Weak martingale solutions to stochastic Navier–Stokes–Cahn–Hilliard system with transport noise

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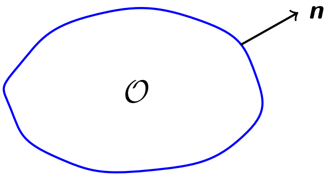
Outline

The geometric setting

Let $\mathcal{O} \subset \mathbb{R}^d$ be a bounded domain, with $d \in \{2, 3\}$ and of $\mathcal{C}^3$-class; let $\boldsymbol{n}$ denote the outward unit normal.



Unknowns are random fields: velocity, pressure and *turbulent pressure*,

$$u(t,x) = u(t,x,\omega), \qquad p(t,x) = p(t,x,\omega), \qquad \bar{p}_k(t,x) = \bar{p}_k(t,x,\omega).$$

An external forcing $f(t,x,omega)$ is an external (deterministic) is given.
The horizon $T > 0$ is fixed;
$\widetilde{W} = (\tilde{w}_k)_{k\geq 1}$ is a sequence of i.i.d. standard Brownian motions.

Navier–Stokes equations with transport noise

$$
\begin{aligned}
&\mathrm{d}u + \Big[\ \operatorname{div}(u \otimes u) - \nu \Delta u + \nabla p \Big]\mathrm{d}t \\
&\quad = \sum_{k=1}^{\infty} \Big[\big(\sigma_k(t,x) \cdot \nabla\big) u - \nabla \bar{p}_k + g_k(t,x;u) \Big] \mathrm{d}\tilde{w}_k .
\end{aligned}
$$

Note that $\operatorname{div}(u \otimes u) = u \cdot \nabla u$.

Proposed by:

- Brzeźniak–Capiński–Flandoli (1991) – simplified form;
- Mikulevicius–Rozovskii (2005) – full form.

The transport-type term $\sigma_k \cdot \nabla u$ models small-scale turbulent advection acting on the velocity.

Two incompressible fluids

Let $\phi(t,x) = \phi(t,x,\omega)$ be a *phase function* (order parameter) distinguishing the two fluids. The model is Model H of Hohenberg–Halperin (1977).

Influence of u on ϕ (Cahn–Hilliard with transport):

$$\boxed{\mathrm{d}\phi + \operatorname{div}(\phi\, u)\,\mathrm{d}t \;=\; \varrho_0\,\Delta\big(-\varepsilon\Delta\phi + \alpha\,\psi'(\phi)\big)\,\mathrm{d}t.}$$

Since $\operatorname{div} u = 0$, we have $\mathrm{d}\phi + \operatorname{div}(\phi\, u)\,\mathrm{d}t = \frac{\partial\phi}{\partial t} + u\cdot\nabla\phi$.

Influence of ϕ on u (Korteweg-type capillary stress, added to the LHS of the NSEs):

$$\boxed{+\,\kappa\,\operatorname{div}\big(\nabla\phi\otimes\nabla\phi\big).}$$

$\varepsilon > 0$: interface thickness;
$\alpha > 0$; $\kappa > 0$: surface tension;
$\varrho_0 > 0$: elastic relaxation time.

Boundary and initial conditions, mass conservation

Dirichlet conditions for u, no-flux conditions for ϕ:

$$u = 0, \qquad \frac{\partial \phi}{\partial \boldsymbol{n}} = \frac{\partial \Delta \phi}{\partial \boldsymbol{n}} = 0 \quad \text{on } (0,T) \times \partial \mathcal{O},$$

$$u(0,\cdot) = u_0, \qquad \phi(0,\cdot) = \phi_0 \quad \text{in } \mathcal{O}.$$

Mass conservation

$$\langle \phi(t) \rangle := ⨍_{\mathcal{O}} \phi(t,x)\,\mathrm{d}x = \langle \phi(0) \rangle =: M_0, \qquad t \in (0,T).$$

Up to a constant shift we assume $\langle \phi(0) \rangle = 0$, hence $\langle \phi(t) \rangle = 0$ for all t. This is what makes the mean-zero spaces below the natural ones.

Capillary stress and chemical potential

Remark.

$$-\operatorname{div}\big(\nabla\phi\otimes\nabla\phi\big) = \tfrac{1}{\varepsilon}\,\mu(\phi)\,\nabla\phi \;-\; \nabla\Big(\tfrac{\alpha}{\varepsilon}\big[\psi(\phi)-\langle\psi'(\phi)\rangle\phi\big] + \tfrac{1}{2}\,|\nabla\phi|^2\Big),$$

where the *chemical potential* is

$$\mu(\phi) := -\varepsilon\Delta\phi + \alpha\big(\psi'(\phi)-\langle\psi'(\phi)\rangle\big),$$

i.e. the *mean-free* chemical potential, so that $\mu(\phi)\in H_1$.

Consequently

$$R_0(\phi,\phi) = -\tfrac{1}{\varepsilon}\,\Pi\big(\mu(\phi)\nabla\phi\big).$$

μ is the variational derivative of the free energy functional

$$\mathscr{F}(\phi) = \int_{\mathcal{O}}\big(\tfrac{\varepsilon}{2}\,|\nabla\phi|^2 + \alpha\psi(\phi)\big)\mathrm{d}x.$$

The potential ψ

We work with the *Landau* (regular) potential

$$\psi(s) = \tfrac{1}{4}(1-s^2)^2, \qquad \psi'(s) = s(s^2-1), \quad \psi''(s) = 3s^2-1,$$

which approximates the Flory–Huggins (logarithmic) potential

$$\psi_{00}(s) = \tfrac{\theta}{2}\left[(1+s)\ln(1+s) + (1-s)\ln(1-s)\right] - \tfrac{\theta_0}{2}\, s^2, \qquad 0<\theta<\theta_0,$$

Class of admissible potentials (Remark 2.x)

All results hold for any $\psi \in C^2(\mathbb{R})$ with $\kappa_1, \kappa_2, \kappa_3 > 0$ such that

$$\psi(s) \geq -\kappa_1, \qquad -\psi''(s) \leq \kappa_2, \qquad |\psi'(s)| \leq \kappa_3\big(|s| + |s|^3\big).$$

Normalisation used from now on: $\varepsilon = \alpha = \varrho_0 = 1$.

Function spaces for the velocity

With $\mathcal{V} := \{v \in C_0^\infty(\mathcal{O}, \mathbb{R}^d) : \operatorname{div} v = 0\}$,

$$H_0 := \overline{\mathcal{V}}^{\mathbb{L}^2(\mathcal{O})}, \qquad V_0 := \overline{\mathcal{V}}^{\mathbb{H}^1(\mathcal{O})}$$

Norms

$$|u|_{H_0}^2 = \int_{\mathcal{O}} |u(x)|^2 \, \mathrm{d}x, \qquad \|u\|_{V_0}^2 = |u|_{H_0}^2 + |\nabla u|_{\mathbb{L}^2}^2.$$

Fix $k_0 > \frac{d}{2} + 1$ and set $\mathrm{U}_0 := \overline{\mathcal{V}}^{\mathbb{H}^{k_0}(\mathcal{O})} = \mathbb{H}^{k_0}(\mathcal{O}) \cap H_0$.

Gelfand triples

$$\mathrm{U}_0 \hookrightarrow V_0 \hookrightarrow H_0 \cong H_0' \hookrightarrow V_0' \hookrightarrow \mathrm{U}_0'.$$

$\Pi : \mathbb{L}^2(\mathcal{O}) \to H_0$ is the Leray–Helmholtz projection;
$\mathcal{A}_0 u = -\Pi \Delta u$ is the Stokes operator.
$\mathrm{A}_1 = -\Delta$ is the Neumann Laplacian on mean-zero functions; $D(\mathrm{A}_1^{1/2}) = H_1$.

Function spaces for the phase

With $H^s_{(0)}(\mathcal{O}) := \{\phi \in H^s(\mathcal{O}) : \langle\phi\rangle = 0\}$,

$$H_1 := H^1_{(0)}(\mathcal{O}) = \left\{ \phi \in H^1(\mathcal{O}) \,:\, \fint_{\mathcal{O}} \phi = 0 \right\},$$

$$V_1 := \left\{ \phi \in H_1 \,:\, \Delta\phi \in H^1(\mathcal{O}),\ \frac{\partial\phi}{\partial \boldsymbol{n}}\Big|_{\partial\mathcal{O}} = 0 \right\} \ (= D(\mathrm{A}_1^{3/2})).$$

Norms

$$|\phi|^2_{H_1} = \int_{\mathcal{O}} |\nabla\phi|^2 \,\mathrm{d}x, \qquad \|\phi\|^2_{V_1} = |\phi|^2_{H_1} + |\Delta\phi|^2_{H_1}.$$

Gelfand triple

$$V_1 \subset H_1 \cong H_1' \subset V_1'.$$

Product spaces

$$\mathcal{U} := \mathrm{U}_0 \times V_1, \qquad \mathbb{H} := H_0 \times H_1, \qquad \mathbb{V} := V_0 \times V_1,$$

with the Gelfand triple

$$\mathcal{U} \hookrightarrow \mathbb{H} \cong \mathbb{H}' \hookrightarrow \mathcal{U}'.$$

The space $\mathcal{U}$ (not $\mathbb{V}$) is the one in which the equation is finally satisfied, and $\mathcal{U}'$ is the space in which the limiting Itô integral is constructed.

Bilinear forms and induced operators

Bilinear forms

$$a_0 : V_0 \times V_0 \ni (u, v) \mapsto \langle u, v \rangle_{V_0} \in \mathbb{R},$$

$$\widetilde{a}_1 : V_1 \times V_1 \ni (\phi, \psi) \mapsto \langle \phi, \psi \rangle_{H_1} + \langle \Delta\phi, \Delta\psi \rangle_{H_1} = \langle \phi, \psi \rangle_{V_1} \in \mathbb{R}.$$

Generate (Lax–Milgram) linear isomorphisms

$$\mathcal{A}_0 : V_0 \to V_0', \qquad \widetilde{\mathcal{A}}_1 : V_1 \to V_1',$$

with explicit form

$$\mathcal{A}_0 u = \Pi(-\Delta u), \qquad \widetilde{\mathcal{A}}_1 \phi = \Delta^2 \phi + \phi.$$

Here Π is the Leray–Helmholtz projector onto divergence-free fields.

A modification for Cahn–Hilliard

We work instead with

$$a_1(\phi,\psi) := \langle \Delta\phi, \Delta\psi\rangle_{H_1}, \qquad \mathcal{A}_1\phi := \widetilde{\mathcal{A}}_1\phi - \phi = \Delta^2\phi.$$

Moreover $\mathcal{A}_1 = \mathrm{A}_3 \circ \mathrm{A}_2$ with $\mathrm{A}_2 : V_1 \ni \phi \mapsto -\Delta\phi \in H_1$ and $\mathrm{A}_3 : H_1 \ni \phi \mapsto -\Delta\phi \in V_1'$.

Linear (uncoupled) system

$$\begin{cases} \dfrac{\partial u}{\partial t} + \nu \mathcal{A}_0 u = 0, \\ \dfrac{\partial \phi}{\partial t} + \mathcal{A}_1\phi = 0. \end{cases}$$

Weak formulation: $\forall (v,\psi) \in V_0 \times V_1$,

$$\frac{d}{dt}\langle u(t), v\rangle_{V_0',V_0} + \nu\, a_0(u(t),v) = 0,$$
$$\frac{d}{dt}\langle \phi(t), \psi\rangle_{V_1',V_1} + a_1(\phi(t),\psi) = 0.$$

Trilinear forms encoding the nonlinearities

$$b_0 : V_0 \times V_0 \times V_0 \ni (u, v, w) \mapsto \sum_{i,j} \int_{\mathcal{O}} u^i \, \partial_i v^j \, w^j \, \mathrm{d}x \ \in \mathbb{R},$$

$$r_0 : \ W^{1,4}(\mathcal{O}) \times W^{1,4}(\mathcal{O}) \times V_0 \ni (\phi, \psi, u) \mapsto \sum_{i,j} \int_{\mathcal{O}} \partial_i \phi \, \partial_j \psi \, \partial_j u^i \, \mathrm{d}x,$$

$$b_1 : V_0 \times \ H^2_{(0)}(\mathcal{O}) \times W^{1,4}(\mathcal{O}) \ni (u, \phi, \psi) \mapsto \sum_{i} \int_{\mathcal{O}} \nabla (u^i \, \partial_i \phi) \cdot \nabla \psi \, \mathrm{d}x.$$

If $u \in V_0$ and ϕ, $\psi \in V_1$, or if $u \in V_0$ and ϕ, $\psi \in H^2_{(0)}(\mathcal{O})$, then

$$b_1(u, \phi, \psi) = \sum_{i} \int_{\mathcal{O}} u^i \, \partial_i \phi \, (-\Delta \psi) \, \mathrm{d}x = - \int_{\mathcal{O}} u \cdot \nabla \phi \, \Delta \psi \, \mathrm{d}x.$$

Fundamental algebraic properties

Antisymmetry / coupling identities

$$b_0(u,v,w) = -b_0(u,w,v), \qquad b_0(u,v,v) = 0, \qquad b_1(u,\phi,\phi) = r_0(\phi,\phi,u).$$

Each trilinear form generates a bilinear map:

$$\begin{aligned}
B_0 :&\quad V_0 \times V_0 \to V_0', &&\text{(from } b_0\text{)}\\[1em]
\widetilde{B_1} :&\quad V_0 \times H_1 \to V_1', &&\text{(from } b_1\text{)}\\
B_1 :&\quad V_0 \times V_1 \to H_1, &&\text{(extension of } \widetilde{B_1}\text{)}\\
R_0 :&\quad H^2_{(0)} \times H^2_{(0)} \to V_0', &&\text{(from } r_0\text{)}
\end{aligned}$$

Both B_0 and R_0 extend: $B_0 : H_0 \times H_0 \to \mathrm{U}_0'$ and $R_0 : H_1 \times H_1 \to \mathrm{U}_0'$, with

$$\|B_0(u,v)\|_{\mathrm{U}_0'} \leq C\,|u|_{H_0}\,|v|_{H_0}\,, \qquad \|R_0(\phi,\psi)\|_{\mathrm{U}_0'} \leq C\,|\phi|_{H_1}\,|\psi|_{H_1}\,.$$

These extensions are exactly what makes the U_0'-tightness argument work.

Nonlinear coupled system

$$\begin{cases} \dfrac{\partial u}{\partial t} + \nu \mathcal{A}_0 u + B_0(u,u) + \kappa\, R_0(\phi,\phi) = 0, \\ \dfrac{\partial \phi}{\partial t} + \mathcal{A}_1 \phi + B_1(u,\phi) = 0. \end{cases}$$

Weak solution – $\forall (v,\psi) \in V_0 \times V_1$:

$$\frac{d}{dt}\langle u(t), v\rangle_{V_0', V_0} + \nu\, a_0(u(t), v) + b_0(u(t), u(t), v) + \kappa\, r_0(\phi(t), \phi(t), v) = 0,$$

$$\frac{d}{dt}\langle \phi(t), \psi\rangle_{V_1', V_1} + a_1(\phi(t), \psi) + b_1(u(t), \phi(t), \psi) = 0.$$

Energy identity (Lions–Magenes)

Testing with u and ϕ, i.e. by the Lions-Magenes Lemma, we infer that

$$\tfrac{1}{2}\frac{d}{dt}\,|u(t)|_{H_0}^2 + \nu\,\|u(t)\|_{V_0}^2 + b_0(u(t),u(t),u(t)) + \kappa\, r_0(\phi(t),\phi(t),u(t)) = 0,$$

$$\tfrac{1}{2}\frac{d}{dt}\,|\phi(t)|_{H_1}^2 + \|\phi(t)\|_{V_1}^2 + \; b_1(u,\phi,\phi) = 0.$$

Using $b_0(u,u,u) = 0$ and $b_1(u,\phi,\phi) = r_0(\phi,\phi,u)$ (Proposition 2.x), the coupling terms cancel:

$$\boxed{\frac{1}{2}\frac{d}{dt}\Big[|u(t)|_{H_0}^2 + \kappa\,|\phi(t)|_{H_1}^2\Big] + \nu\,\|u(t)\|_{V_0}^2 + \kappa\,\|\phi(t)\|_{V_1}^2 = 0.}$$

This identity is essential, but not yet the final one — we still need the *potential*.

Adding the Cahn–Hilliard potential

With $\varepsilon = \alpha = \varrho_0 = 1$, the Cahn–Hilliard equation with potential reads

$$\frac{\partial \phi(t)}{\partial t} + \mathcal{A}_1 \phi(t) + B_1(u(t), \phi(t)) = \Delta\big(\psi'(\phi)\big), \qquad \text{i.e.} \quad \partial_t \phi + \operatorname{div}(\phi u) = \Delta \mu(\phi).$$

This equation is structurally *similar to a gradient flow*: testing yields

$$\begin{aligned} &\tfrac{1}{2}\frac{d}{dt}\left[|\phi(t)|_{H_1}^2 + 2\int_{\mathcal{O}} \psi(\phi(t))\,\mathrm{d}x \right] + \int_{\mathcal{O}} u(t) \cdot \nabla \phi(t)\, \mu(\phi(t))\,\mathrm{d}x \\ &= - \int_{\mathcal{O}} |\nabla \mu(\phi(t))|^2 \,\mathrm{d}x = - \int_{\mathcal{O}} \big|\nabla\big(-\varepsilon \Delta \phi(t) + \psi'(\phi(t))\big)\big|^2 \,\mathrm{d}x. \end{aligned}$$

This dissipative term is the cornerstone of all a priori estimates.

Equivalently, the natural Lyapunov functional is the free energy

$$\mathscr{E}(u, \phi) = \tfrac{1}{2}\,|u|_{H_0}^2 + \kappa\Big(\tfrac{1}{2}\,|\phi|_{H_1}^2 + \int_{\mathcal{O}} \psi(\phi(x))\,\mathrm{d}x\Big).$$

Symbolically:

$$\boxed{\begin{aligned} &\text{(NS)} \quad \mathrm{d}u + \left[\nu \mathcal{A}_0 u + B_0(u,u) + \kappa\, R_0(\phi,\phi)\right]\mathrm{d}t = \sum_{k=1}^{\infty}\left[\Sigma_k(t)u + G_k(t,u)\right]\mathrm{d}\tilde{w}_k, \\ &\text{(CH)} \quad \frac{\partial \phi}{\partial t} + \mathcal{A}_1\phi + B_1(u,\phi) = \Delta\big(\psi'(\phi)\big), \end{aligned}}$$

where

$$\Sigma_k(t)u = \Pi\big([\sigma_k(t)\cdot\nabla]u\big), \qquad G_k(t,u) = \Pi\big(g_k(t,u)\big),$$

and, to keep the stochastic part divergence-free, the turbulent pressures are determined by

$$\nabla \bar{p}_k(t) = (I - \Pi)\left[(\sigma_k(t)\cdot\nabla)u(t) + g_k(t,u(t))\right].$$

Note the noise is in the *velocity* equation only (transport plus multiplicative parts).

Probabilistic setup and standing assumptions

Let $\widetilde{W} = (\tilde{w}_k(t))_{k\geq 1}$ be an ℓ^2-cylindrical Wiener process on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$, $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ satisfying the usual conditions.

Boundedness

$$\sum_{k=1}^{\infty} |\sigma_k(t,x)|^2 \leq C_1, \qquad \sum_{k=1}^{\infty} |\operatorname{div} \sigma_k(t,x)|^2 \leq C_2, \qquad (t,x) \in [0,T] \times \mathcal{O}.$$

Coercivity assumption

There exists $\delta_0 > 0$ such that, for every $(t,x) \in [0,T] \times \mathcal{O}$ and every $\xi \in \mathbb{R}^d$,

$$\sum_{i,j=1}^{d} \Big[\nu\, \delta_{ij} - \tfrac{1}{2} \sum_{k=1}^{\infty} \sigma_k^i(t,x)\, \sigma_k^j(t,x) \Big] \xi_i\, \xi_j \;\geq\; \delta_0\, |\xi|^2 .$$

This compensates the Itô-to-Stratonovich correction of the transport noise and guarantees parabolicity. It also implies the boundedness condition above.

Assumption on the multiplicative coefficient g

With $\mathrm{Y} := \ell^2(\mathbb{R}^d)$, the map $g : [0,T] \times \mathcal{O} \times \mathbb{R}^d \ni (t,x,z) \mapsto (g_k(t,x,z))_{k\geq 1} \in \mathrm{Y}$ is measurable, continuous in z, and there are $C_g > 0$ and $\widetilde{H} \in L^2([0,T] \times \mathcal{O})$ with

$$\|g(t,x,z)\|_{\mathrm{Y}} \ \leq \ C_g \ |z| + \widetilde{H}(t,x), \qquad (t,x,z) \in [0,T] \times \mathcal{O} \times \mathbb{R}^d.$$

Only *linear growth* and *continuity* in z are required — no Lipschitz condition. Lipschitz continuity in z is imposed only for the 2D uniqueness result.

Σ and G are then $\ell^2(H_0)$-valued: $\Sigma(t)u = (\Sigma_k(t)u)_{k\geq 1}$, $G(t,u) = (G_k(t,u))_{k\geq 1}$.

Product spaces and Itô integral

Recall

$$\mathbb{H} := H_0 \times H_1, \qquad \mathbb{V} := V_0 \times V_1, \qquad \mathcal{U} := \mathrm{U}_0 \times V_1.$$

The Itô integral $\int_0^t \xi(s)\,\mathrm{d}\widetilde{W}(s)$ is an $\mathbb{H}$-valued martingale provided $\xi(s) \in \gamma(\ell^2; \mathbb{H})$ = Hilbert–Schmidt operators.

Useful identification

$$\gamma(\ell^2; \mathbb{H}) \;\cong\; \ell^2(\mathbb{H}), \qquad \gamma(L^2(\cdot); \mathbb{H}) \;\cong\; L^2(\cdot; \mathbb{H}).$$

Hence we may also write $\int_0^t \langle \xi(s), \mathrm{d}\widetilde{W}(s)\rangle$.

Abstractly, with $F = F_1 + F_2$ and $G_0(t, X) = (\Sigma(t)u + G(t, u), 0)$,

$$\mathrm{d}X = F(X)\,\mathrm{d}t + G_0(t, X)\,\mathrm{d}\widetilde{W}(t), \qquad X = (u, \phi), \quad X(0) = (u_0, \phi_0).$$

Definition of weak (martingale) solution

A weak solution of the Stochastic (NS)–(CH) Equations is a $\mathbb{V}$-valued progressively measurable process $X = (u, \phi)$ with $\mathbb{H}$-valued, weakly continuous trajectories, such that:

(i) $\mu(t) = -\Delta\phi(t) + \psi'(\phi(t)) - \langle\psi'(\phi(t))\rangle \in L^2(0, T; H_1)$ a.s., and

$$\mathbb{E}\int_0^T |\mu(s)|^2_{H_1}\, \mathrm{d}s < \infty;$$

(ii) $\mathbb{E}\int_0^T \|F(X(s))\|_{\mathbb{V}'}\, \mathrm{d}s < \infty, \quad \mathbb{E}\int_0^T \|G_0(s, X(s))\|^2_{\ell^2(\mathbb{H})}\, \mathrm{d}s < \infty;$

(iii) (NS)–(CH) Equation holds in $\mathcal{U}'$, in integrated form, a.s. for every $t \in [0, T]$.

Main theorem

Let $\beta = \frac{3}{2}$ if $d = 3$ and $\beta = 2$ if $d = 2$.

Theorem (Existence of a martingale solution)

Let $\mathcal{O} \subset \mathbb{R}^d$, $d \in \{2,3\}$, be bounded of $\mathcal{C}^3$-class, $X_0 = (u_0, \phi_0) \in \mathbb{H}$. Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ be a probability space with right-continuous filtration carrying *two i.i.d. copies* of ℓ^2-cylindrical $\mathbb{F}$-adapted Wiener processes. Then there exist an ℓ^2-cylindrical $\mathbb{F}$-adapted Wiener process W and processes (u, ϕ), μ on *that same space*, such that (W, u, ϕ, μ) is a weak martingale solution, with

$$\mathbb{E} \sup_{s \le T} \|X(s)\|_{\mathbb{H}}^2 + \mathbb{E} \int_0^T \|X(s)\|^2_{V_0 \times H^2_{(0)}} \, \mathrm{d}s < \infty, \qquad \mathbb{E} \int_0^T \|\phi(s)\|^\beta_{V_1} \, \mathrm{d}s < \infty.$$

No change of probability space is needed — it suffices that the given space be rich enough.

Uniqueness in dimension two

Theorem (Pathwise uniqueness, $d = 2$)

Under the assumptions of the previous theorem, if $d = 2$ then the $\mathbb{H}$-valued process X is *strongly* continuous in t, and pathwise uniqueness holds: if (u^1, p^1, ϕ^1) and (u^2, p^2, ϕ^2) are weak solutions on the same filtered probability space driven by the same Brownian motions, then almost surely $u^1 = u^2$, $\phi^1 = \phi^2$, $\nabla(p^1 - p^2) = 0$ and $\nabla(p^1_k - p^2_k) = 0$.

For this result the coefficients g_k are additionally assumed Lipschitz in the third variable.

Idea of the proof

Idea. A *unified framework combining* the approach of Mikulevicius–Rozovskii (2005) with that of Brzeźniak–Motyl (2013). Neither method alone covers the coupled system with transport noise.

Two comments.

(a) We do *not* use the Jakubowski–Skorokhod representation theorem to build a new probability space. Instead we use a *generalisation of the Prohorov theorem* (based on a discussion in Jakubowski 1998, p. 174) and identify the canonical process on $\boldsymbol{Z}_T$ as a martingale solution.

(b) The Brzeźniak–Motyl route does not seem to apply: because of the coupling, no a priori estimate is available for moments of u of order higher than 2.

This allows a *stronger* result — weaker assumptions on the external force:

$$\int_0^T \|f(t)\|_{V_0'}^2 \,\mathrm{d}t < \infty \quad \text{(only } L^2\text{)}, \qquad \text{not} \qquad \int_0^T \|f(t)\|_{V_0'}^p \,\mathrm{d}t < \infty,\ p > 2.$$

Proof: strategy

1. **Finite-dimensional (Galerkin) approximations** X_n.
2. **A priori estimates** via the Lyapunov functional

$$\mathscr{E}(u,\phi) := \tfrac{1}{2}\,|u|_{H_0}^2 + \kappa\Big(\tfrac{1}{2}\,|\phi|_{H_1}^2 + \int_{\mathcal{O}} \psi(\phi(x))\,\mathrm{d}x\Big).$$

   Yields global existence of X_n and the uniform bound

$$\sup_n \mathbb{E}\Big[\sup_{s\le T}\big(|u_n|_{H_0}^2 + \kappa\,|\phi_n|_{H_1}^2 + \kappa\,\|\psi(\phi_n)\|_{L^1}\big) + \delta_0\,\|u_n\|_{L^2(V_0)}^2 + \kappa\,\|\mu_n\|_{L^2(H_1)}^2 + \|\phi_n\|_{L^\beta(V_1)}^\beta\Big] < \infty.$$

3. **Tightness** of the laws $\mathbb{P}_n := \mathrm{Law}(X_n)$ on the non-metrisable space $\boldsymbol{Z_T}$.
4. **Passage to the limit** (generalised Prohorov theorem) and **martingale identification**, then martingale representation.

The tightness space

With $\beta = \frac{3}{2}$ $(d = 3)$, $\beta = 2$ $(d = 2)$, set $\boldsymbol{Z}_T := \boldsymbol{Z}_{T,1} \times \boldsymbol{Z}_{T,2}$ where

$$\boldsymbol{Z}_{T,1} = C([0,T]; \mathrm{U}_0') \cap C([0,T]; H_{0,w}) \cap L^2_w(0,T;V_0) \cap L^2(0,T;H_0),$$
$$\boldsymbol{Z}_{T,2} = C([0,T]; V_1') \cap C([0,T]; H_{1,w}) \cap L^\beta_w(0,T;V_1) \cap L^\beta(0,T;H_1).$$

Set $\mathbb{P}_n := \mathrm{Law}(X_n)$ on $\boldsymbol{Z}_T$ (a product measure $\mathbb{P}^1_n \otimes \mathbb{P}^2_n$).

Theorem (Tightness)

The sequence $(\mathbb{P}_n)_{n\in\mathbb{N}}$ is tight on $\boldsymbol{Z}_T$.

Proof. Aldous–Rebolledo criterion in U_0' resp. V_1', plus compactness of $V_1 \hookrightarrow H_1$.

Jakubowski–Prohorov and uniform energy

Applying the Prohorov theorem in the form of Jakubowski (1998, Theorem 2):

Proposition (generalised Prohorov)

There exist a Borel probability measure $\mathbb{P}_T$ on $\boldsymbol{Z}_T$ and a subsequence (still denoted $\mathbb{P}_n$) converging weakly to $\mathbb{P}_T$, such that for every bounded measurable $f : \boldsymbol{Z}_T \to \mathbb{R}$ which is *sequentially* continuous $\mathbb{P}_T$-a.s.,

$$\int_{\boldsymbol{Z}_T} f \, \mathrm{d}\mathbb{P}_n \longrightarrow \int_{\boldsymbol{Z}_T} f \, \mathrm{d}\mathbb{P}_T.$$

No new probability space, and no a.s. convergent copies, are constructed.

Combined with uniform bounds, we obtain the crucial continuity / integrability result:

$$\int_{\boldsymbol{Z}_T} \left[\sup_{s \in [0,T]} \|X(s)\|_{\mathbb{H}}^2 + \int_0^T \|X(s)\|_{\mathbb{V}}^2 \, \mathrm{d}s \right] \mathrm{d}\mathbb{P}_T < \infty.$$

Martingale identification: the proposition

Proposition

Assume that in $\boldsymbol{Z}_T$ we have $X_n = (u_n, \phi_n) \to X = (u, \phi)$. For each test element $z \in \mathcal{U}$, set

$$M_t^{n,z} : \boldsymbol{Z}_T \ni X \longmapsto e^{\,i\langle X(t), z\rangle} - \int_0^t e^{\,i\langle X(r), z\rangle} \; \mathcal{L}_n^z(r, X)\, \mathrm{d}r \;\in\; \mathbb{C},$$

where

$$\mathcal{L}_n^z(r, X) := i\langle F_n(X(r)),\, z\rangle - \tfrac{1}{2} \left\| \langle G_{0,n}(r, X(r)),\, z\rangle \right\|_{\ell^2}^2,$$

and similarly M_t^z for the limit. Then:

- $(M_t^{n,z})$ is equicontinuous and $M_t^{n,z} \to M_t^z$ for every $t \in [0, T]$;
- $M_t^{n,z}$ is a martingale on $(\boldsymbol{Z}_T, \mathbb{P}_n)$.

The notation φ has been changed to $\mathcal{L}$ to avoid a clash with the phase function ϕ.

Limit martingale and Stroock–Varadhan

Theorem

For every z in the test space $\mathscr{V}$, M_t^z is a continuous local martingale on $(\boldsymbol{Z}_T, \mathbb{D}, \mathbb{P}_T)$, where $\mathbb{D}$ is the augmented filtration generated by the canonical process.

The following classical result of Stroock–Varadhan (Lemma 11.1.2) is used to make the previous passage to the limit rigorous through stopping times.

Lemma 1

For $R > 0$, the hitting time

$$\widetilde{\tau}_R : C([0,T], \mathbb{R}^2) \ni \omega \longmapsto \inf\{\, t \in [0,T] : \ |\omega(t)| \geq R \,\}$$

is lower-semicontinuous. If $\mathbb{P}$ is a Borel probability measure on $C([0,T], \mathbb{R}^2)$, then there exists $\mathcal{R} \subset (0, \infty)$, of at most countable complement, and a full-measure set $\Omega' \subset C([0,T], \mathbb{R}^2)$, such that for every $R \in \mathcal{R}$, $\widetilde{\tau}_R$ is continuous at every $\omega \in \Omega'$.

Representation theorem: from martingale to Itô integral

Theorem (Martingale representation)

Let $\mathcal{U} \subset \mathbb{H} \subset \mathcal{U}'$ be a Gelfand triple, and let the underlying filtered probability space carry two i.i.d. copies of ℓ^2-cylindrical Wiener processes. Let M_t be a $\mathcal{U}'$-valued continuous local martingale such that, for every $u \in \mathcal{U}$,

$$\langle M_t, u\rangle^2_{\mathcal{U}',\mathcal{U}} - \int_0^t \int_E \langle \sigma_s(x), u\rangle^2 \, \widetilde{\kappa}(\mathrm{d}x)\, \mathrm{d}s$$

is a continuous (real-valued) local martingale, for some $\sigma : [0, T] \times \Omega \times E \to \mathcal{U}'$.

Then there exists a $K = L^2(E, \widetilde{\kappa})$-cylindrical Wiener process W on the *same* space such that

$$M_t = \int_0^t \sigma_s \, \mathrm{d}W_s, \qquad t \in [0, T].$$

Applied to our limit martingale M^z, this produces the noise of the limit solution, completing the proof.

Summary

- Coupled stochastic Navier–Stokes / Cahn–Hilliard system in dimension $d \in \{2, 3\}$, with transport noise on the velocity.
- Nonlinear capillary coupling $\operatorname{div}(\nabla\phi \otimes \nabla\phi)$ on the fluid side; Cahn–Hilliard with polynomial (Landau) potential ψ on the phase side, and a mean-free chemical potential $\mu(\phi)$.
- Lyapunov / energy structure survives noise thanks to a coercivity assumption on σ_k (which already encodes the viscosity ν).
- Existence of a martingale solution via Galerkin + Aldous–Rebolledo tightness + a generalised Prohorov theorem + martingale representation, *on the given probability space*.
- A unified Mikulevicius–Rozovskii / Brzeźniak–Motyl framework: weaker assumptions on the forcing than Brzeźniak–Motyl.
- Pathwise uniqueness and strong continuity of paths in dimension 2.

Outlook

The methods developed in the paper allow one to study:

(i) unbounded domains $\mathcal{O}$;

(ii) a Cahn–Hilliard equation which itself depends on the noise;

(iii) singular potentials, e.g. the Flory–Huggins potential.

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Thank you!

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