Workshop on Stochastic Analysis: irregularity, mean-fields and related topics

Quantitative uniform-in-time propagation of chaos for stochastic particle systems interacting through L^p kernels

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Summary

1. From Macroscopic to Microscopic : Mathematical Introduction
   - From PDEs to Systems of Interacting Particles
   - Typical questions and methods
2. Moderately Interacting Particles : Framework and Results
   - Another viewpoint : moderate interaction
   - Results via a semigroup approach
3. Uniform in Time Propagation of Chaos : Ongoing Work
   - Motivation and methods
   - Ongoing work : Result
   - Ongoing work : Sketch of the proof

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From PDEs to Systems of Interacting Particles I

On $[0, T] \times \mathbb{R}^d$, consider the non-linear Fokker-Planck PDEs with singular kernel K

$$\begin{cases} \partial_t u_t(x) = \Delta u_t(x) - \nabla \cdot \big(u_t(x)\,(K * u_t(x))\big), \\ u_{t=0}(x) = u_0(x). \end{cases}$$

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Krylov-Röckner (or LPS type) interaction kernels

- $b \in L^q([0,T]; L^p(\mathbb{R}^d))$ with $\frac{d}{p} + \frac{2}{q} < 1$.
- In our case, $K \in L^p(\mathbb{R}^d)$ for some $p > d$.
- For example, $K(x) :\simeq \frac{x}{|x|^s} B_1, \quad s < 2$.

From PDEs to Systems of Interacting Particles II

Goal: Approximate solutions of the PDE using a probabilistic interpretation.

Nonlinear Fokker-Planck PDE:

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with $K : \mathbb{R}^d \to \mathbb{R}^d$ being a singular kernel.

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McKean-Vlasov SDE:

$$dX_t = K * u_t(X_t)dt + \sqrt{2}dW_t, \quad \forall t \in [0, T]$$

$$X_t \overset{(d)}{=} u_t$$

However: Unknown u_t in the drift term $\to$ replace it by the empirical measure $\mu_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{X_t^i}$.

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System of Interacting Particles:

$$dX_t^{i,N} = K * \mu_t^N(X_t^{i,N})\, dt + \sqrt{2}dW_t^i, \quad i \in \{1,\dots,N\}$$

$$= \frac{1}{N}\sum_{j=1}^N K(X_t^{i,N} - X_t^{j,N})\, dt + \sqrt{2}dW_t^i$$

Propagation of chaos

Fix $T > 0$. We say that the system of particles propagates chaos if for any $k \in \mathbb{N}$, the k-marginal $\mathcal{L}_{X_\cdot^{1,N},\dots,X_\cdot^{k,N}} \in \mathcal{P}(\mathcal{C}([0,T),\mathbb{R}^d)^k)$ satisfies

$$\mathcal{L}_{X_\cdot^{1,N},\dots,X_\cdot^{k,N}} \xrightarrow[N\to\infty]{} \mathcal{L}_{X_\cdot}^{\otimes k} = u_\cdot^{\otimes k}$$

in $\mathcal{P}(\mathcal{C}([0,T),(\mathbb{R}^d)^k))$ or in $\mathcal{C}([0,T),\mathcal{P}(\mathbb{R}^d)^k)$.

$\rightarrow$ *"When N is large, any group of k particles is close to be independent."*

From PDEs to Systems of Interacting Particles III

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Lemma for exchangeable particles

When the particles are exchangeable, propagation of chaos is equivalent to saying that the empirical measure process converges in law towards the deterministic measure u, i.e.

$$\forall \Phi \in C_b(\mathcal{P}(\mathcal{C}([0,T];\mathbb{R}^d))), \quad \mathbb{E}\left[\Phi\left(\mu^N\right)\right] \to \Phi(u).$$

Typical questions and methods

Goal: To show

$$\mu^N_\cdot \to u_\cdot \quad \text{or} \quad u^{k,N}_\cdot \to u^{\otimes k}_\cdot \quad \text{as } N \to \infty,$$

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Some methods:

- **Coupling methods** (Sznitman, Lindvall, Rogers, Eberle, ...): Define

$$W_p(\mu, \nu) = \inf_{X \sim \mu,\, Y \sim \nu} \mathbb{E}(|X - Y|^p)^{\frac{1}{p}}.$$

  Show

$$W_p\left(u_t^{k,N},\, u_t^{\otimes k}\right) \to 0.$$

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- **Energy/Entropy estimates** (Serfaty, Jabin-Wang,...) :
  Consider a "good" quantity (e.g., energy, relative entropy), and prove that it is decreasing.

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- **BBGKY hierarchies** (Lacker, Han, Bresch-Jabin-Soler,...) :
  The joint law of k particles depends on the joint law of $k+1$ particles. Use this to find iterative bounds on relative entropy or similar quantities.

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- **Tightness/Consistency** (Rogers-Shi, Cépa-Lépingle, Fournier-Hauray-Mischler...) :
  Prove tightness of the sequence of measures, consistency of the limit and uniqueness of the limit.

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Motivated by singular attractive kernels for which either :

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Moderately interacting particles:

$$dX_t^{i,N} = \frac{1}{N}\sum_{j=1}^{N} K * V^N(X_t^{i,N} - X_t^{j,N})\, dt + \sqrt{2} dW_t^i,\ i \in \{1, ..., N\}$$

where:

- $V_N(x) = N^{d\alpha} V(N^\alpha x)$, $\alpha > 0$ and V a smooth probability density;
- Some references: [Oelschläger '85], [Méléard & Roelly '87], [Jourdain & Méléard '98].

Results via a semigroup approach

- Consider $r \geq 1$, $T > 0$ and $u \in C([0,T]; L^1 \cap L^r(\mathbb{R}^d))$, such that

$$u_t = g_2(t,\cdot) * u_0 - \int_0^t \nabla \cdot [g_2(t-s,\cdot) * (u_s\, K * u_s)]\, ds, \quad 0 \leq t \leq T$$

where $g_2(t,x) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x|^2}{4t}}$ is the heat kernel on $\mathbb{R}^d$.

[2] N. Cazacu, *Stochastic numerical approximation for nonlinear Fokker-Planck equations with singular kernels*, Stoch PDE : Anal Comp (2026).

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- Consider the family of mollified empirical measures

$$\{\tilde{\mu}_t^N = V^N * \mu_t^N,\ t \in [0,T]\}_{N \in \mathbb{N}}$$

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Theorem (Olivera-Richard-Tomasevic, Ann Sc Norm Super Pisa'23)

Under some hypothesis on K, u_0 and α, we have the estimate :

$$\forall m \geq 1,\ \mathbb{E}\left[\sup_{s\in[0,t]} \left\|\tilde{\mu}_s^N - u_s\right\|_{L^1\cap L^r}^m\right]^{1/m} \lesssim \|\tilde{\mu}_0^N - u_0\|_{L^m(\Omega; L^1\cap L^r(\mathbb{R}^d))} + N^{-v},$$

where $v := v(K,\alpha,d,r) > 0$.

- Numerical approximation result using similar techniques[2]

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Motivation and setting

Motivation : Understand the long time behaviour of the PDE via the particles system.

$$\partial_t u_t = \Delta u_t + \nabla \cdot (u_t(\nabla U + K * u_t)),\ t > 0, x \in \mathbb{R}^d. \quad \text{(NLFP)}$$

$$dX_t = -\nabla U(X_t)\,dt - (K * u_t)(X_t)\,dt + \sqrt{2}\,dW_t. \quad \text{(McKean-Vlasov SDE)}$$

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Related questions : Ergodicity of the McKean-Vlasov SDE and limit PDE...

Ongoing work : Result

Theorem (C., Richard, Tomasevic 26+)

Under assumptions below, there exists $M > 0$ such that if $\|K\|_{L^k} \leq M$, then

$$\mathbb{W}_2(\mathcal{L}(X_t^N), u_t^{\otimes N}) \leq c \left(e^{-dt} \mathbb{W}_2(\mathcal{L}(X_0^N), u_0^{\otimes N}) + N^{-v}(1 + \log(N)\mathbf{1}_{d=2}) \right),$$

$$u_0 \in \mathcal{P}_2(\mathbb{R}^d),\ \mathcal{L}(X_0^N) \in \mathcal{P}_2(\mathbb{R}^{dN}),\ t \geq 0,$$

for some constants $c, d, v \in (0, \infty)$.

Assumptions :

- U is strongly convex and ∇U is Lipschitz.
- $K \in L^k(\mathbb{R}^d)$ for some $k > d$.
- $u_0 \in \mathcal{P}_2(\mathbb{R}^d)$, $\mathcal{L}(X_0^N) \in \mathcal{P}_2(\mathbb{R}^{dN})$, and if $d \geq 2$, for some p
$$\sup_{N \in \mathbb{N}} \max_{1 \leq i \leq N} \mathbb{E}\left[|X_0^{i,N}|^p \right] < \infty.$$
- $0 < \alpha < \frac{1}{d \vee 2}$.

X. Huang, F.-Y. Wang, *Exponential Ergodicity for McKean-Vlasov SDEs with Singular Interactions*, arXiv 2025.

C. Olivera, A. Richard, M. Tomasevic, *Quantitative particle approximation of nonlinear Fokker-Planck equations with singular kernel*, 2023.

Step 1 : Synchronous coupling

$$\begin{cases} dX_t^{i,N} = -\nabla U(X_t^{i,N})dt - K * V^N * \mu_t^N(X_t^{i,N})\,dt + \sqrt{2}\,dW_t^i, \\ dX_t^i = -\nabla U(X_t^i) - (K * u_t)(X_t^i)\,dt + \sqrt{2}\,dW_t^i, \\ u_t = \mathrm{Law}(X_t^i). \end{cases}$$

$$\mathbb{W}_2^2(\mathcal{L}(X_t^N), u_t^{\otimes N}) \leq \frac{1}{N}\sum_{i=1}^{N} \mathbb{E}(|X_t^{i,N} - X_t^i|^2).$$

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Using Itô's formula, for $t \in (0,1]$, if $K * u_t$ is $C_{Lip}(t)$-Lipschitz, then we have for any $i \in [1, N]$,

$$\begin{aligned} d|X_t^{i,N} - X_t^i|^2 &= -2\left(\nabla U(X_t^{i,N}) + K * u_t^N(X_t^{i,N}) - \nabla U(X_t^i) - K * u_t(X_t^i)\right) \cdot (X_t^{i,N} - X_t^i)dt \\ &\leq -2\lambda|X_t^{i,N} - X_t^i|^2 dt - 2(K * u_t^N(X_t^{i,N}) - K * u_t(X_t^i)) \cdot (X_t^{i,N} - X_t^i)dt \\ &\leq (C_{Lip}(t)\|K\|_{L^k} - 2\lambda)|X_t^{i,N} - X_t^i|^2 dt + 2\|K\|_{L^k}\|u_t^N - u_t\|_{L^{\bar{k}}}|X_t^{i,N} - X_t^i|dt. \end{aligned}$$

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Gronwall's differential inequality gives for all $t \in (0,1]$,

$$\begin{aligned} |X_t^{i,N} - X_t^i|^2 e^{\int_0^t 2\lambda - C_{Lip}(s)\|K\|_{L^k} ds} &\leq |X_0^{i,N} - X_0^i|^2 \\ &+ 2\|K\|_{L^k}\int_0^t e^{\int_0^s (2\lambda - C_{Lip}(u)\|K\|_{L^k})du}\|u_s^N - u_s\|_{L^{\bar{k}}}|X_s^{i,N} - X_s^i|ds \end{aligned}$$

Step 2 : Estimates for the mild formulation

Take f in $L^k(\mathbb{R}^d)$, and consider $\hat{P}_t$ the semigroup associated to the SDE $dX_t = -\nabla U(X_t)dt + \sqrt{2}dW_t$.

$$u_t(f) = \hat{P}_t^* u_0(f) + \int_0^t \hat{P}_{t-s}^* \nabla \left(u_s(K * u_s)\right)(f)ds,$$

$$u_t^N(f) = \hat{P}_t^* u_0^N(f) + \int_0^t \hat{P}_{t-s}^* \nabla \cdot \left\langle \mu_s^N, V^N(x-\cdot)\, K * u_s^N(\cdot) \right\rangle (f)\, ds$$

$$+ \int_0^t \hat{P}_{t-s}^* \nabla \cdot \langle \mu_s^N, V^N(x-\cdot)\, (\nabla U(x) - \nabla U(\cdot))\rangle(f)\, ds - \frac{\sqrt{2}}{N} \sum_{i=1}^N \int_0^t \left(\int_{\mathbb{R}^d} \nabla \hat{P}_{t-s} f(x)\, V^N(x - X_s^{i,N}) dx \right) \cdot dW_s^i$$

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$$+ \int_0^t \hat{P}_{t-s}^* \nabla \cdot \langle \mu_s^N, V^N(x - \cdot)\ (\nabla U(x) - \nabla U(\cdot))\rangle(f)\, ds - \frac{\sqrt{2}}{N} \sum_{i=1}^N \int_0^t \left(\int_{\mathbb{R}^d} \nabla \hat{P}_{t-s} f(x)\, V^N(x - X_s^{i,N}) dx \right) \cdot dW_s^i$$

Hence,

$$\|u_t^N - u_t\|_{L^{\bar{k}}} = \sup_{\|f\|_k \leq 1} |u_t^N(f) - u_t(f)| \leq \|\hat{P}_t^*(u_0^N - u_0)\|_{L^{\bar{k}}} + E_t^{1,N} + E_t^{2,N} + M_t^N,$$

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Take f in $L^k(\mathbb{R}^d)$, and consider $\hat{P}_t$ the semigroup associated to the SDE $dX_t = -\nabla U(X_t)dt + \sqrt{2}dW_t$.

$$u_t(f) = \hat{P}_t^* u_0(f) + \int_0^t \hat{P}_{t-s}^* \nabla \left(u_s(K * u_s) \right)(f) ds,$$

$$u_t^N(f) = \hat{P}_t^* u_0^N(f) + \int_0^t \hat{P}_{t-s}^* \nabla \cdot \left\langle \mu_s^N, V^N(x - \cdot)\, K * u_s^N(\cdot) \right\rangle (f)\, ds$$

$$+ \int_0^t \hat{P}_{t-s}^* \nabla \cdot \langle \mu_s^N, V^N(x - \cdot)\ (\nabla U(x) - \nabla U(\cdot)) \rangle (f)\, ds - \frac{\sqrt{2}}{N} \sum_{i=1}^N \int_0^t \left(\int_{\mathbb{R}^d} \nabla \hat{P}_{t-s} f(x)\, V^N(x - X_s^{i,N}) dx \right) \cdot dW_s^i$$

Hence,

$$\| u_t^N - u_t \|_{L^{\bar{k}}} = \sup_{\|f\|_k \le 1} |u_t^N(f) - u_t(f)| \le \| \hat{P}_t^* (u_0^N - u_0) \|_{L^{\bar{k}}} + E_t^{1,N} + E_t^{2,N} + M_t^N,$$

where

$$E_t^{1,N} := \sup_{\|f\|_k \le 1} \left| \int_0^t \int_{\mathbb{R}^d} \hat{P}_{t-s} f(x) \nabla \cdot \langle \mu_s^N, V^N(x - \cdot)\ (\nabla U(x) - \nabla U(\cdot)) \rangle\, dx\, ds \right|$$

$$E_t^{2,N} := \sup_{\|f\|_k \le 1} \left| \int_0^t \int_{\mathbb{R}^d} \hat{P}_{t-s} f(x) \nabla \cdot \left(\left\langle \mu_s^N, V^N(x - \cdot)\, K * u_s^N(\cdot) \right\rangle - u_s(K * u_s)(x) \right)\, dx\, ds \right|$$

$$M_t^N := \sup_{\|f\|_k \le 1} \left| \frac{\sqrt{2}}{N} \sum_{i=1}^N \int_0^t \left(\int_{\mathbb{R}^d} \nabla \hat{P}_{t-s} f(x)\, V^N(x - X_s^{i,N}) dx \right) \cdot dW_s^i \right|$$

Step 2 : Estimates for the mild formulation

Take f in $L^k(\mathbb{R}^d)$, and consider $\hat{P}_t$ the semigroup associated to the SDE $dX_t = -\nabla U(X_t)dt + \sqrt{2}dW_t$.

$$u_t(f) = \hat{P}_t^* u_0(f) + \int_0^t \hat{P}_{t-s}^* \nabla \left(u_s(K * u_s)\right)(f) ds,$$

$$u_t^N(f) = \hat{P}_t^* u_0^N(f) + \int_0^t \hat{P}_{t-s}^* \nabla \cdot \left\langle \mu_s^N, V^N(x - \cdot)\, K * u_s^N(\cdot) \right\rangle (f)\, ds$$

$$+ \int_0^t \hat{P}_{t-s}^* \nabla \cdot \langle \mu_s^N, V^N(x - \cdot)\ (\nabla U(x) - \nabla U(\cdot))\rangle(f)\, ds - \frac{\sqrt{2}}{N} \sum_{i=1}^N \int_0^t \left(\int_{\mathbb{R}^d} \nabla \hat{P}_{t-s} f(x)\, V^N(x - X_s^{i,N}) dx \right) \cdot dW_s^i$$

Hence,

$$\|u_t^N - u_t\|_{L^{\bar{k}}} = \sup_{\|f\|_k \le 1} |u_t^N(f) - u_t(f)| \le \|\hat{P}_t^*(u_0^N - u_0)\|_{L^{\bar{k}}} + E_t^{1,N} + E_t^{2,N} + M_t^N,$$

where

$$E_t^{1,N} := \sup_{\|f\|_k \le 1} \left| \int_0^t \int_{\mathbb{R}^d} \hat{P}_{t-s} f(x) \nabla \cdot \langle \mu_s^N, V^N(x - \cdot)\ (\nabla U(x) - \nabla U(\cdot))\rangle\, dx\, ds \right|$$

$$E_t^{2,N} := \sup_{\|f\|_k \le 1} \left| \int_0^t \int_{\mathbb{R}^d} \hat{P}_{t-s} f(x) \nabla \cdot \left(\left\langle \mu_s^N, V^N(x - \cdot)\, K * u_s^N(\cdot) \right\rangle - u_s(K * u_s)(x) \right) dx\, ds \right|$$

$$M_t^N := \sup_{\|f\|_k \le 1} \left| \frac{\sqrt{2}}{N} \sum_{i=1}^N \int_0^t \left(\int_{\mathbb{R}^d} \nabla \hat{P}_{t-s} f(x)\, V^N(x - X_s^{i,N}) dx \right) \cdot dW_s^i \right|$$

Estimating each term conclude via Gronwall's lemma that for all $t \in (0,1]$,

$$\|u_t^N - u_t\|_{k^*} \le C(t^{-\frac{1}{2} - \frac{d}{2k}} + t^{-\frac{d}{k}})(\mathbb{W}_1(u_0^N, u_0) + CN^{-\alpha}) + M_t^N + C \int_0^t (t-s)^{-\frac{1}{2} - \frac{d}{2k}} M_s^N ds.$$

Step 3: From $t \in (0,1)$ to uniform in time estimates

Inject the bounds on $\|u_t^N - u_t\|_{k*}$ into the bound on $|X_t^{i,N} - X_t^i|^2$ to get for all $t \in (0,1]$, and some $c, v > 0$

$$\mathbb{W}_2(\mathcal{L}X_t^N, u_t^{\otimes N}) \leq \left(\sqrt{\frac{e^{-\int_0^t 2\lambda - C_{Lip}(s)\|K\|_{L^k} \mathrm{d}s}}{1 - \|K\|_{L^k}}} + C\sqrt{\frac{\|K\|_{L^k}}{1 - \|K\|_{L^k}}} \right) \mathbb{W}_2(\mathcal{L}X_0^N, u_0^{\otimes N}) + C\sqrt{\frac{\|K\|_{L^k}}{1 - \|K\|_{L^k}}} N^{-v}.$$

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Choosing $\|K\|_{L^k}$ small enough such that

$$A(1) := \sqrt{\frac{e^{-\int_0^1 2\lambda - C_{Lip}(s)\|K\|_{L^k} \mathrm{d}s}}{1 - \|K\|_{L^k}}} + C\sqrt{\frac{\|K\|_{L^k}}{1 - \|K\|_{L^k}}} < 1,$$

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we obtain, Using the semigroup property, for all $n \in \mathbb{N}$,

$$\begin{aligned}
&\mathbb{W}_2\left(P_{n+1}^{N,*}\mathcal{L}X_0^N, P_{n+1}^{*}u_0^{\otimes N}\right)^2 = \mathbb{W}_2\left(P_1^{N,*}\mathcal{L}X_n^N, P_1^{*}u_n^{\otimes N}\right)^2 \\
&\leq A(1)\mathbb{W}_2\left(\mathcal{L}X_n^N, u_n^{\otimes N}\right)^2 + CN^{-v} \\
&\leq A(1)^n\mathbb{W}_2\left(\mathcal{L}X_0^N, u_0^{\otimes N}\right)^2 + CN^{-v}\sum_{k=0}^{n-1} A(1)^k.
\end{aligned}$$

Step 3: From $t \in (0,1)$ to uniform in time estimates

Inject the bounds on $\|u_t^N - u_t\|_{k*}$ into the bound on $|X_t^{i,N} - X_t^i|^2$ to get for all $t \in (0,1]$, and some $c, v > 0$

$$\mathbb{W}_2(\mathcal{L}X_t^N, u_t^{\otimes N}) \leq \left(\sqrt{\frac{e^{-\int_0^t 2\lambda - C_{Lip}(s)\|K\|_{L^k} \mathrm{d}s}}{1 - \|K\|_{L^k}}} + C\sqrt{\frac{\|K\|_{L^k}}{1 - \|K\|_{L^k}}} \right) \mathbb{W}_2(\mathcal{L}X_0^N, u_0^{\otimes N}) + C\sqrt{\frac{\|K\|_{L^k}}{1 - \|K\|_{L^k}}} N^{-v}.$$

Choosing $\|K\|_{L^k}$ small enough such that

$$A(1) := \sqrt{\frac{e^{-\int_0^1 2\lambda - C_{Lip}(s)\|K\|_{L^k} \mathrm{d}s}}{1 - \|K\|_{L^k}}} + C\sqrt{\frac{\|K\|_{L^k}}{1 - \|K\|_{L^k}}} < 1,$$

we obtain, Using the semigroup property, for all $n \in \mathbb{N}$,

$$\begin{aligned}
&\mathbb{W}_2\left(P_{n+1}^{N,*}\mathcal{L}X_0^N, P_{n+1}^* u_0^{\otimes N}\right)^2 = \mathbb{W}_2\left(P_1^{N,*}\mathcal{L}X_n^N, P_1^* u_n^{\otimes N}\right)^2 \\
&\leq A(1)\mathbb{W}_2\left(\mathcal{L}X_n^N, u_n^{\otimes N}\right)^2 + CN^{-v} \\
&\leq A(1)^n \mathbb{W}_2\left(\mathcal{L}X_0^N, u_0^{\otimes N}\right)^2 + CN^{-v}\sum_{k=0}^{n-1} A(1)^k.
\end{aligned}$$

Hence,

$$\mathbb{W}_2(\mathcal{L}(X_t^N), u_t^{\otimes N}) \leq c\left(e^{-dt}\mathbb{W}_2(\mathcal{L}(X_0^N), u_0^{\otimes N}) + N^{-v}\right),$$

$$u_0 \in \mathcal{P}_2(\mathbb{R}^d),\ \mathcal{L}(X_0^N) \in \mathcal{P}_2(\mathbb{R}^{dN}),\ t \geq 0,$$

for some constants $c, d, v \in (0, \infty)$.

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Appendix I : Quantitative particle approximation: a semigroup approach

(HK):

(i) $K \in L^p(B_1)$, for some $p \in [1, +\infty]$;

(ii) $K \in L^q(B_1^c)$, for some $q \in [1, +\infty]$;

(iii) There exist $r \geq \max(p_0, q_0)$ and $\zeta \in (0, 1]$ such that:

$$N^\zeta(K * f) \lesssim \|f\|_{L^1 \cap L^r(\mathbb{R}^d)}, \quad \forall f \in L^1 \cap L^r(\mathbb{R}^d),$$

where N^ζ denotes the Hölder semi-norm of order ζ.

(Hα): The parameters α and r satisfy

$$0 < \alpha < \frac{1}{d + 2d \max\left(\frac{1}{2} - \frac{1}{r}, 0\right)}.$$

(H0): Fix r from (HK)(iii). For any $m \geq 1$,

$$\sup_{1 \leq i \leq N,\ N \in \mathbb{N}} \mathbb{E}\left[|X_{i,N}^0|^m\right] < \infty \quad \text{and} \quad \sup_{N \in \mathbb{N}} \mathbb{E}\left[\left\|\mu_N^0 * V^N\right\|_{L^r(\mathbb{R}^d)}^m\right] < \infty.$$

A sufficient condition for **(H0)** to hold is that the particles are initially i.i.d. with a law in L^r, and that $\alpha d < 1$.

Appendix II : Quantitative particle approximation: a semigroup approach

Proposition

Let K satisfy $(HK)(i)$–(ii), and $u_0 \in L^1 \cap L^r(\mathbb{R}^d)$ with $r \geq \max(p, q)$. There exists $T > 0$ and a unique function $u : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ such that

$$u \in C([0, T]; L^1 \cap L^r(\mathbb{R}^d))$$

and

$$u_t = g_2(t, \cdot) * u_0 - \int_0^t \nabla \cdot [g_2(t-s, \cdot) * (u_s\, K * u_s)]\, ds, \quad 0 \leq t \leq T$$

where $g_2(t, x) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x|^2}{4t}}$ is the heat kernel on $\mathbb{R}^d$.

Theorem (Olivera-Richard-Tomasevic, Ann Sc Norm Super Pisa'21)

Let T as above, and assume (HK), (Hα) and (H0) hold. Then for the family $\{\tilde{\mu}_t^N = V^N * \mu_t^N,\ t \in [0, T]\}_{N \in \mathbb{N}}$ we have the estimate:

$$\forall \varepsilon > 0,\ \forall m \geq 1,\ \|\tilde{\mu}^N - u\|_{L^m(\Omega; L^\infty([0,T]; L^1 \cap L^r(\mathbb{R}^d)))} \lesssim \|\tilde{\mu}_0^N - u_0\|_{L^m(\Omega; L^1 \cap L^r(\mathbb{R}^d))} + N^{-\rho+\varepsilon},$$

where

$$\rho = \min\left\{\alpha\zeta,\ \frac{1}{2}\left(1 - \alpha\left(d + d(1 - \tfrac{2}{r}) \vee 0\right)\right)\right\}.$$

Semigroup approach

Key idea: Convolution with the heat kernel [3,4]

Mild formulations

$$u_t(x) = g_2(t,\cdot) * u_0(x) - \int_0^t \nabla \cdot \left(g_2(t-s,\cdot) * (u_s K * u_s)\right)(x)\, ds$$

$$\tilde{\mu}_t^{N,h}(x) = g_2(t,\cdot) * \tilde{\mu}_0^{N,h}(x) + M_t^{N,h}(x)$$

$$- \frac{1}{N}\sum_{i=1}^{N} \int_0^t \nabla \cdot g_2(t-s,\cdot) * V^N(x - X_s^{i,N,h}) F_A(\tilde{\mu}_{\tau_s^h}^{N,h} * K(X_{\tau_s^h}^{i,N,h}))\, ds$$

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[3] C. Olivera, A. Richard, M. Tomasevic, *Quantitative particle approximation of nonlinear Fokker-Planck equations with singular kernel*, 2023.

[4] B. Jourdain, S. Menozzi, *Convergence rate of the Euler-Maruyama scheme applied to diffusion processes with L^q–L^ρ drift coefficient and additive noise*, 2024.

Appendix III : Key Ideas of Proof

Error decomposition

$$\tilde{\mu}_t^{N,h}(x) - u_t(x) = \text{initial error} + \text{Grönwall term} + E_t^{N,h}(x) + M_t^{N,h}(x) + H_t^{N,h}(x)$$

Transport Error:

$$E_t^{N,h}(x) = \int_0^t \nabla \cdot g_2(t-s,\cdot) * \langle \mu_s^{N,h}, V^N(x-\cdot) \cdot \Big[F_A(K * \tilde{\mu}_s^{N,h}(x)) - F_A(K * \tilde{\mu}_s^{N,h}(\cdot)) \Big] \rangle \, ds = \mathcal{O}_\omega(N^{-\alpha\zeta})$$

where ζ is the Hölder exponent of $K * f$ and α is the scale of molification in V^N.

Stochastic Integral:

$$M_t^{N,h}(x) = \frac{-1}{\sqrt{2}N} \sum_{i=1}^N \int_0^t g_2(t-s,\cdot) * \nabla V^N(x - X_s^{i,N,h}) \cdot dW_s^i = \mathcal{O}_\omega(N^{-\rho+\epsilon})$$

where $\varepsilon > 0$ and ρ depend on the assumptions on the kernel K and on α.

Time-Discretization Error:

$$H_t^{N,h}(x) = \frac{1}{N} \sum_{i=1}^N \int_0^t \nabla \cdot g_2(t-s,\cdot) * V^N(x - X_s^{i,N,h}) \cdot \Big[F_A(K * \tilde{\mu}_s^{N,h}) - F_A(K * \tilde{\mu}_{\tau_s^h}^{N,h}) \Big] \, ds = \mathcal{O}_\omega(N^{\frac{2d\alpha}{r}} h^{\frac{\zeta}{2}})$$

where r depends on the integrability of the kernel K.

1.Examples

Consider for $d \geq 2$ and $s \in [0, d-2]$ the following kernels derived from Riesz potentials $\boldsymbol{K_s := \pm \nabla V_s}$, where

$$V_s(x) = \begin{cases} |x|^{-s} & \text{if } s \in (0, d-2] \\ -\log|x| & \text{if } s = 0 \end{cases}, \quad x \in \mathbb{R}^d.$$

Convergence rate

$$\mathcal{O}\left(N^{-\frac{1}{2(d+1)}^{-}} + N^{\frac{d}{2(d+1)}^{+}} h^{\frac{1}{2}^{-}} \right).$$

Remark : For the $2D$ Keller-Segel kernel, we have $\mathcal{O}(N^{-\frac{1}{6}^{-}} + N^{\frac{1}{3}^{+}} h^{\frac{1}{2}^{-}})$.
For bounded Lipschitz kernels, we have $\mathcal{O}(N^{-\frac{1}{d+2}^{-}} + h^{\frac{1}{2}})$.

Appendix V : From PDEs to Systems of Interacting Particles II

$$\begin{cases} \partial_t u_t(x) = \Delta u_t(x) - \nabla \cdot \big(u_t(x) \, (K * u_t(x)) \big), \;\; t \in (0, T], x \in \mathbb{R}^d \\ u_{t=0}(x) = u_0(x), \; x \in \mathbb{R}^d. \end{cases} \tag{NLFP}$$

Some properties:

- **Diffusion and Interaction** : the drift term $-\nabla \cdot (uK * u)$ is non-local, hence the difficulty to approximate the solution u.
- **Existence & uniqueness** in weak/mild sense for u in $\mathcal{C}([0, T]; L^p(\mathbb{R}^d))$ or Sobolev spaces...
- **Non-negativity Principle**: if $u_0 \geq 0$, then $u_t(x) \geq 0$ for all $t \geq 0$ and $x \in \mathbb{R}^d$
- **Mass Conservation**: $\int_{\mathbb{R}^d} u_t(x)dx = \int_{\mathbb{R}^d} u_0(x)dx$ for all $t \geq 0$.