Workshop on Stochastic Analysis

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Weighted Helmholtz-Hodge decompositions, Lyapunov functions, and invariant measures

Abstract

We study weighted Helmholtz–Hodge decompositions of drift vector fields associated with second-order diffusion operators on $\mathbb{R}^d$, $d \geq 2$. Given a decomposition of the form $\mathbf{G} = A\nabla\Phi + \mathbf{B}$, we relate the weighted divergence-free condition $\operatorname{div}_\mu(\mathbf{B}) = 0$, where $\mu = e^{2\Phi}dx$, to infinitesimal invariance of μ for the operator

$$\frac{1}{2}\mathrm{trace}(A\nabla^2) + \langle \mathbf{G}, \nabla\cdot \rangle.$$

We compare weighted, orthogonal, and strictly orthogonal Helmholtz–Hodge decompositions and show that uniqueness of the infinitesimally invariant measure yields uniqueness of the corresponding weighted decomposition, and hence a canonical potential. For linear vector fields, we characterize Gaussian infinitesimally invariant measures by an algebraic Riccati equation together with a trace condition. In the Ornstein–Uhlenbeck case, this gives a structural proof of the classical criterion that a finite invariant measure exists if and only if the drift matrix is Hurwitz, and it identifies the associated strictly orthogonal

decomposition. Finally, we treat nonlinear polynomial perturbations that preserve a given potential and obtain explicit classes of drifts for which the invariant measure and the weighted decomposition remain unique. The results clarify the relation between Lyapunov-type potentials, non-reversible perturbations, and invariant measures for diffusion semigroups. This is joint work with Haesung Lee (Kumoh National Institute of Technology, South Korea).