The logistic Keller–Segel equation and its particle approximation

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Keller–Segel model: Motivation

- System of PDE describing the collective motion of cells (typically bacteria), attracted by a chemical substance (chemoattractant) that they themselves emit.
- The parabolic-elliptic form is

$$\begin{cases} \partial_t \rho = \Delta \rho - \chi \nabla \cdot (\rho \nabla c), & t > 0, x \in \mathbb{R}^d \\ -\Delta c = \rho, & t > 0, x \in \mathbb{R}^d, \end{cases}$$

  where $\chi > 0$ is the intensity of chemoattractant.
- Solving Poisson's equation for c gives $\nabla c = K * \rho$, where $K(x) \propto -\frac{x}{|x|^d}$, and so

$$\partial_t \rho = \Delta \rho - \chi \nabla \cdot (\rho K * \rho).$$

- Competition between diffusion and attraction.
- For large χ solutions blow up in finite time (mass concentration), e.g. $\chi > \frac{8\pi}{m}$ in $d = 2$ [Per04].

Keller–Segel with logistic source

- There are several mechanisms to suppress or delay blow up, one of which is a logistic source term modelling births and deaths.
- Introduced by Tello and Winkler [TW07] on bounded domains and recently extended to $\mathbb{R}^d$ by Cavallazzi, Richard and Tomašević [CRT25]:

$$\partial_t \rho = \Delta \rho - \chi \nabla \cdot (\rho K * \rho) + \nu \rho - \mu \rho^2,$$

where $\nu \geq 0$ is the birth rate and $\mu \rho \geq 0$ is the death rate.

[CRT25, Thm. 1] (Well-posedness)

For $d \geq 2$, for any $\nu \geq 0$, if

$$\mu > \frac{d-2}{d} \chi,$$

there exist a unique global mild solution.

Quantitative approximation by particles I

- A central question is whether the logistic Keller–Segel equation can be derived from a system of interacting particles.
- We consider a logistic particle system starting with N particles using a Poisson process:
  - particles branch at rate ν;
  - particles die at a rate proportional to the local density.
- Between birth and death events, particles evolve according to

$$dX_t^k = \chi F_A\left(\frac{1}{N}\sum_{j\in I_t} K * \theta^N(X_t^k - X_t^j)\right) dt + \sqrt{2}\, dB_t^k,$$

  where I_t= particles alive at time t.
- The interaction is regularised by a cutoff F_A and mollifier θ^N, Oelschläger [Oel85]. Used for technical reasons.

Quantitative approximation by particles II

- Let $\mu_t^N := \frac{1}{N} \sum_{k \in I_t} \delta_{X_t^k}$ denote the empirical measure and $\rho_t^N := \theta^N * \mu_t^N$ its mollification.

[CRT25, Thm. 2] (Mean field limit)

Whenever $\mu > \frac{d-2}{d}\chi$, for any $T, \epsilon > 0$ and initial number of particles N, there exists an explicit rate $\Gamma > 0$ such that

$$\sup_{t \in [0,T]} \|\rho_t^N - \rho_t\| \lesssim \|\rho_0^N - \rho_0\| + N^{-\Gamma+\varepsilon}.$$

- "Large population" result.
- The same convergence rate holds for the empirical measure itself (in Kantorovich–Rubinstein distance).

Current work: Removing moderate interactions

- Remark: moderate interacting particles were used because the well-posedness of the singular system particle is not known.
- In dimension $d = 2$, the classical supercritical Keller–Segel interacting particle system $\chi > \frac{8\pi}{m}$ around the time of blow up was analysed by Fournier and Tardy [FT25]

$$dX_t^k = \frac{\chi}{N} \sum_{j \neq k} \frac{X_t^j - X_t^k}{\|X_t^j - X_t^k\|^2} \, dt + \sqrt{2} \, dB_t^k,$$

- They show that at the time of explosion, there is a critical proportion $k_0 = \lceil N \frac{8\pi}{m\chi} \rceil$ of particles which collide together.
- Our aim is to analyse the logistic particle system, and show that the killing mechanism prevents there from being a critical proportion of particles close together.

Current results: The particle system I

- A difference in the microscopic and macroscopic system.
- No blow up of the PDE, but it is possible with low probability that the killing mechanism does not kill aggregating particles.

Theorem (No critical collision with high prob.)

Let $d = 2$, fix terminal time $T > 0$ and arbitrary killing rate $\mu > 0$.
For any $\epsilon > 0$, there exists an initial number of particles N sufficiently large such that

$$\mathbb{P}(\text{particle system has a critical collision before time } T) < \epsilon.$$

Current results: proof of no critical collisions

Proof idea

- Introduce a functional $F^\gamma(\mu_t^N)$ measuring the proportion of particles that are γ distance apart.
- Itô formula on the particles tells us that if proportion exceeds level $k_0/2$, the killing mechanism dominates and generates a strong negative drift:

$$F_t^\gamma \lesssim \frac{k_0}{2} + \frac{1}{N}M_t - Nc_{\nu,\mu},$$

  with $c_{\nu,\mu} > 0$ due to the killing mechanism.
- The negative drift overwhelms the martingale fluctuations as $N \to \infty$, preventing F^γ from reaching the critical threshold k_0 with high probability.

Current results II: Towards a complete mean field limit

Theorem (Mean field limit, $d = 2$)

- The empirical measures $((\mu_t^N)_{t\in[0,T]})_N$ are tight.
- Every sub-sequential limit is a "very weak" solution of the logistic Keller–Segel equation, in the sense of Tardy [Tar24].

Next steps

- Establish uniqueness of the limiting law.
  - Show that the limiting measure admits a density.
  - Upgrade the limit to a mild solution of the PDE.
  - Invoke uniqueness of mild solutions [CRT25].
- Upgrade subsequential convergence to convergence of the full sequence.
- Establish stronger modes of convergence (e.g. pathwise/trajectorial convergence).

Thank you for your attention, Obrigado :)

Any questions?

References I

[CRT25] Thomas Cavallazzi, Alexandre Richard, and Milica Tomasevic. Quantitative approximation of a Keller–Segel PDE by a branching moderately interacting particle system and suppression of blow-up. *arXiv preprint arXiv:2512.20504*, 2025.

[FT25] Nicolas Fournier and Yoan Tardy. Collisions of the supercritical Keller-Segel particle system. *J. Eur. Math. Soc. (JEMS)*, 27(10):4083–4158, 2025.

[Oel85] Karl Oelschläger. A law of large numbers for moderately interacting diffusion processes. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 69(2):279–322, 1985.

[Per04] Benoît Perthame. PDE models for chemotactic movements: parabolic, hyperbolic and kinetic. *Appl. Math.*, 49(6):539–564, 2004.

[Tar24] Yoan Tardy. Weak convergence of the empirical measure for the Keller-Segel model in both subcritical and critical cases. *Electronic Journal of Probability*, 29:1–35, 2024.

References II

[TW07] J Ignacio Tello and Michael Winkler. A chemotaxis system with logistic source. *Communications in Partial Differential Equations*, 32(6):849–877, 2007.