NESS for KPZ

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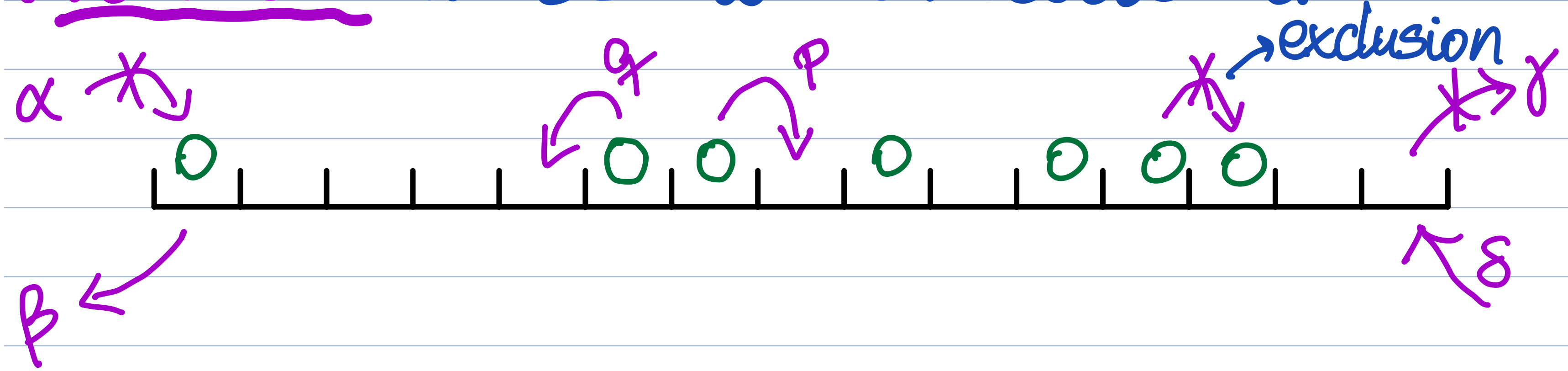
* Model 1: WASEP with reservoirs.



KPZ scaling: $p = 1 + \frac{\upsilon}{\sqrt{n}}, \; q = 1 - \frac{\upsilon}{\sqrt{n}}$

$$\alpha = \left(\frac{1}{2} + \frac{a}{\sqrt{n}}\right) p, \quad \beta = \left(\frac{1}{2} - \frac{a}{\sqrt{n}}\right) q$$

$$\gamma = \left(\frac{1}{2} - \frac{b}{\sqrt{n}}\right) p, \quad \delta = \left(\frac{1}{2} + \frac{b}{\sqrt{n}}\right) q$$

$\Lambda_n = \{1, \dots, n-1\}$, $\Omega_n = \{0,1\}^{\Lambda_n}$

$(\eta(t); t \geq 0)$: Markov chain WASEP(v, a, b, n)

Irreducible $\rightsquigarrow$ invariant measure π^n.

Question: Scaling limit of density of particles under π^n.

$\rho_* :=$ triple-point density

$\hookrightarrow = \frac{1}{2}$ for WASEP

* Density fluctuation field:

$$X^n(f) := \frac{1}{\sqrt{n}} \sum_{x \in \Lambda_n} (\eta(t) - \rho_*) f\left(\frac{x}{n}\right) \quad ; f \in \mathcal{C}_c^\infty((0,1))$$

Thm. (Corwin-Knizel, Bryc et al.)

The field $X^n(f)$ has a very complicated limit. Moreover, it is a stationary solution of the stochastic Burgers equation.

* SBE: $\partial_t X = D(\rho_*)\Delta X + \nu\lambda(\rho_*)\nabla X^2 + \sqrt{\chi(\rho_*)}\,\nabla\dot{W}$

* KPZ: $X = \nabla H$; $\partial_t H = D\Delta H + \nu\lambda(\nabla H)^2 + \sqrt{\chi}\,\dot{W}$

* SHE: $Z = e^{\gamma H}$; $\partial_t Z = D\Delta Z + \sqrt{\chi}\,Z\dot{W}$

Einstein relation [Gon-J]: $\lambda = \partial_\rho^2(\chi D)$

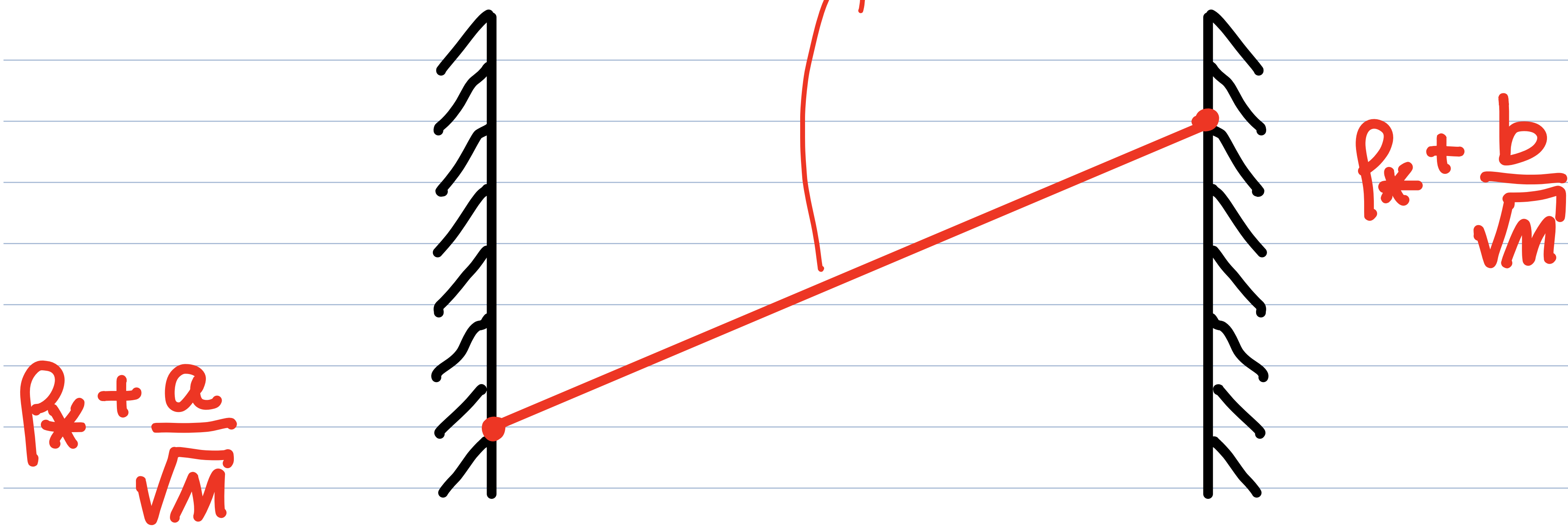
__Universality:__ Is the scaling limit of weakly asymmetric, conservative systems given by stationary solutions of SBE?? NESS for KPZ

__Thm.:__ (Arroyave-J.) For gradient (quasilinear) systems, if $|b-a| \le \varepsilon_0$, then X^n converges to NESS of SBE.

* Step 1: A priori bound.

f_t: density of $\eta(tn^2)$ wrt

$$\mu_p := \bigotimes_{x\in\Lambda_n} \mathrm{Ber}\left(p_* + \frac{a+(b-a)x}{\sqrt{n}}\right)$$



Lemma: If $|b-a| \le \varepsilon_0$, then

$$\frac{d}{dt} \int f_t^2 \, d\mu_\rho \le -\delta_0 \int f_t^2 \, d\mu_\rho + C$$

In particular,

$$\int \left(\frac{d\pi}{d\mu_\rho} \right)^2 d\mu_\rho \le C/\delta_0$$

$\rightsquigarrow$ Quantitative Hydrodynamics

<u>Scheme of proof:</u> L_n: generator of $(\eta(t);\, t\geq 0)$

Γ_n: carré du champ of L_n

L_n^*: adjoint of L_n wrt μ_ρ

a) Fokker-Planck:

$$\frac{d}{dt}\underbrace{\int f_t^2\, d\mu_\rho}_{=:\|f_t\|^2} = -n^2\int \Gamma_n f_t\, d\mu_\rho + \int n^2 L_n^* \mathbb{1}\, f_t^2\, d\mu_\rho$$

b) Main lemma (J.-Menezes):

$$\int n^2 L_n^* \mathbb{1}\, f^2\, d\mu_\rho \leq \beta n^2 \int \Gamma_n f_t\, d\mu_\rho + \frac{C|b-a|\,\|f\|^2}{\beta}\left(H\left(\frac{f^2}{\|f\|^2}\right)+1\right)$$

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c) log-Sobolev + Poincaré:

$$\frac{d}{dt}\|f_t\|^2 \leq -\frac{n^2}{C}(1 - C|b-a|)\|f_t\|^2 + C$$

★ **Step 2**: Variational norms.

$$Q(V) := \sup_{\|f\|=1} \left\{ \int V f^2 \, d\mu_p - \frac{1}{4} n^2 \int \Gamma_n f \, d\mu_p \right\}$$

Lemma: Under the conditions of the previous lemma,

$$\log \mathbb{E}\left[e^{\int_0^t V_s \, ds} \right] \leq Ct + \int_0^t Q(V_s) \, ds$$

* Step 3: Gartner-Cole-Hopf transform.

$J(t)$: current of particles between the system and the left boundary

Gartner-Cole-Hopf

$$\xi_x(t) := \exp\left\{ \frac{D}{v\lambda} \cdot \frac{1}{\sqrt{n}} \left(J(t) + \sum_{y=1}^{x} \left(\eta_y(n^2 t) - \rho_* \right) \right) - C n t \right\}$$

Exponential field:

$$Z_t^n(F) := \frac{1}{n} \sum_{x \in \Lambda_n} F\left(\frac{x}{n}\right) \xi_x(t)$$

Thm.: The field $\mathcal{Z}^n_t$ converges to a weak solution of SHE with Robin boundary conditions.

Obs.: For general models, we need a non-linear replacement to show that non-linear terms in the evolution of $\mathcal{Z}^n_t$ vanish in the limit.

$\hookrightarrow$ Boltzmann-Gibbs principle [Gonçalves, J.]