Workshop on Stochastic Analysis

irregularity, mean-fields and related topics

# On foliated optimal transport


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Joint work (in progress) with Diego S. Ledesma and Christian S. Rodrigues


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## Introduction

- There is a nontrivial relationship between the geometry of a given space $X$ and the space $\mathscr{P}(X)$ of probability measures on $X$.

- Felix Otto, Cédric Villani, John Lott, Karl-Theodor Sturm, et.al.

  If $X$ is Riemannian manifold, then $(\mathscr{P}_2(X), W_2)$ is Riemannian-like.

  $\mathrm{Ric}(X) \geq K \iff$ convexity properties of functionals on $\mathscr{P}_2(X)$.

· R. Possobon and C. S. Rodrigues. **Geometric properties of disintegration of measures.** Ergodic Theory and Dynamical Systems, 45, 1619–1648, 2025.

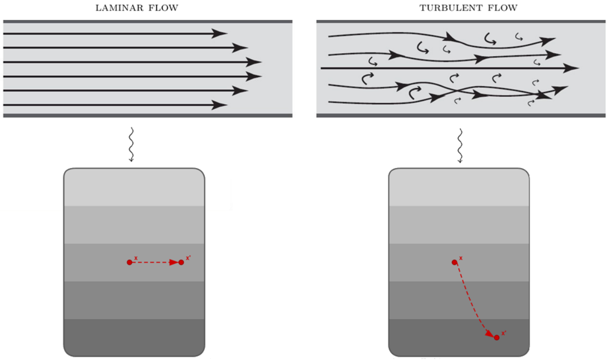
F. Münch, R. Possobon and C. S. Rodrigues. **Characterization of foliations via disintegration maps.** Calculus of Variations and Partial Differential Equations 65, 148, 2026.

- R. Possobon and C. S. Rodrigues. **Geometric properties of disintegration of measures.** Ergodic Theory and Dynamical Systems, 45, 1619–1648, 2025.

  F. Münch, R. Possobon and C. S. Rodrigues. **Characterization of foliations via disintegration maps.** Calculus of Variations and Partial Differential Equations 65, 148, 2026.

- How can we understand the relationship between $X$ and $\mathscr{P}(X)$ when $X$ is "sliced" into layers?

- Can global transport be characterized by studying transport within each layer?


LAMINAR FLOW
TURBULENT FLOW
x
x'
x
x'

OPTIMAL TRANSPORT

$\rightsquigarrow$

geometric constraint

$\rightsquigarrow$

FOLIATED OPTIMAL TRANSPORT

Classical (Monge-Kantorovich Problem) / dynamic (Benamou–Brenier formulation)

## Optimal transport

Main idea: to transport mass from one point to another at minimal cost.


Transport plan: $\gamma \in \mathscr{P}(X \times Y)$ such that $d\gamma(x, y)$ is the sent from $x$ to $y$.

Admissible set: $\Pi(\mu, \nu) := \{\gamma \in \mathscr{P}(X \times Y) : (\mathrm{proj}_1)_*\gamma = \mu,\ (\mathrm{proj}_2)_*\gamma = \nu\}$.

Monge-Kantorovich: Radon spaces $X$ and $Y$ , $\mu \in \mathscr{P}(X)$, $\nu \in \mathscr{P}(Y)$, and fixed Borel cost function $c : X \times Y \to [0, \infty]$ , minimize

$$\gamma \mapsto \int_{X \times Y} c(x, y) \, d\gamma(x, y)$$

among all measures $\gamma \in \Pi(\mu, \nu)$.

**Definition: Wasserstein distance**

Let $(X, d)$ be a separable complete metric space. Consider $\mu$ and $\nu$ probabilities on $X$ and $p \in [1, \infty)$. The **Wasserstein distance** of order $p$ between $\mu$ and $\nu$ is

$$W_p(\mu, \nu) := \left( \inf_{\gamma \in \Pi(\mu,\nu)} \int d(x_1, x_2)^p \; d\gamma(x_1, x_2) \right)^{\frac{1}{p}}.$$

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We do not specify how the mass is transported. Let's do it.

## Foliated optimal transport

Main idea: to transport the mass at minimal cost in such a way that it is restricted to each leaf of a foliation of $X$.

- $(X, d)$ Riemannian manifold;
- $\mathcal{F}$ smooth foliation, that is, there exists a standard Borel space $X^*$ and a Borel map $\pi : X \to X^*$ such that $\pi(x) = \pi(y) \iff x, y$ belong to the same leaf of $\mathcal{F}$;
- $\mathcal{L}_x$ the leaf of $\mathcal{F}$ which contains $x \in X$.

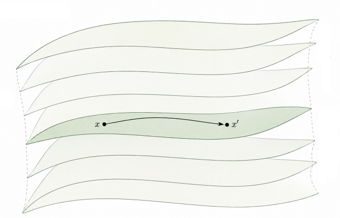
## Definition

Given $\mu, \nu \in \mathscr{P}(X)$, we define the set of **foliated transport plans** by

$$\Pi_{\mathcal{F}}(\mu,\nu) = \left\{ \gamma \in \Pi(\mu,\nu),\ \operatorname{supp}(\gamma) \subset \bigcup_{y \in X^*} (\mathcal{L}_y \times \mathcal{L}_y) \right\}.$$

## Proposition

If $\Pi_{\mathcal{F}}(\mu,\nu) \neq \varnothing$, then $\pi_*\mu = \pi_*\nu$. Moreover, if the leaves are closed, then the reverse holds true.




Mass is transported within individual leaves!

**Definition**

We define the total cost of the foliated optimal transport between $\mu$ and $\nu$ by

$$W^2_{2,\mathcal{F}}(\mu,\nu) := \inf_{\gamma\in\Pi_{\mathcal{F}}(\mu,\nu)} \int_{X\times X} d^2_{\mathcal{L}}(x,x')\, d\gamma,$$

where $d_{\mathcal{L}}$ is the Riemannian distance restrict to the leaf $\mathcal{L}_{p(x)} = \mathcal{L}_{p(x')}$.

$\{\mu^y\}_{y\in X^*}$ disintegration of $\mu$ w.r.t. $\pi_*\mu$.

$\{\nu^y\}_{y\in X^*}$ disintegration of $\nu$ w.r.t. $\pi_*\nu$.

**Proposition**

Given $\gamma \in \Pi_{\mathcal{F}}(\mu,\nu)$, there exists a measure $\sigma$ on $X^*$ and a disintegration of $\{\gamma^y\}_{y\in X^*}$ of $\gamma$ w.r.t. $\sigma$, where $\gamma^y \in \mathscr{P}(\mathcal{L}_y \times \mathcal{L}_y)$ satisfying $(\mathrm{proj}_1)_*\gamma^y = \mu^y$, $(\mathrm{proj}_2)_*\gamma^y = \nu^y$ for $\sigma$-a.e. $y \in Y$.

**Theorem**

Under the previous hypothesis, we can write

$$W^2_{2,\mathcal{F}}(\mu,\nu) = \int_{X^*} \left( \inf_{\gamma^y \in \Pi(\mu^y,\nu^y)} \int_{\mathcal{L}_y\times\mathcal{L}_y} d^2_{\mathcal{L}}(x,x')\, d\gamma^y \right) d\sigma.$$

Note that

$$W_{2,\mathcal{F}}^2(\mu,\nu) = \int_{X^*} \widetilde{W_2^2}(\mu^y,\nu^y)\,d\sigma,$$

where $\widetilde{W_2^2}(\mu^y,\nu^y)$ is the solution of the optimal transport problem on the leaf $\mathcal{L}_y$, with cost function $d_{\mathcal{L}}^2$. Therefore, the foliated optimal transport consists in a measurable family of independent transport problems on each leaf.

## Benamou-Brenier formulation

Benamou-Brenier formulation is a **dynamic approach** to optimal transport that minimizes kinetic energy over evolving densities governed by a continuity equation.

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$$W_2^2(\mu_0, \mu_1) = \inf_{(\mu_t)} \inf_{v_t(x)} \left\{ \int_0^1 \int_{\mathbb{R}^n} |v_t(x)|^2 \; d\mu_t(x) dt \; : \; \frac{\partial \mu}{\partial t} + \nabla \cdot (v\mu) = 0 \right\}$$

where the first inf is taken among all paths $(\mu_t)_{0 \leq t \leq 1}$ satisfying certain regularity conditions.

## Foliated Benamou-Brenier

The velocity must be tangent to the leaf: $v_t(x) \in T_x\mathcal{L}_x$

**Lemma**

Let $(\mu_t, v_t)$ be a solution of $\partial_t \mu_t + \mathrm{div}(\mu_t v_t) = 0$ such that $v_t(x) \in T_x\mathcal{L}(x)$ for $\mu_t$-a.e. $x$, a.e. $t$. Then $\pi_*\mu_t = \pi_*\mu_0$ for every $t \in [0, 1]$.

Given $(\mu_t, v_t)$, we define

$$\mathcal{A}(\mu_t, v_t) := \int_0^1 \int_X \|v_t(x)\|_d^2 \, d\mu_t(x) \, dt = \int_0^1 \int_{X^*} \int_{\mathcal{L}_y} \|v_t(x)\|_d^2 \, d\mu_t^y(x) \, d\sigma(y) \, dt$$

$$\mathcal{W}_{2,\mathcal{F}}^2(\mu_0, \mu_1) := \inf_{(\mu_t, v_t)} \left\{ \mathcal{A}(\mu_t, v_t) : \partial_t \mu_t + \mathrm{div}_{\mathcal{F}}(\mu_t v_t) = 0 \right\}$$

**Theorem**

$$W_{2,\mathcal{F}}^2(\mu_0, \mu_1) = \mathcal{W}_{2,\mathcal{F}}^2(\mu_0, \mu_1).$$

## Ongoing work

- Using the foliated optimal transport framework to better understand the interplay between the geometry of foliations on $X$ and properties of $\mathscr{P}(X)$.

## References


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Thank you! :)