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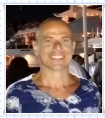
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PLANE WAVE FOR A FREE RELATIVISTIC PARTICLE $(\hbar = c = 1)$

$$\exp\left[i p_x x + i p_y y + i p_z z - i\sqrt{p^2+m^2}\, t\right]$$

WHERE m IS THE MASS OF THE PARTICLE ($m = 0.5$ MeV FOR THE ELECTRON)

AND $\quad p = \sqrt{p_x^2 + p_y^2 + p_z^2}$

$$\boxed{\int dp\, dp_x\, dp_y \; e^{-(p-p_0)^2 w_0^2/4}\; e^{-(p_x^2+p_y^2)w_0^2/4}\; e^{i\left(p_x x + p_y y + \sqrt{p^2-p_x^2-p_y^2}\, z - \sqrt{p^2+m^2}\, t\right)}}$$

THE PREVIOUS INTEGRAL DOES NOT HAVE AN ANALYTICAL SOLUTION!!!

1st CASE $\quad p_x = p_y = 0 \quad (w \to \infty)$

$$E = E_0 + E_0'(p-p_0) + E_0''(p-p_0)^2/2 + O[(p-p_0)^3]$$

$$E' = \frac{p}{E} \quad \text{and} \quad E'' = \frac{m^2}{E^3} \quad \Rightarrow \quad \boxed{E = E_0 + \frac{p_0}{E_0}(p-p_0) + \frac{m^2}{2E_0^3}(p-p_0)^2}$$

$$e^{i(p_0 z - E_0 t)} \int dp\; e^{-\frac{(p-p_0)^2 w_0^2}{4}}\; e^{i\left(z - \frac{p_0}{E_0}t\right)(p-p_0)}\; e^{-\frac{i m^2}{2E_0^3}(p-p_0)^2 t}$$

$$\int du\; e^{-au^2+bu} = e^{b^2/4a}\sqrt{\frac{\pi}{a}}$$

$$a = \frac{w_0^2}{4} + \frac{i\, m^2 t}{2E_0^3} \qquad b = i\left(z - \frac{p_0}{E_0}t\right)$$

$$\varphi(z,t;p_0) := N\; e^{-\frac{(z-p_0 t/E_0)^2/w_0^2}{1+2i\, m^2 t/E_0^3 w_0^2}} \Big/ \sqrt{1+2i\, m^2 t/E_0^3 w_0^2}$$

$$|\varphi(z,t;p_0)|^2 := N^2 \exp\left[-2\,\frac{(z-p_0 t/E_0)^2/w_0^2}{1+4\left(\frac{m^2 t}{E_0^3 w_0^2}\right)^2}\right] \Big/ \sqrt{1+4\left(\frac{m^2 t}{E_0^3 w_0^2}\right)^2}$$

$$\int_{-\infty}^{+\infty} dz\; |\varphi(z,t,w_0)|^2 = 1 \quad \Rightarrow \quad N^2 = \sqrt{2/\pi}$$

$$\sqrt{\frac{2}{\pi}}\ \exp\left[-2\frac{\left(z-\frac{p_0}{E_0}t\right)^2/w_0^2}{1+4\left(\frac{m^2t}{E_0^3w_0^2}\right)^2}\right]\Bigg/\sqrt{1+4\left(\frac{m^2t}{E_0^3w_0^2}\right)^2}$$

$$\gamma=\left(z-\frac{p_0}{E_0}t\right)/w_0 \qquad \tau=\frac{m^2t}{E_0^3w_0^2}$$

$$\boxed{\sqrt{\frac{2}{\pi}}\ e^{-2\frac{\gamma^2}{1+4\tau^2}}\Big/\sqrt{1+4\tau^2}}$$

$p_0 = 1\,\mathrm{KeV}$ $(m=0.5\,\mathrm{MeV})$ $w_0 = 200\,\mu m$ $(0.2\,\mu m = 1\,eV^{-1})$

$p_0 w_0 = 10^6$ $\gamma \to 0 \Rightarrow z=\frac{p_0}{E_0}t \sim \frac{p_0}{m}t = p_0 w_0 \frac{t}{m w_0}$

$$\frac{z}{w_0}\sim p_0 w_0\,\tau$$

$\tau=5$ $\frac{z}{w_0}=5\cdot10^6 \Rightarrow z = 5\cdot10^6\cdot 200\,\mu m = 1\,\mathrm{Km}$

$\tau=5 \Rightarrow t = 5\,m w_0^2 = 5\ 0.5\,\mathrm{MeV}\,(10^3 eV^{-1})^2 = \frac{5}{2}10^{12}\,eV^{-1}$

$\frac{1}{eV}=\frac{0.2\,\mu m}{c}=\frac{2}{3}10^{-15}\,\mathrm{sec}$

$t=\frac{5}{3}10^{-3}\,\mathrm{sec}$

$\frac{p_0}{m}=2\times10^{-3}\,c$

$\frac{p_0 t}{m}=2\times10^{-3}\times3\times10^5\frac{km}{sec}\ \frac{5}{3}10^{-3}\,\mathrm{sec} = 1\,\mathrm{Km}$ ✓ controllo

$p_0 = 1\,\mathrm{keV}$ $w_0=200\,\mu m$ $\tau=5$ $(t=\frac{5}{3}10^{-3}\,\mathrm{sec})$ $z=1\,\mathrm{Km}$

$w_0 \to w_0\sqrt{1+4\tau^2}\sim 10\,w_0$

$p_0 = m$ $E_0=\sqrt{2}\,m$ $p_0/E_0 = 1/\sqrt{2}$ $\tau = t/\sqrt{2}\,m w_0^2$

$w_0=200\,\mu m$

$\tau=5$

$t=\sqrt{2}\ \frac{5}{3}10^{-3}\,\mathrm{sec}$

$z=\frac{1}{\cancel{\sqrt{2}}}c\ \cancel{\sqrt{2}}\ \frac{5}{3}10^{-3}\,\mathrm{sec}$

$=\cancel{3}\cdot10^5\frac{km}{\cancel{sec}}\ \frac{5}{\cancel{3}}10^{-3}\,\cancel{sec}$

$= 500\,\mathrm{km}$

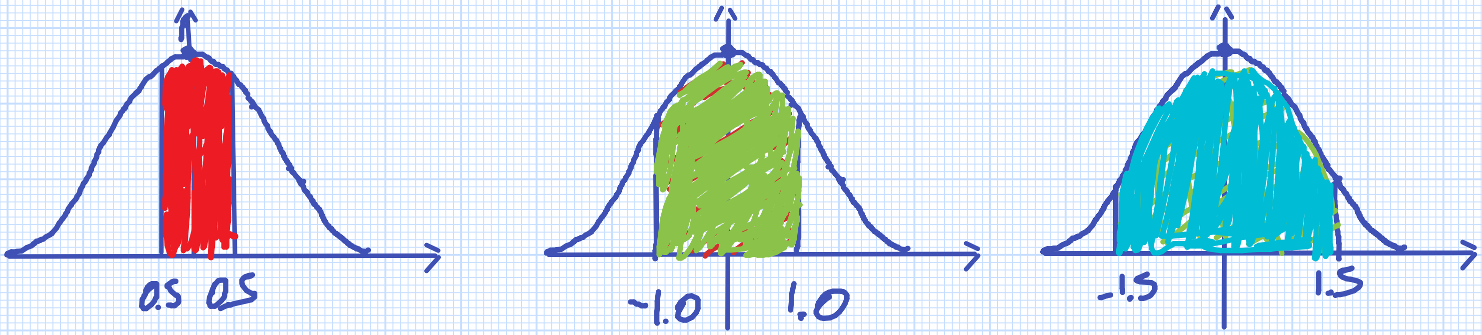
$w_0=200\,\mu m$ $p_0=1\,\mathrm{KeV}$ $z=1\,\mathrm{km}$ $t=\frac{5}{3}10^{-3}\,\mathrm{sec}$ $(w_0\to10\,w_0)$

$w_0=200\,\mu m$ $p_0=0.5\,\mathrm{MeV}$ $=500\,\mathrm{KeV}$

$z=500\,\mathrm{km}$ $t=\sqrt{2}\ \frac{5}{3}10^{-3}\,\mathrm{sec}$ $(w_0\to10\,w_0)$

LET US ANALYSE

$$\sqrt{\frac{2}{\pi}} \int_{-\alpha\sqrt{1+4\tau^2}}^{\alpha\sqrt{1+4\tau^2}} d\gamma \; e^{-2\gamma^2/(1+4\tau^2)} \Big/ \sqrt{1+4\tau^2} \qquad \frac{\gamma}{\sqrt{1+4\tau^2}} = u \qquad \Rightarrow \quad F(\alpha) = \sqrt{\frac{2}{\pi}} \int_{-\alpha}^{\alpha} du \, e^{-2u^2}$$



F(0.5) ?%

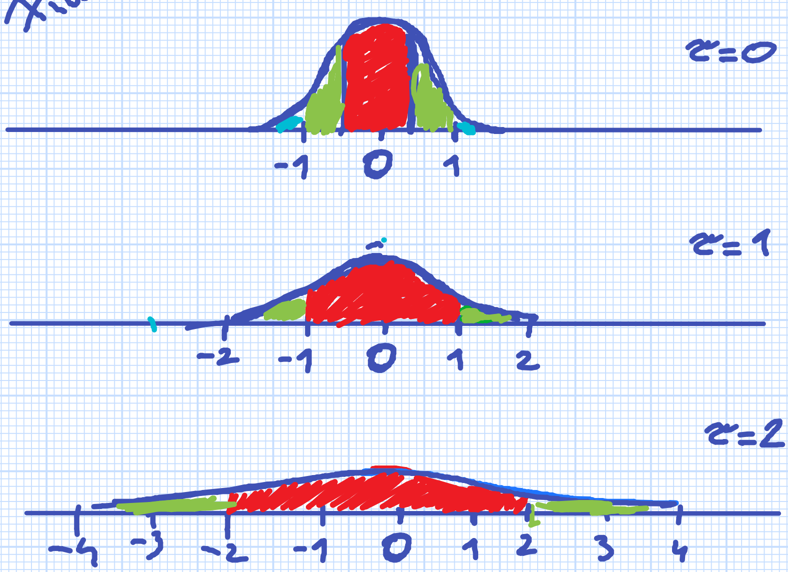
F(1.0) ?%

F(1.5) ?%

$$F(\alpha) = 15\%,\ 30\%,\ 45\%,\ 60\%,\ 75\%,\ 90\%$$

DETERMINE α

Fig 1 $\quad \alpha = \frac{1}{2}, 1, \frac{3}{2}$ $\qquad$ $\alpha(\ \ ), \ldots, \alpha(90\%)$



$\Rightarrow$?

Fig 2

$\rightarrow$?

$\rightarrow$?

Fig 3

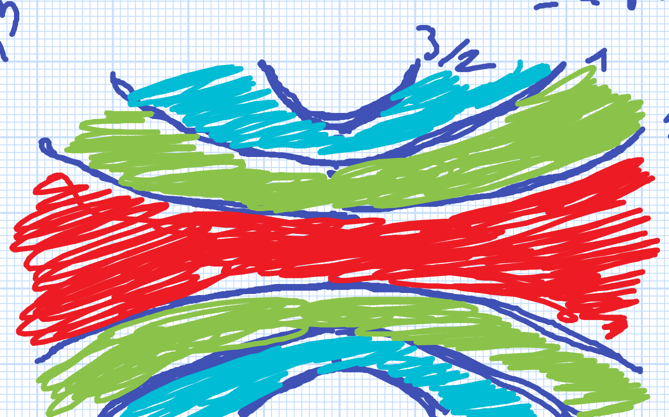


Fig 4

$\Rightarrow$

$\alpha(15\%)$ $\alpha(60\%)$

$\alpha(30\%)$ $\alpha(75\%)$

$\alpha(45\%)$ $\alpha(90\%)$

$p = p_0$ $\int dp_x\, dp_y\, e^{-(p_x^2+p_y^2)w^2/4}\, e^{i(p_x x + p_y y + \sqrt{p_0^2 - p_x^2 - p_y^2}\, z)}\, e^{-iE_0 t}$

$$\sqrt{p_0^2 - p_x^2 - p_y^2} \sim p_0 - \frac{p_x^2 + p_y^2}{2p_0}$$

$$e^{i(p_0 z - E_0 t)} \int dp_x\, e^{-p_x^2 w^2/4}\, e^{i(p_x x - \frac{p_x^2}{2p_0} z)} \int dp_y\, e^{-p_y^2 w^2/4}\, e^{i(p_y y - \frac{p_y^2}{2p_0} z)}$$

$$\int dp_x\, e^{-\left(\frac{w^2}{4} + i\frac{z}{2p_0}\right)p_x^2 + i p_x x} \Rightarrow \frac{e^{-\frac{x^2/w^2}{1+2iz/p_0 w^2}}}{\sqrt{1+2iz/p_0 w^2}}$$

$$\sqrt{\frac{2}{\pi}}\, \frac{e^{-\frac{2x^2/w^2}{1+4z^2/p^2w^4}}}{\sqrt{1+4z^2/p^2w^4}} \Rightarrow \sqrt{\frac{2}{\pi}}\, \frac{e^{-2\delta^2/1+4\tau^2}}{\sqrt{1+4\tau^2}}$$

$\delta = \frac{x}{w}$ $\tau = z/pw^2$

$p_0 = 1$ keV $w = 200\ \mu m$ $= 10^3\ eV^{-1}$ $\tau = 5 \Rightarrow \frac{z}{w} = 5\, p_0 w = 5 \times 10^6$

$p_0 w = 10^6$ $z = 5 \times 10^6 \times 200\mu m = 1$ km

$p_0 = \frac{1}{2}$ MeV $= 500$ keV $p_0 w = 500\ 10^6$ $z = 500$ km

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$\lambda = 628$ nm $= 0.628\ \mu m$ $0.2\ \mu m = 1/eV$

$\mu m = 5/eV$

$$p_0 = \frac{2\pi}{\lambda} = \frac{6.28}{0.628\ \mu m} = \frac{10}{\mu m} - \frac{10}{5} eV = 2\ eV$$

$p_0 = 1$ keV $w = 200\ \mu m$ $p_0 w = 10^6$

$p_0 = 2$ eV $= 1$ keV$/500$ $p_0 w = 10^6/500$

$\tau = 5 \Rightarrow z = 1$ km$/500 = 2$ m

$p w = 1$ $\quad w = 200\,\mu m = 10^3\, eV^{-1} \Rightarrow p = 10^{-3}\, eV$

$$\frac{p}{m} = 2\cdot 10^{-9}\, c = 0.6\ m/sec = 2.16\ Km/h$$

$\tau = 5$ $\quad 200\,\mu m \xrightarrow{10} 2\, cm$ $\quad \tau = t/m w^2$

$z_{MAX} = \frac{p t}{m}$ $\quad \frac{z_{MAX}}{w} = p w \frac{t}{m w^2} = 1 \cdot \tau$ $\quad z_{MAX} = \tau w$

$\tau = \frac{z/w}{p w}$ $\quad p w = 1 \Rightarrow z = \tau w$