-flat connection; then*

$$A \textit{ is irreducible} \Rightarrow \ker\left(d_A : \Omega^0(\mathfrak{g}_E) \to \Omega^1(\mathfrak{g}_E)\right) = \{0\}.$$

Proof *Suppose $0 \neq f \in \ker d_A \subset \Omega^0(\mathfrak{g}_E$. Since $\mathfrak{g}_E = \mathfrak{su}(n)$ are the traceless skew-symmetric matrices, in a local trivialisation over $U \subset M$ the section f is pointwise diagonalisable and admits a local basis of smooth (unit) eigenvectors $\left\{e^i \in \Gamma(U)\right\}$:*

$$\begin{cases} fe^i &= \left(\mathbf{i}\lambda^i\right).e^i \\ \left\|e^i\right\| &= 1 \end{cases} \tag{A.4}$$

where the λ^i are smooth real functions on U such that $\left(\sum \lambda^i\right)|_U = 0$. I claim that the λ^i are constant; to see this, apply d_A to equations (A.4):

$$\begin{cases} \underbrace{(d_A f)}_{0}e^i + f\left(d_A e^i\right) &= \mathbf{i}\left(d\lambda^i\right)\otimes e^i + \left(\mathbf{i}\lambda^i\right).d_A e^i \\ \mathrm{Re}\left\langle d_A e^i, e^i\right\rangle &= 0. \end{cases} \tag{A.5}$$

Taking the inner product with e^i and then the imaginary part in the first line gives:

$$\begin{aligned} d\lambda^i &= \mathrm{Im}\left\langle f\left(d_A e^i\right), e^i\right\rangle - \lambda^i \underbrace{\mathrm{Re}\left\langle d_A e^i, e^i\right\rangle}_{0} \\ &= -\mathrm{Im}\left\langle d_A e^i, fe^i\right\rangle = -\lambda^i \mathrm{Re}\left\langle d_A e^i, e^i\right\rangle = 0. \end{aligned} \tag{A.6}$$

Hence the functions λ^i are constant and the corresponding eigenspaces are globally defined. Since $\sum \lambda^i = 0$, the $\left\{e^i\right\}$ split the bundle E into at least two distinct (distributions of) eigenspaces. Furthermore, replacing (A.6) in (A.5) shows that d_A preserves each eigenspace:

$$f\left(d_A e^i\right) = \left(\mathbf{i}\lambda^i\right).d_A e^i$$

which leads to a contradiction when A is irreducible. ■

A.3.4 Bound on the norm of the multiplication operator

The next result, from [Fin], is essentially the generalisation to Banach spaces of the fact that the determinant of a linear map is continuous.

Lemma A.88 *Let $D : B_1 \to B_2$ be a bounded invertible linear map of Banach spaces with bounded inverse Q. If $L : B_1 \to B_2$ is another linear map with*

$$\|L - D\| \leq (2\|Q\|)^{-1},$$

then L is also invertible with bounded right inverse P satisfying

$$\|P\| \leq 2\|Q\|.$$

For its application in the proof of *Proposition 1.23* we are also going to need the following *Lemma*, saying that the norm of the operator "multiplication by a function" on L^2 (in fact, a similar claim holds for general L^p) is controlled by a suitable Sobolev norm of that function.

Lemma A.89 *On a compact $n-$manifold M, fix $f \in L_k^2(M)$ with $k \geq \frac{n}{2}$; then there exists a constant $c = c(M)$ such that*

$$\|L_f\| \leq c\|f\|_{L_k^2},$$

where

$$\begin{array}{rcl} L_f : L^2(M) & \to & L^2(M) \\ g & \mapsto & f.g \end{array}.$$

Proof *This is an immediate consequence of Sobolev's embedding theorem, since $\|f.g\|_{L^2} = \left(\int_M |f|^2 . |g^2| \, d\mu\right)^{\frac{1}{2}} \leq \|f\|_{C^0} . \|g\|_{L^2}$ and so*

$$\|L_f\| = \sup_{\|g\|_{L^2}=1} \|f.g\|_{L^2} \leq \|f\|_{C^0}.$$

■

A.4 Gaussian upper bounds for the heat kernel

The following instance of [Gri, Theorem 1.1] stems from a long series[1] of generalised 'Gaussian' upper bounds (i.e., given by a Gaussian exponential on the geodesic distance r) for the heat kernel K_t of a Riemannian manifold.

Theorem A.90 *Let M be an arbitrary connected Riemannian $n-$manifold, $x, y \in M$ and $0 \leq T \leq \infty$; if there exist suitable [see below] real functions f and g satisfying the 'diagonal' conditions*

$$K_t(x,x) \leq \frac{1}{f(t)} \quad \textit{and} \quad K_t(y,y) \leq \frac{1}{g(t)}, \quad \forall t \in (0,T),$$

then, for any $C > 4$, there exists $\delta = \delta(C) > 0$ such that

$$\boxed{K_t(x,y) \leq \frac{(cst.)}{\sqrt{f(\delta t) g(\delta t)}} \exp\left\{-\frac{r(x,y)^2}{Ct}\right\}, \quad \forall t \in (0,T)}$$

where $(cst.)$ depends on M and its metric only.

For all purposes in the present text one may assume simply $f(t) = g(t) = t^{\frac{n}{2}}$, but in fact f and g can be *much* more general, provided they have subpolynomial or superpolynomial growth in the sense of the following conditions [Op. cit. p.37]:

- $f, g : (0,T) \to \mathbb{R}^+$ are monotonically increasing;
- $\exists\ A, B \geq 1,\ \alpha, \beta > 1$ such that, $\forall\ 0 < t_1 < t_2$,

$$\frac{f(\alpha t_1)}{f(t_1)} \leq A \frac{f(\alpha t_2)}{f(t_2)} \quad \text{and} \quad \frac{g(\beta t_1)}{g(t_1)} \leq B \frac{g(\beta t_2)}{g(t_2)}.$$

[1] going back to J. Nash (1958) and D. Aronson (1971) [Op. cit., pp.1-2].

APPENDIX B
ADDENDUM: PROOF OF *CONJECTURE 3.71*

The original thesis sets up a general framework for the study of G_2-instantons on suitable (i.e. asymptotically stable) holomorphic bundles over exponentially asymptotically cylindrical (EAC) Calabi-Yau $3-$folds, as constructed by A. Kovalev [Kov_1, Kov_2]. An existence theorem is announced [*Theorem 3.76*], based on smooth $t \to \infty$ convergence of a nonlinear 'heat flow' $\{H_t\}$ on the space of Hermitian metrics, for which a necessary uniform C^0-bound is left pending on the validity of *Conjecture 3.71*.

This Addendum provides a rigorous restatement as well as a proof of that necessary result, therefore establishing the existence of a Hermitian Yang-Mills (HYM) metric on any asymptotically stable holomorphic bundle over such an EAC Calabi-Yau $3-$fold.

B.1 Preliminaries

In *B.1.1* I will derive a certain number of analytical facts to be used in the main proof. As the relevance of some of these may seem unclear at first, the reader might prefer to skip this part and refer back to it at a later stage.

Then, in *B.1.2*, I will provide a proper statement of the desired result and give some intuition on the proof to follow.

B.1.1 Analytical lemmas

Definition B.91 *Given a Lipschitz function $f : W \to \mathbb{R}$, smooth away from a set of codimension at least 3, and $\beta \in \mathbb{R}$, denote*

$$\Delta f \overset{w}{\leq} \beta \quad \Leftrightarrow \quad \int_W f \Delta\varphi \leq \beta \, \|\varphi\|_{C^0(W)}, \quad \forall \varphi \in C_c^\infty(W).$$

The constant β will be referred to as a weak bound *of the Laplacian.*

Recalling that $\xi_t \in \Gamma(\text{End}\,\mathcal{E})$ is defined by $H_t = H_0 e^{\xi_t}$ (hence is self-adjoint with respect to both metrics), write $\bar{\lambda}_t$ for its highest eigenvalue; then:

Lemma B.92 *The Laplacian of* $\bar{\lambda}$ *admits a weak bound* $\beta > 0$*:*

$$\Delta\bar{\lambda}_t \overset{w}{\leq} \beta.$$

Proof *This is a direct consequence of the weak inequality [Don$_3$, p.243]:*

$$\Delta\bar{\lambda} \leq 2\left(\left\|\hat{F}_{H_t}\right\|_{H_t} + \left\|\hat{F}_{H_0}\right\|_{H_0}\right),$$

in view of the time-uniform bound on $\hat{F}_{H_t}$ *[*Corollary 3.45*].* ■

Lemma B.93 (weak maximum principle) *In the terms of* Definition B.91*, if* $f \overset{w}{\leq} 0$ *over a (bounded open) domain* $U \subset W$*, then* $f|_U \leq \max_{\partial U} f$*.*

Proof *Let* $M \doteq \max_{\partial U} f$ *and suppose, by contradiction, that a local maximum occurs at (an interior point)* $q \in U$*. Then, working in local coordinates, take then a (small) neighbourhood* $U \supset V \ni q$ *such that the gradient field* ∇f *'points inwards' to* q*, and choose a smooth positive bump function* $\varphi \in C_c^\infty(W)$*, peaking at* q*, with* $\operatorname{supp}(\varphi) \subset V$ *and such that* $(\nabla f, \nabla\varphi) > 0$ *almost everywhere on* V*. Then*

$$0 \geq \int_W f\Delta\varphi = \int_V f\Delta\varphi \overset{(*)}{=} \int_V (\nabla f, \nabla\varphi) \geq 0$$

which either iterates all over U *to imply* $f \equiv M$ *or contradicts the assumption that* q *is an interior point. Note that step* $(*)$ *is only rigorous under the assumption that* ∇f *is defined away from a singular set of high enough codimension that an* $\varepsilon-$*neighbourhood* N_ε *of it has* $d\mathrm{Vol}_\omega|_{\partial N_\varepsilon} = O(\varepsilon^2)$*. This guarantees that boundary terms in the integration by parts over* $V \setminus N_\varepsilon$ *vanish when* $\varepsilon \to 0$*, since* f *is Lipschitz [Don$_3$, p.244].* ■

Lemma B.94 *Let* (D, g) *be a compact Riemannian manifold,* $f \in L^\infty(D, \mathbb{R}^+)$*,* $p > 1$ *and* $x > 0$*; then there exists a constant* $k_p = k_p(D, g) > 0$ *such that*

$$\|f\|_p \geq \frac{k_p}{F^x}\left\|f^{1+x}\right\|_1$$

with $\|.\|_q \doteq \|.\|_{L^q(D,g)}, 1 < q \leq \infty$*, and* $F \doteq \|f\|_\infty$*.*

Proof *It suffices to write*

$$\|f\|_p \doteq \left(\int_D f^p \; d\mathrm{Vol}_g\right)^{\frac{1}{p}} \geq \left(\int_D f^p \left(\frac{f}{F}\right)^{xp} d\mathrm{Vol}_g\right)^{\frac{1}{p}} = F^{-x} \left\|f^{1+x}\right\|_p$$

then apply Hölder's inequality, finding $k_p = (\mathrm{Vol}_g D)^{(\frac{1}{p}-1)}$. ∎

B.1.2 Restatement of the *Conjecture* and partial proof

The original statement of *Conjecture 3.71*, although heuristically plausible, turns out not to hold *ipsis litteris*, in the sense that the claimed lower bound on L^2-norms of $\hat{F}_H$ over transverse slices only holds in a *weak* sense, i.e., its integral along a tubular segment is bounded below by the tube's length. Nonetheless, this is exactly what is necessary to obtain the C^0-bound in *Proposition 3.75*.

Recalling that discussion, one would like to derive, for small enough $|z|$, a time-uniform *lower* bound on the 'energy density' given by the $\omega-$trace of the restriction of curvature $F_{H_t}|_{D_z}$:

$$\widehat{F_{t|z}} \doteq \widehat{F_{H_t}|_{D_z}} = \left(F_{H_t}|_{D_z}, \omega|_{D_z}\right),$$

in the sense that its L^2-norm over a cylindrical segment Σ far enough down the tubular end is bounded below by a scalar multiple of $\mathrm{Vol}\,\Sigma$. Moreover, the length of such a cylinder Σ can be assumed roughly proportional to $L_t \doteq \sup_W \bar{\lambda}_t$, so that $L_t \gg 0$ implies a large 'energy' contribution. Explicitly:

Conjecture B.95 (rephrased *3.71*) *There are constants* $c, c', c'' > 0$ *independent of* t *and* z *such that, for every* $t \in\,]0, \infty[$, *there exists an open set* $A_t \subset \tau(W_\infty) \subset \mathbb{C}P^1$ *of parameters along the tubular end satisfying:*

1. $\forall z \in A_t$, $|z| < \delta$ *as in* Definition 3.40, *i.e.,* $\mathcal{E}|_{D_z}$ *is stable;*

2. *in the measure* μ_∞ *induced on* $\tau(W_\infty)$ *by the cylindrical metric* (2.5), *with* $z = e^{-s+\mathrm{i}\alpha}$,

$$\boxed{\int_{A_t} \|\widehat{F_{t|z}}\|^2_{L^2(D_z)} ds \wedge d\alpha \;\geq \frac{c}{2}\mu_\infty\left(A_t\right)\left(1 - \frac{c'}{L_t}\right)^2} \quad ;$$

3. *moreover, when* $L_t \gg 0$, *one has*

$$\mu_\infty(A_t) \approx c''\sqrt{L_t}.$$

Following *Subsection 3.2.3*, suppose that the following claim could be made rigorous:

> "In the terms of *Conjecture B.95*, there exists a set A_t, 'proportional' to $c''\sqrt{L_t}$, such that
>
> $$\|\xi_t\|_{L^{\frac{4}{3}}(D_z)} \geq c_2\, L_t \qquad \forall t \in]0, \infty[\,,$$
>
> where c_2 is independent of t and z."

Then, together with *Lemma 3.72* and *Lemma 3.73*, this would prove *Conjecture B.95*, with

$$c = 2\left(\frac{c_2}{c_1} \inf_{|z|<\delta} c_z\right)^2 \quad \text{and} \quad c' = \frac{1}{c_2}.$$

Now, in what follows, this *will not* be, strictly speaking, the right formulation, due to some analytical idiosyncrasies, but it conveys the relevant intuition behind the proof.

B.2 Completion of the proof

The strategy consists, on one hand, of using the weak control over the Laplacian from *Lemma B.92* to show that, around the furthest point down the tube where $L_t = \max_W \bar{\lambda}_t$ is attained, the slicewise supremum of $\bar{\lambda}_t$ is always on top of a certain concave parabola P_t.

On the other hand, the integral along the tube of the slicewise norms $\|\bar{\lambda}\|_{L^{\frac{4}{3}}(D_z)}$, which bounds below $\mathcal{N}_{D_z}(H_t)$ [*Lemma 3.72*], can be shown to be itself bounded below by those slicewise suprema, using again the weak bound on the Laplacian to apply Moser's estimate on 'balls' of a standard shape, which fill essentially 'half' of the corresponding tubular volume.

Since $\mathcal{N}_{D_z}(H_t)$ is controlled above by $\hat{F}_{H_t}$, in the sense of *Lemma 3.73*, this leads to the desired minimal 'energy' contribution, 'proportional' to the length (roughly $\sqrt{L_t}$) of the tubular segment underneath the parabola P_t.

B.2.1 A concave parabola as lower bound

In tubular coordinates $|z| = e^{-s}$, the supremum of $\bar{\lambda}_t$ over a transversal slice along W_∞ defines a smooth function

$$\begin{aligned} \ell_t \quad &: \quad \mathbb{R}^+ \to \mathbb{R}^+ \\ \ell_t(s) \quad &\doteq \quad \sup_{\partial W_s} \bar{\lambda}_t. \end{aligned}$$

Moreover, for each $t > 0$, denote S_t the 'furthest length' down the tube at which L_t is attained, i.e., $S_t = \max_{L_t = \ell_t(s)} s$, and set

$$I_t \doteq \left[S_t, S_t + \delta_t^+\right] \subset \mathbb{R}^+$$

with $\delta_t^+ \doteq \frac{1}{2}\left(\sqrt{1 + \frac{8}{\beta}L_t} - 1\right)$ and β as in *Lemma B.92*.

Lemma B.96 *For each $t > 0$, the transversal supremum ℓ_t is bounded below by the concave parabola $P_t(s - S_t) \doteq L_t - \frac{\beta}{2}(s - S_t)(s - S_t + 1)$ over I_t, i.e.,*

$$\ell_t(s) \geq P_t(s), \quad \forall s \in I_t.$$

Proof *Fix $t > 0$, $\delta_t^+ \geq \delta > 0$ and set $J_t(\delta) \doteq]S_t - 1, S_t + \delta[$ (I suppress henceforth the t subscript everywhere, for clarity). The parabola takes the value $P(-1) = P(0) = L$ at the points $S - 1$ and S, and its concavity is precisely the weak bound on $\Delta\bar{\lambda}$ [*Lemma B.92*], so we have $\Delta\left(\bar{\lambda} - P\right) \overset{w}{\leq} 0$ on $\overline{J(\delta)}$ and* Lemma *B.93 gives*

$$\begin{aligned} \left(\bar{\lambda} - P\right)\big|_{J(\delta)} &< \max\left\{\sup_{s=S-1}\left(\bar{\lambda} - P\right), \sup_{s=S+\delta}\left(\bar{\lambda} - P\right)\right\} \\ \Rightarrow 0 = \ell(S) - P(S) &< \ell(S+\delta) - P(S+\delta), \quad \forall 0 < \delta < \delta^+ \end{aligned}$$

since by assumption $\ell(S-1) \leq L = P(-1)$. Hence $\ell \geq P, \quad \forall s \in I$. ■

Remark B.97 *Fixing $0 < \epsilon < 1$ and setting $\delta_{\epsilon,t}^+ \doteq \frac{1}{2}\left(\sqrt{1 + \frac{8}{\beta}(1-\epsilon)L_t - 1}\right)$,*

$$\ell_t(s) \geq \epsilon L_t, \quad \forall s \in I_{\epsilon,t} \doteq \left[S_t, S_t + \delta_{\epsilon,t}^+\right]. \tag{B.1}$$

Moreover, for $L_t \gg 0$, one has $\delta_{\epsilon,t}^+ \approx c''_\epsilon\sqrt{L_t}$, with $c''_\epsilon = \sqrt{\frac{2(1-\epsilon)}{\beta}}$.

B.2.2 Moser's estimate over transversal slices

The goal of this *Subsection* is to establish the following inequality:

Lemma B.98 *Given $t > 0$ and $0 < \epsilon < 1$, let $\Sigma_t(\epsilon) \doteq I_{\epsilon,t} \times D \simeq \overline{W_{S_t+\delta^+_{\epsilon,t}} \setminus W_{S_t}}$ be the finite cylinder along W, under the parabola P_t of* Lemma B.96*, determined by the interval of length $\delta^+_{\epsilon,t}$ on which (B.1) holds, and suppose $2\pi\delta^+_{\epsilon,t} \in \mathbb{N}$; then, for each $x > 0$, there exists a uniform constant $k_{x,\epsilon} > 0$ such that*

$$\int_{\Sigma_t(\epsilon)} \bar{\lambda}_t^{1+x} \, d\mathrm{Vol}_{\omega_\infty} \geq k_{\epsilon,x}.\delta^+_{\epsilon,t}.L_t^{1+x}. \tag{B.2}$$

Again let me suppress the t subscript, for tidiness, and work all along in the cylindrical metric ω_∞. For each $s \in I_\epsilon$, let $p_s \in \partial W_s \simeq D_s \times S^1$ be a point on the corresponding transversal slice such that the maximum $\ell(s) = \bar{\lambda}(p_s)$ is attained, and form the 'unit' open cylinder $B_s \subset \Sigma_\epsilon$ of length[1] $\frac{1}{2\pi}$, centered on p_s, such that

$$\mathrm{Vol}\, B_s = \mathrm{Vol}\,(B_s \cap D_s) = \frac{1}{2}\mathrm{Vol}\, D,$$

where $\mathrm{Vol}\, D \equiv \mathrm{Vol}\, D_s$ denotes the (same) four-dimensional volume of (all) D_s.

Under the weak bound $\Delta\bar{\lambda} \overset{w}{\leq} \beta$ [*Lemma B.92*], Moser's estimate [Mos] over B_s gives

$$\frac{1}{\mathrm{Vol}\, B_s} \int_{B_s} \bar{\lambda}^{1+x} \geq k'_x \left(\max_{B_s} \bar{\lambda} \right)^{1+x} \geq k'_x \, (\epsilon L)^{1+x} \tag{B.3}$$

where $k'_x > 0$ is a uniform constant (as all B_s are congruent by translation) and the second inequality comes from property (*B*.1). In particular, one can choose at most $2\pi\delta^+_\epsilon \in \mathbb{N}$ values $s_j \in I_\epsilon$ such that the corresponding B_{s_j} are necessarily disjoint, and form their union

$$B\,(\epsilon) \doteq \coprod_{j=1}^{2\pi\delta^+_\epsilon} B_{s_j}.$$

Clearly $\mathrm{Vol}\, B(\epsilon) \geq \frac{1}{2}\,(2\pi\delta^+_\epsilon)\,\mathrm{Vol}\, D$. Now, the statement about averages (*B*.3) goes over to the disjoint union, which proves the *Lemma*, with $k_{x,\epsilon} \doteq \pi k'_x \, \epsilon^{1+x}\, \mathrm{Vol}\, D$:

$$\int_{\Sigma(\epsilon)} \bar{\lambda}^{1+x} \geq \int_{B(\epsilon)} \bar{\lambda}^{1+x} \geq \mathrm{Vol}\, B(\epsilon)\, k'_x \, (\epsilon L)^{1+x}.$$

[1] so that the volume integral over B_s along the $S^1 \times I_\epsilon$ directions is 1.

B.2.3 End of proof

It is now just a matter of putting together the previous results. At the end of *Subsection B.1.2* we knew

$$L\|\hat{F}\|_2 \geq k'\left(\|\bar{\lambda}\|_{\frac{4}{3}} - 1\right)$$

over each D_z sufficiently far down the tube, for a uniform constant $k' > 0$. Choosing $0 < \epsilon < 1$ and $x > 0$, integrating over $A_\epsilon \doteq I_\epsilon \times S^1$ and applying *Lemma B.94* we have

$$\int_{A_\epsilon} \|\hat{F}\|_2 \, ds \wedge d\alpha \geq k'' \frac{1}{L^{1+x}} \int_{A_\epsilon} \left\|\bar{\lambda}^{1+x}\right\|_1 ds \wedge d\alpha - k' \frac{\delta_\epsilon^+}{L}$$

where $k'' \doteq k'.k_{\frac{4}{3}}$ is still a uniform constant. Moreover, by *Lemma B.98*, the integral term is bounded below by $k''.k_{\epsilon,x}.\delta_\epsilon^+$, so Hölder's inequality gives

$$(\mathrm{Vol}\, A_\epsilon)^{\frac{1}{2}} \left(\int_{A_\epsilon} \|\hat{F}\|_2^2 \, ds \wedge d\alpha\right)^{\frac{1}{2}} \quad \geq \quad \delta_\epsilon^+ \left(k''.k_{\epsilon,x}. - \frac{k'}{L}\right).$$

Since the interval I_ϵ has length precisely δ_ϵ^+, we finally have

$$\int_{A_\epsilon} \|\hat{F}\|_2^2 \, ds \wedge d\alpha \quad \geq \quad \frac{\delta_\epsilon^+}{2\pi} \left(k''.k_{\epsilon,x}. - \frac{k'}{L}\right)^2$$

which yields the *Conjecture*, choosing e.g. $\epsilon = \frac{1}{2}$, $x = 1$ and [cf. *Remark B.97*]

$$c = \left(\frac{k''.k_{\frac{1}{2},1}}{\sqrt{2\pi}}\right)^2, \quad c' = \frac{k'}{k''.k_{\frac{1}{2},1}} \quad \text{and} \quad c'' = 2\pi\sqrt{\tfrac{1}{\beta}}.$$

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