Entropy chaos on path space

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A general interacting particle system

We consider an N-dimensional interacting diffusions of the type

$$X_N(t) = b_N(X_N(t), t)t + W_N(t), \tag{1}$$

where W_N is a Nd-dimensional Wiener process, and let

$$X(t) = b(X(t), t)t + W(t), \tag{2}$$

where W a d-dimensional Wiener process.
We will work on the path space under the hypothesis of *finite relative entropy condition* (w.r.t the Wiener measure) or, equivalently, of the square-integrability property of the drift coefficient

$$\mathbb{E}_{\mathbb{P}_N}\left[\int_0^T b_N^2(t)dt\right] = \int_0^T \int_{\mathbb{R}^{Nd}} |b_N(x,t)|^2 \mu_t^N(dx)dt < \infty, \tag{3}$$

where μ_t^N denotes the marginal law of $X_N(t)$ for a fixed time $t \in [0, T]$.

Schedule of presentation

1. Propagation of chaos on path space via entropy methods
2. A criterium for proving propagation of chaos on path space for a general interacting particle system under finite entropy conditions starting from the weak entropic chaos
3. The additional condition w.r.t. the weak entropic chaos permitting to obtain a strong form of propagation of chaos
4. Application to stochastic description of Bose-Einstein Condensation

Boltzmann entropy

Let us denote by $\mathcal{P}(E)$ the set of probability measures $\mathbb{P}$ on a Polish space E, equipped with its Borel $\sigma-$field.

Definition (Boltzmann entropy)

Given a measure μ on $E = \mathbb{R}^d$ with density ρ with respect to the Lebesgue measure on E, if ρ has a finite moment of order $k > 0$, then the Boltzmann entropy $H(\mu) \equiv H(\rho)$ is defined as

$$H(\rho) := \int_{\mathbb{R}^k} \rho(x) \log(\rho(x)) dx.$$

See, e.g., Hauray-Mischler: On Kac chaos and related topics, 2014.

Relative entropy

The relative entropy between two probability measures, differently from the Boltzmann entropy, is always well-defined.

Definition

Relative entropy Given $\mu, \nu \in \mathcal{P}(E)$,, the relative entropy between the two measures is defined, when μ is *absolutely continuous* with respect to ν as

$$H(\mu|\nu) := \int_{\mathbb{R}^d} \log\left(\frac{d\mu}{d\nu}\right) d\mu,$$

where $\frac{d\mu}{d\nu}$ is the Radon-Nikodym derivative, and $H(\mu|\nu) := \infty$ when μ is not absolutely continuous with respect to ν.

The relative entropy is lower semi-continuous and, even thought it is not a distance, it controls the total variation (TV) distance thank to the celebrated Csiszar-Kullback-Leibler-Pinsker inequality

$$||\mu - \nu||^2_{TV} \leq 2H(\mu|\nu). \tag{4}$$

Finite relative entropy and Föllmer intrinsic drift

Let $\Omega = C([0,T];\mathbb{R}^d)$, let $X_t(\omega) = \omega(t)$ be the canonical process, and let $\mathbb{W}$ be the Wiener measure on Ω, endowed with the canonical filtration $(\mathcal{F}_t)_{0\le t\le T}$. Let $\mathbb{Q}$ be a probability measure on Ω.
Then the following conditions are equivalent:

(i)
$$\mathcal{H}(\mathbb{Q} \mid \mathbb{W}) < +\infty.$$

(ii) $\mathbb{Q} \ll \mathbb{W}$ and there exists a progressively measurable process $b^{\mathbb{Q}}$ s.t.
$$B^{\mathbb{Q}}_t := X_t - \int_0^t b^{\mathbb{Q}}_s \, ds$$
is a Brownian motion under $\mathbb{Q}$.

In this case, $b^{\mathbb{Q}}$ is unique $\mathbb{Q}\otimes dt$-a.e. and
$$\mathcal{H}(\mathbb{Q} \mid \mathbb{W}) = \frac{1}{2}\mathbb{E}_{\mathbb{Q}}\left[\int_0^T |b^{\mathbb{Q}}_t|^2 \, dt\right].$$

Some remarks

- We stress that the above proposition says that the finiteness of relative entropy is a stronger condition w.r.t. the absolutely continuous property.
- The result can be derived by a general *finite-entropy Girsanov theorem* for continuous semimartingales, see Leonard 2012.
- The original construction goes back to Follmer1985,Follmer1986 on the time reversibility of diffusion processes.
  For a recent deep generalization of the above Föllmer framework see Cattiaux, Conforti, Gentil, Leonard 2021.

Weak uniqueness of the $SDE(b^{\mathbb{Q}}, 1)$

Proposition

Assume that there exists at least one weak solution of

$$dX_t = b_t^{\mathbb{Q}}(X)\, dt + dB_t.$$

Then there exists at most one law $\mathbb{Q}$ of a weak solution satisfying

$$H(\mathbb{Q} \mid \mathbb{W}) < \infty.$$

Since in the following all the involved probability measures are assumed to be absolutely continuous w.r.t. Wiener measure, then for every $t \in (0, T]$ the marginals $\mu_t(dx)$ are absolutely continuous w.r.t. Lebesgue measure, which means that the probability densities always exists, i.e. $\mu_t(dx) = \rho_t(x)dx$, even though not necessarily regular.

Entropic chaos versus standard Kac chaos at fixed time

Definition (Propagation of chaos or Kac chaos)

We say that $\rho_N^{(n)}(t)$ is $\bar{\rho}(t)$-*chaotic* if, for any fixed $k \geq 1$,

$$\rho_N^{(k)}(t) \xrightarrow[N\to\infty]{} \bar{\rho}^{\otimes k}(t),$$

where $\rho_N^{(n)}(t)$ denotes the $k-$th marginal of $\rho_N(t)$, $\bar{\rho}^{\otimes k}$ denotes the $k-$th product measure, and the convergence is understood in the sense of weak convergence of probability measures.

We stress that original Kac chaos property is a property concerning the marginal at a fixed time t of the laws of the process.

Entropy chaos at fixed time

There exists a stronger notion of chaoticity, denominated entropy chaos, introduced by Carlen-Carvalho-Roux-Villani 2010.

Definition (Entropy chaos)

We say that ρ_N is $\rho-$entropic chaotic if for any fixed $t \in [0, T]$

1. $\rho_N^1(t) \to \rho(t)$ weakly.
2. $\frac{1}{N} H(\rho_N(t)) \to H(\rho(t))$.

The entropy chaos at fixed time t is stronger that Kac chaos and implies it. More recently, some authors call entropy chaos the following stronger chaoticity property

$$\mathcal{H}\left(\rho_N(t) \mid \bar{\rho}^{\otimes N}(t)\right) = \frac{1}{N} H\left(\rho_N(t) \mid \bar{\rho}^{\otimes N}(t)\right) \xrightarrow[N\to\infty]{} 0.$$

See Jabin-Wang 2018, Lim-Lu-Nolen 2024 and Lacker 2023 for a quantitative approach to relative entropy methods.

Entropy chaos on path space

Definition (Chaoticity properties on path space)

Let $\mathbb{P}_N$ be a sequence of (symmetric) probability measures on Ω^N and let $\mathbb{P}$ be a probability measure on $\Omega = C([0,T],\mathbb{R}^d)$. Denote by $\mathbb{P}_N^{(k)}$ the $k-$th marginal measure of $\mathbb{P}_N$.

1. We say that $\mathbb{P}_N$ is $\mathbb{P}$-Kac chaotic if, for any $k \in \mathbb{N}$, $\mathbb{P}_N^{(k)}$ converges weakly to $\mathbb{P}^{\otimes k}$.
2. We say that $\mathbb{P}_N$ is (weakly) $\mathbb{P}$-entropy chaotic (w.r.t. Wiener measure) if

$$\lim_{N\to+\infty} \mathcal{H}\left(\mathbb{P}_N^{(k)}|\mathbb{W}^{\otimes k}\right) = \mathcal{H}\left(\mathbb{P}^{(k)}|\mathbb{W}^{\otimes k}\right).$$

3. We say that $\mathbb{P}_N$ is strongly $\mathbb{P}$-entropy chaotic if

$$\lim_{N\to+\infty} \mathcal{H}\left(\mathbb{P}_N^{(k)}|\mathbb{P}^{\otimes k}\right) = 0.$$

A criterium for Kac chaos via weak entropic chaos

Theorem (Borasi,De Vecchi,Ugolini 2025)

Let X_N the family of N $d-$dim. diffusions with law $\mathbb{P}_N$ and let X a d-dim. diffusion with law $\mathbb{P}$ and drift c. Let us assume:

(i) $H(\mathbb{P}_N|\mathbb{W}^{\otimes N}) < +\infty$, *for all* $N \in \mathbb{N}$ *and* $H(\mathbb{P}|\mathbb{W}) < +\infty$ *and*

$$\limsup_{N\to+\infty} \mathcal{H}(\mathbb{P}_N|\mathbb{W}^{\otimes N}) \leq H(\mathbb{P}|\mathbb{W}).$$

(ii) *The marginals* μ_t^N *of* $X_N(t)$, *and* μ_t *of* $X(t)$ *exist* $\forall t \in [0,T]$ *and satisfy* $\lim_{N\to\infty} \mu_t^N = \mu_t$, $\forall t \in [0,T]$, *weakly for* t *Leb. a.s.*

(iii) *For every* $K : \mathbb{R}^d \times [0,T] \to \mathbb{R}^d$ *continuous b.f. we have*

$$\lim_{N\to\infty} \mathbb{E}\left[\int_0^T K(X_N(t),t)\cdot dX_N(t)\right] = \mathbb{E}\left[\int_0^T K(X(t),t)\cdot dX(t)\right],$$

then, for any $k \in \mathbb{N}$, $\mathbb{P}_N^{(k)}$ *converges to* $\mathbb{P}^{\otimes k}$ *weakly.*

Some remarks

(i) Since the relative entropy is lower semi-continuous under weak convergence of the probability measures, then assumption (i) together with (iii) implies that

$$\lim_{N\to\infty} \mathcal{H}(\mathbb{P}_N|\mathbb{W}^{\otimes N}) = H(\mathbb{P}|\mathbb{W}),$$

which is the weak entropic chaos we have introduced.

(ii) We stress that we do not require any regularity assumptions on the drift coefficient, only the relative entropy boundedness condition in (i), implying, by Girsanov theory, the L^2-type condition on the drift.

(iii) The hypothesis (ii) and (iii) have to be proved according to the specific model at hand. In the stochastic description of time-dependent Bose Einstein Condensation we use a compactness argument and some quantum energy bounds (Borasi, De Vecchi, Ugolini 2025).

Useful results

We first recall some general results concerning weak convergence of stochastic integrals and Markovian projections.

Proposition (De Vecchi, Rigoni, 2024)

Consider a sequence of diffusion laws $\mathbb{P}_M$, $M \in \mathbb{N}$, on the same filtered space $(\Omega, \mathcal{F}_{[0,T]})$, where each $\mathbb{P}_M$ is absolutely continuous w.r.t. the Wiener measure $\mathbb{W}$ on Ω, weakly converging to a probability measure $\tilde{\mathbb{P}}$ with $\sup_{M\in\mathbb{N}} H(\mathbb{P}_M|\mathbb{W}) < +\infty$. Then, for any bounded continuous function $K : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ we have

$$\lim_{M\to\infty} \mathbb{E}_{\mathbb{P}_M}\Big[\int_0^T K(X(t),t)\cdot dX(t)\Big] = \mathbb{E}_{\tilde{P}}\Big[\int_0^T K(X(t),t)\cdot dX(t)\Big].$$

where $X(t), t \in [0,T]$ denotes the canonical process on Ω.

Useful results

Proposition (De Vecchi, Rigoni, 2024)

Given the canonical process X with law $\mathbb{P}$, consider a measurable function $\mathcal{D} : \Omega \times [0, T] \to$ such that $\mathbb{E}_{\mathbb{P}}[\int_0^T |\mathcal{D}(\cdot, t)|^2 dt] < +\infty$. Then there exists a measurable function $D : \mathbb{R}^d \times [0, T] \to \mathbb{R}$ that satisfies the following properties, $\forall f \in C_b([0, T] \times \mathbb{R}^d)$

1. $\int_0^T \int_{\mathbb{R}^d} |D(x, t)|^2 \mu_t(dx) dt < +\infty$;
2. $$\int_0^T \int_{\mathbb{R}^d} D(x, t) f(x, t) \mu_t(dx) = \mathbb{E}_{\mathbb{P}}\Big[\int_0^T \mathcal{D}(\cdot, t) f(X_t(\cdot), t) dt \Big]$$
3. $$\int_0^T \int_{\mathbb{R}^d} |D(x, t)|^2 \mu_t(dx) dt \leq \mathbb{E}_{\mathbb{P}}[\int_0^T |\mathcal{D}(\cdot, t)|^2 dt]$$

   where the equality holds if and only if $\mathcal{D}(\cdot, t) = D(X_t(\cdot), t)$.

Sketch of the proof of the criterium

1) By (i) and using the property of compactness of the level sets of the relative entropy w.r.t. the weak convergence of probability measures there exists a convergent subsequence $\mathbb{P}_{k(N)} \to \tilde{\mathbb{P}}$.
2) By lower semicontinuity of relative entropy $H(\tilde{\mathbb{P}}|\mathbb{W}) \leq H(\mathbb{P}|\mathbb{W}) < \infty$
3) $\exists!$ Föllmer intrinsic drift $\tilde{b}$ such that the canonical process under $\tilde{\mathbb{P}}$ weakly solves $SDE(\tilde{b}, 1)$ not necessarily Markovian.
4) By (ii), i.e. from the weak convergence of the fixed time marginals, we have, for any $F \in C_b(\mathbb{R}^d \times [0, T])$,

$$\mathbb{E}_{\tilde{\mathbb{P}}}\left[\int_0^T F(X(t), t)dt\right] = \mathbb{E}_{\mathbb{P}}\left[\int_0^T F(X(t), t)dt\right],$$

which implies that $\tilde{\mu}_t = \mu_t, \quad dt - a.e.$.

Sketch of the proof of the criterium

5) By (iii) and weak convergence of stochastic integrals, for any $K \in C_b(\mathbb{R}^d \times [0, T]; \mathbb{R}^d)$,

$$\mathbb{E}_{\mathbb{P}}\left[\int_0^T K(X(t), t) \cdot dX(t)\right] = \mathbb{E}_{\tilde{\mathbb{P}}}\left[\int_0^T K(X(t), t) \cdot dX(t)\right].$$

6) Since $\tilde{\mathbb{P}}$ is the law of the non Markovian $SDE(\tilde{b}, 1)$ and $\mathbb{P}$ is the law of the Markovian $SDE(c, 1)$ and passing to the Markovian projection γ of $\tilde{b}$ we get

$$\int_0^T \int_{\mathbb{R}^d} K(y, t) \cdot c(y, t)\mu_t(y)dydt = \int_0^T \int_{\mathbb{R}^d} K(y, t) \cdot \gamma(y, t)\tilde{\mu}_t(y)t \quad (5)$$

Since by point 4) above $\tilde{\mu}_t = \mu_t$, then we get $\gamma = c$, $\tilde{\mu}_t(dy)dt$-a.e.

Sketch of the proof of the criterium

7) By the minimization property of the Markovian projection

$$\int_0^T \int_{\mathbb{R}^d} |c(y,t)|^2 \mu_t(dy)dt = \int_0^T \int_{\mathbb{R}^d} |\gamma(y,t)|^2 \mu_t(dy)dt \leq \mathbb{E}_{\tilde{P}} \left[\int_0^T |\tilde{b}(\cdot,t)|^2 dt \right]$$

which gives $H(\tilde{\mathbb{P}}|\mathbb{W}) \geq H(\mathbb{P}|\mathbb{W})$. Since at point 2) we have proved the opposite inequality, then $H(\tilde{\mathbb{P}}|\mathbb{W}) = H(\mathbb{P}|\mathbb{W})$, which implies, by the minimizing property of the Markovian projection, that $\tilde{b}$ is Markovian and equal to c.
8) Since $W(t) = X(t) - X(0) - \int_0^t c(X(t),t)dt$ is a Brownian motion both under $\mathbb{P}$ and under $\tilde{\mathbb{P}}$, this means that both probability measures have the same Föllmer intrinsic drift c and they have finite relative entropy w.r.t. Wiener measure. By the uniqueness result we have that $\mathbb{P} = \tilde{\mathbb{P}}$.

Strong entropy chaos via weak one

We finally provide the condition under which, starting from the weak entropic chaos, the strong entropic chaos holds in our general stochastic model.

Theorem

Suppose that in addition to the hypotheses of Theorem (Weak criterium Thm) we have that the limit drift b is continuous and, for any $t \in [0, T]$, $|b(x,t)| \lesssim |x|^2 + 1$.
Then if $\mathbb{P}_N$ is weakly $\mathbb{P}$-entropy chaotic it is also strongly $\mathbb{P}$-entropy chaotic.

Proof: from weak to strong entropy chaos

By Girsanov theory we get

$$\mathcal{H}(\mathbb{P}_N|\mathbb{P}^{\otimes N}) = \int_0^T \int_{\mathbb{R}^{dN}} |b_N(x_1, ..., x_N, t) - b(x_1, t)|^2 \, \rho_N(x_1, ..., x_N, t) dxdt$$

$$= \int_0^T \int_{\mathbb{R}^{dN}} |b_N|^2 \rho_N(x_1, ..., x_n, t) dxdt$$

$$- 2 \int_0^T \int_{\mathbb{R}^{dN}} b(x_1, t) \cdot b_N(x_1, ..., x_N, t) \rho_N(x_1, ..., x_N, t) dxdt +$$

$$+ \int_0^T \int_{\mathbb{R}^{dN}} |b(x_1, t)|^2 \rho_N(x_1, ..., x_N, t) dxdt.$$

Since b is regular, the second term converges by assumption (iii) in Thm (Criterium) and the third term convergences by weak convergence of the marginal laws. The convergence of $\mathcal{H}(\mathbb{P}_N|\mathbb{P}^{\otimes N})$ to zero is guaranteed by the weak entropic chaos deriving by assumption (i) in Thm (Criterium).

Applications

We note that the model here analyzed includes also the usual mean field form of the drift given by

$$b_N(x_N, t) = \frac{1}{N}\sum_{j=1}^{N} \nabla\Phi(X_i - X_j),$$

with

$$\mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\nabla\Phi(x-y)|^2 d\mu_t(dx) d\mu_t(dy)\right] < \infty,$$

where μ_t denotes the marginal law of the single component X_j at time t. All the interacting diffusion systems in which the drift depends on all the N variables are included in our stochastic scheme.

Bose Einstein Condensation

A central problem in mathematical physics is *to derive a non-linear one-particle model starting from the linear N-particle one via an infinite particle limit.*

1. Time dependent BEC (we provide the convergence of the fixed-time $k-$marginal of the joint distribution of the N Bose particles, in the sense of weak convergence of probability measures, to the corresponding tensorized objects. Furthermore, we prove the weak convergence on the path-space of the overall process probability law to the limit process law associated to the nonlinear Schrödinger equation via Nelson map)
2. Stationary BEC (considering the ground state wave function, the minimizer of the energy functional, we describe the condensation phenomenon by proving the propagation of chaos in the infinite particle limit)

N-particles Schrödinger equation

Let $\Psi_N = \Psi_N(x_1, \ldots, x_N)$ be the wave function of N interacting Bose particles which evolve in time according to the following $N-$ particles Schrödinger equation on $\mathbb{R}^{Nd}$:

$$i\hbar\partial_t \Psi_N = -\frac{\hbar^2}{2} \sum_{1\leq j\leq N} \Delta_{x_j}\Psi_N + \frac{1}{N} \sum_{1\leq j<k\leq N} V(|x_j - x_k|)\Psi_N \qquad \text{(SE)}$$
$$\Psi_N(t=0) = \Psi_N^0(x_1, x_2, ..., x_N)$$

where the term $\frac{1}{N}$ before the potential V reveals the "weak coupling scaling" and where the interacting potential V is assumed to be real-valued and bounded from below.

Non-linear Schrödinger equation

Starting from the linear $N-$particle Schrödinger equation, under a suitable mean-field infinite particle limit, one obtains the self-interacting one-particle nonlinear Schrödinger equation on $\mathbb{R}^d$:

$$i\hbar\partial_t\psi(x,t) = -\frac{\hbar^2}{2}\Delta_x\psi(x,t) + \int V(|x-z|)|\psi(z,t)|^2\, dz \cdot \psi(x,t), \tag{NLSE}$$

N-interacting diffusions system by Nelson map

Let ψ_N be the solution to the N-particle Schrödinger equation for Bose particles and consider $(\Omega^N, \mathcal{F}^N, \mathcal{F}^N_t, \hat{X}_t)$, with $\Omega = C([0,T], \mathbb{R}^{dN})$, $\hat{X}_t(\omega) = \omega(t)$ and $\mathcal{F}^N_t = \sigma(\hat{X}_s, s \leq t)$.

Theorem (Carlen theorem,1985)

If $\|\nabla \Psi^0_N\|^2_{L^2} < +\infty$, then there exists an unique Borel probability measure $\mathbb{P}_N$ such that
i) $(\Omega^N, \mathcal{F}^N, \mathcal{F}^N_t, \hat{X}_t, \mathbb{P}_N)$ is a Markov process;
ii) the image of $\mathbb{P}_N$ under $\hat{X}_t$ has density $\rho_N(x^N, t) = |\Psi_N(x^N, t)|^2$;
iii) $\hat{W}_t := \hat{X}_t - \hat{X}_0 - \int_0^t b_N(\hat{X}_s, s)ds$
where

$$b_N = u_N + v_N, \quad u_N = \frac{\nabla \rho_N}{2\rho_N} \quad v_N = Im[\frac{\nabla \Psi^N_t}{\Psi^N_t}]$$

is a $(\mathbb{P}_N, \mathcal{F}^N_t)$ standard Brownian motion on $\mathbb{R}^{dN}$.

Convergence theorem in time-dependent BEC

On the path-space $\Omega = C([0,T],\mathbb{R}^{Nd})$, we denote by

- $\mathbb{P}_N$ the law of the process X_N
- $\mathbb{P}_N^k$ the k-marginal law.
- $X_{\infty,k} = (X_1, X_2, \ldots, X_k)$, with law $\mathbb{P}_{\infty,k}$, where each $X_i \equiv X$, being the Nelson diffusion corresponding to the one-particle nonlinear Schrödinger equation.

Theorem

Let us suppose that $\Psi_N(0)$ satisfies the property

(i) $\Psi_N^0 = (\psi^0)^N \in H^1(\mathbb{R}^{Nd})$, $\psi^0 \in H^1(\mathbb{R}^d)$,

Then $\mathbb{P}_{N,k}$ weakly converges to $\mathbb{P}_{\infty,k}$ on the path space Ω.

Convergence theorem (sketch of the proof)

We plan to verify the assumptions of the Theorem (Criterium) in order to prove propagation of chaos in BEC.

- By a well-known result (see, e.g., Föllmer 1986) we have

$$\mathcal{H}(\mathbb{P}_N|\mathbb{W}^{\otimes N}) = \mathbb{E}[\int_0^t |b_N(X^N(t),t)|^2 dt],$$

$$H(\mathbb{P}|\mathbb{W}) = \mathbb{E}[\int_0^t |b(X(t),t)|^2 dt].$$

- We can establish the following relative entropy decomposition

$$\mathcal{H}(\mathbb{P}_N|\mathbb{W}^{\otimes N}) = \int_0^T \mathcal{K}_N(t)dt + H(\rho_N(0)) - H(\rho_N(T)),$$

$$\mathcal{H}(\mathbb{P}|\mathbb{W}) = \int_0^T \mathcal{K}(t)dt + H(\rho(0)) - H(\rho(T)),$$

where $\mathcal{K}_N(t), \mathcal{K}(t)$ are the quantum kinetic energies.

- In particular $\mathcal{H}(\mathbb{P}_N|\mathbb{W}^{\otimes N}) < +\infty$, $\mathcal{H}(\mathbb{P}|\mathbb{W}) < +\infty$ and as $N \uparrow \infty$

$$\frac{1}{N}\mathcal{K}_N(t) = \int_{\mathbb{R}^{dN}} |u_N|^2 + |v_N|^2)(t)\rho_N(t)dx^N$$
$$\to \mathcal{K}(t) = \int_{\mathbb{R}^d} (|u|^2 + |v|^2)(t)\rho(t)dx$$

  via energy conservation by proving the potential energy convergence.
- By using quantum results we can prove that the marginal probability densities $\rho_N^k(t)$ converge weakly to $\rho^{\otimes k}(t)$, for all $t \in [0, T]$, therefore the assumption (ii) of the Theorem (Criterium) holds.
- We establish that

$$H(\rho) \leq \liminf_N \frac{1}{N} H(\rho_N),$$

  and finally that

$$\limsup_N \mathcal{H}(\mathbb{P}_{N,k}|\mathbb{W}^{\otimes k}) \leq \mathcal{H}(\mathbb{P}^{\otimes k}|\mathbb{W}^{\otimes k}),$$

  which implies that assumption (i) of the Theorem (Criterium), that is the weak entropic chaos, is satisfied.

- Finally, for any $K : \mathbb{R}^d \times [0, T] \to \mathbb{R}^d$, continuous bounded function

$$\lim_{N\to\infty} \mathbb{E}[\int_0^t b_{N,k}(X_{N,k}(s), s) \cdot K(X_{N,k}, s)ds] =$$

$$\mathbb{E}[\int_0^t b_\infty(X_{\infty,k}(s), s) \cdot K(X_{\infty,k}, s)ds],$$

  where $b_{\infty,k} = (b, b, \dots b)$ and $X_{\infty,k} = (X_1, X_2, \dots, X_k)$.
  Thus, assumption (iii) of the Theorem (Criterium) is satisfied.
  Applying the Theorem we get the propagation of chaos result.

Stationary scenario of BEC

The stationary scenario of BEC is quite different from the previous one, and it is also very interesting from the convergence problem point of view. Let

$$E[\Psi] = \sum_{i=1}^{N} \int_{\mathbb{R}^{3N}} |\nabla_i \Psi|^2 d\mathbf{r}_1 \cdots d\mathbf{r}_N +$$

$$+ \sum_{i=1}^{N} \int_{\mathbb{R}^{3N}} V(\mathbf{r}_i)|\Psi|^2 d\mathbf{r}_1 \cdots d\mathbf{r}_N + \frac{1}{2} \sum_{i=2}^{N} \int v(\mathbf{r}_1 - \mathbf{r}_i)|\Psi|^2 d\mathbf{r}_1 \cdots d\mathbf{r}_N$$

be the mean quantum mechanical energy associated with Ψ.

$$E[\Psi_N] := \inf\{E(\Psi) : \int |\Psi|^2 = 1\}$$

is called *ground state energy*.

Mean-field, Intermediate and GP scaling limits

We assume
h1) $V(|\mathbf{r_i}|)$ is locally bounded, positive and going to infinity when $|\mathbf{r_i}|$ goes to infinity.
h2) v_0 is smooth, compactly supported, non negative, spherically symmetric.
h3)

$$v_N(r) = \frac{N^{3\beta}}{N-1} v_0(N^\beta \mathbf{r}), \quad 0 \leq \beta < 1$$

($\beta = 1$: Gross-Pitaevskii scaling)

Non-linear Schrödinger model

The BE condensate is usually described by the minimizer $\phi_{nlS} \in L^2(\mathbb{R}^3)$ of the non-linear Schrëdinger (energy) functional

$$E_{nlS}[\phi] = \int (\frac{\hbar^2}{2m}|\nabla\phi(\mathbf{r})|^2 + V(r)|\phi(\mathbf{r})|^2 + g|\phi(\mathbf{r})|^4)d\mathbf{r} \tag{6}$$

under the condition $\int_{\mathbb{R}^3} |\phi^{GP}(\mathbf{r})|^2 d\mathbf{r} = 1$, where

$$g = \int v_0(\mathbf{r})\mathbf{dr} \quad (\textit{mean-field}) \quad \mathbf{g} = \mathbf{4}\pi\mathbf{Na} \quad (\textit{GP case})$$

The minimizer ϕ_{nlS} solves the stationary cubic non-linear equation

$$-\frac{\hbar^2}{2m}\triangle\phi_{nlS} + V\phi_{nlS} + 2g|\phi_{nlS}|^2\phi_{nlS} = \mu_{nlS}\phi_{nlS} \tag{7}$$

μ_{nlS} the real-valued Lagrange multiplier of the normalization constraint.

Theorem (Energy components convergence, Lieb et coll. 2005)

Under the same (quantum) hypothesis h1), h2) ,h3) it holds that:

$$\lim_{N\uparrow\infty}\int_{\mathbb{R}^3}\int_{\mathbb{R}^{3N-3}}|\nabla_1\Psi_N(\mathbf{r}_1,\dots,\mathbf{r}_N)|^2 d\mathbf{r}_1\cdots d\mathbf{r}_N = \int_{\mathbb{R}^3}|\nabla\phi_{GP}(\mathbf{r})|^2 d\mathbf{r} + gs\int_{\mathbb{R}^3}|\phi_{GP}(\mathbf{r})|^4 d\mathbf{r} \quad (8)$$

In entropic terms this means that weak entropic chaos does not hold(!):

$$\lim_{N\uparrow\infty}\mathcal{H}(\mathbb{P}_N|\mathbb{W}^N) = H(\mathbb{P}|\mathbb{W}) + gs\int_{\mathbb{R}^3}|\phi_{GP}(\mathbf{r})|^4 d\mathbf{r}$$

$$\lim_{N\uparrow\infty}\int_{\mathbb{R}^3}\int_{\mathbb{R}^{3N-3}} V(\mathbf{r}_1)\rho_N(\mathbf{r}_1,...,\mathbf{r}_N)d\mathbf{r}_1\cdots d\mathbf{r}_N=\int_{\mathbb{R}^3} V(\mathbf{r})\rho_{GP}(\mathbf{r})d\mathbf{r} \quad (9)$$

and

$$\lim_{N\uparrow\infty}\frac{1}{2}\sum_{j=2}^{N}\int_{\mathbb{R}^3}\int_{\mathbb{R}^{3N-3}} v(|\mathbf{r}_1-\mathbf{r}_j|)\rho_N(\mathbf{r}_1,...,\mathbf{r}_N)d\mathbf{r}_1\cdots d\mathbf{r}_N=$$

$$(1-s)g\int_{\mathbb{R}^3}|\rho_{GP}(\mathbf{r})|^2 d\mathbf{r} \quad (10)$$

Some results

Since the extra term $gs \int_{\mathbb{R}^3} |\phi_{GP}(\mathbf{r})|^4 d\mathbf{r}$ is *finite* and furthermore it is supported in a domain having zero Lebesgue measure (because asymptotically the interaction is a Dirac delta function), by introducing suitable stopping time it is possible to prove by entropy methods the following weak convergence result.

Theorem (Albeverio, De Vecchi, Ugolini, 2017)

The one-particle probability measure $\mathbb{P}^1_N$ weakly converges on the path space to the law $\mathbb{P}_{GP}$ associated to the asymptotic Gross-Pitaevski equation.

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