On the moderate interactions model with common noise

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based on joint work with A. Souza and C. Olivera (Unicamp)

Workshop on Stochastic Analysis:
irregularity, mean-fields and related topics

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Introduction
Quick example

From interacting particles to a macroscopic density

- Consider at a microscopic level N diffusive particles:

$$dX_t^i = \frac{1}{N}\sum_{j=1}^{N} K(X_t^i - X_t^j)dt + dB_t^i, \qquad i = 1, ..., N$$

where B_t^i are indep. B.M. on $\mathbb{R}^d$, $X_0^i \sim \rho_0(x)\,dx$ indep.

- Collective behavior of the particles is captured by the empirical measure

$$S_t^N := \frac{1}{N}\sum_{i=1}^{N} \delta_{X_t^i}, \quad t \in [0, T]$$

Under suitable conditions on K, there is convergence in law

$$S_t^N \longrightarrow \mu_t = \rho_t(x)dx$$

Characterizing the limit.

$$dX_t^i = K * S_t^N(X_t^i)dt + dB_t^i$$

In the limit:

$$dX_t^i = K * \rho_t(X_t^i)dt + dB_t^i$$

with $\rho(t,x)$ being the solution of a Fokker-Planck equation:

$$\partial_t \rho(t,x) + \nabla \cdot (\rho(t,x)\, K * \rho_t(x)) = \frac{1}{2}\Delta \rho(t,x),$$

with initial condition ρ_0.

From interacting particles to a macroscopic density

$$dX_t^i = \frac{1}{N}\sum_{j=1}^{N} K(X_t^i - X_t^j)dt + dB_t^i$$

$$\partial_t \rho(t,x) + \nabla \cdot (\rho(t,x)\, K * \rho_t(x)) = \frac{1}{2}\Delta\rho(t,x),$$

Goals:

- Consider singular kernels
- Introduce common noise
- Quantify the convergence $S^N \to \rho$

Hypotheses over K:

(i) $K \in L^{\boldsymbol{p}}(\mathcal{B}_1)$, for some $\boldsymbol{p} \in [1, +\infty]$
(ii) $K \in L^{\boldsymbol{q}}(\mathcal{B}_1^c)$, for some $\boldsymbol{q} \in [1, +\infty]$
(iii) $\exists \boldsymbol{r} \geq \max(\boldsymbol{p}', \boldsymbol{q}'), \gamma \in (0, 1]$ and $C > 0$ s.t. $\forall f \in L^1 \cap L^{\boldsymbol{r}}(\mathbb{R}^d)$

$$[K * f]_{C^\gamma} \leq C \|f\|_{L^1 \cap L^r(\mathbb{R}^d)}.$$

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$$[K * f]_{C^\gamma} \leq C\|f\|_{L^1 \cap L^r(\mathbb{R}^d)}.$$

Our assumption derives from this, but on the torus $\mathbb{T}^d$:

$$\|K * f\|_{C^\gamma} \leq C\|f\|_{L^q}, \quad \text{for } q > d.$$

Assumption: $\|K * f\|_{C^\gamma} \leq C\|f\|_{L^q}$, for $q > d$.

Examples: (periodisation of the following)

1. Coulomb kernel, $K_C(x) = \nabla V_{d-2}(x)$ where

$$V_s(x) := \begin{cases} |x|^{-s} & \text{if } s \in (0, d) \\ -\log|x| & \text{if } s = 0 \end{cases}$$

2. Parabolic-elliptic Keller-Segel models. $K_{KS}(x) = -\chi \dfrac{x}{|x|^d}$.

3. Biot-Savart kernel, $d = 2$: $K_{BS}(x) = \dfrac{1}{\pi} \dfrac{x^\perp}{|x|^2}$.

For all of the above, $\gamma = 1 - \frac{d}{q}$ works.

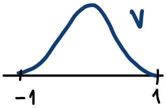
Interacting Particle System

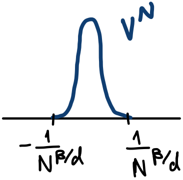
$$dX_t^i = K * S_t^N(X_t^i)\,dt + dB_t^i$$

Regularize the interaction and use cut-off F_A.

$$dX_t^i = F_A(K * S_t^N * V^N(X_t^i))\,dt + dB_t^i$$

where $V^N(\cdot) = N^\beta V(N^{\frac{\beta}{d}}\cdot), \beta \in (0,1)$,
V smooth, $V \equiv 0$ for $\|x\| \geq 1$.





Remark: $\beta = 0$: strong interaction, mean-field.
$\beta = 1$: weak interaction, comparable with nearest-neighbor (1d)

Moderate Mean Field interaction $\quad V^N(\cdot) = N^\beta V(N^{\frac{\beta}{d}} \cdot)$

We use the scaling function V^N to regularize also the empirical measure:

$$S_t^N := \frac{1}{N} \sum_{i=1}^{N} \delta_{X_t^i} \quad \text{and} \quad \rho_t^N := V^N * S_t^N$$

$$\rho_t^N(x) = \frac{1}{N} \sum_{i=1}^{N} V^N(x - X_t^i)$$

$\boxed{\text{Goal: } \rho^N \longrightarrow \rho \text{ solution of PDE.}}$

Equation for ρ_t^N

Ito's formula gives you

$$\mathrm{d}\rho_t^N(x) = -\frac{1}{N}\sum_{i=1}^{N} \nabla V^N(x - X_t^i)\cdot F_A\big(K * \rho_t^N(X_t^i)\big)\,dt$$
$$+ \frac{1}{2}\Delta\rho_t^N(x)dt + dM_t^N(x) \qquad M_t^N : \text{martingale.}$$

Recall the limiting PDE:

$$\partial_t\rho_t(x) = \frac{1}{2}\Delta\rho_t(x) - \nabla\cdot\big(\rho_t(x)K * \rho_t(x)\big)$$

Theorem (ORT'21)

$$\Big\| \sup_{t\in[0,T]} \big\|\rho_t^N - \rho_t\big\|_{L^1\cap L^r(\mathbb{R}^d)} \Big\|_{L^m(\Omega)} \leq \Big\| \big\|\rho_0^N - \rho_0\big\|_{L^1\cap L^r(\mathbb{R}^d)} \Big\|_{L^m(\Omega)} + CN^{-\varrho+\varepsilon},$$

Particles with common noise

Consider the following SDE on the torus $\mathbb{T}^d$, for any d.

$$dX_t^i = F_t\Big(X_t^i, \frac{1}{N}\sum_{j=1}^{N}(K * V^N)(X_t^i - X_t^j)\Big)dt + dB_t^i + \sigma_t\, dB_t^0$$

$$X_0^i \sim \rho_0(x)\, dx \quad \text{independent}$$

The limiting equation is now stochastic

$$\begin{aligned} \mathrm{d}\rho_t = \frac{1}{2}\Delta\rho_t\, dt + \frac{1}{2}\sum_{i,j=1}^{d} D_{ij}\rho_t\, (\sigma_t\sigma_t^\dagger)_{ij}\, dt \\ - \nabla\cdot\big(\rho_t\, F_t(x, K * \rho_t)\big)\, dt - \nabla\rho_t\cdot\sigma_t\, dB_t^0 \end{aligned}$$

- Lipschitz interact.: Coghi, Flandoli (AAP'16); Coghi, Gess (NA'19)
- Hegselmann-Krause: Chen, Nikolaev and Promel (Trans. AMS'25).
- Biot-Savart, 2D: Shao, Zhao in $\mathbb{T}^2$ (ArXiv'24),
  de Souza in $\mathbb{R}^2$ (Arxiv'26)
- $k \in L^2 \cap L^\infty$: Nikolaev (SIAM'25).
- $\nabla \cdot K = 0, K = \nabla \cdot K_0$: Olivera, de Souza (Stoch. and PDE'26)

The last four prove convergence via relative entropy.

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The last four prove convergence via relative entropy.

Our setting

Assume: K s.t. $\|K * f\|_\gamma \lesssim \|f\|_q$ on $\mathbb{T}^d$, σ spatial invariant.

Theorem - K., Olivera, de Souza - JDE'26

$$\left(\mathbb{E}\left[\sup_{t\in[0,T]}\left\|\rho_t - \rho_t^N\right\|_{L^q(\mathbb{T}^d)}^m\right]\right)^{1/m} \leq C\left(\mathbb{E}\left[\left\|\rho_0 - \rho_0^N\right\|_{L^q(\mathbb{T}^d)}^m\right]\right)^{1/m} + CN^{-\kappa}$$

where

$$\kappa = \min\left(\frac{\beta}{d}\gamma, \frac{1}{2} - \beta\Big(1 + \frac{1}{d} - \frac{1}{q}\Big)\right).$$

Corollary

$$\left\|\sup_{t\in[0,T]}\left\|S_t^N - \rho_t\right\|_1\right\|_{L^m(\Omega)} \leq C\left\|\|\rho_0^N - \rho_0\|_{L^q}\right\|_{L^m(\Omega)} + CN^{-\kappa},$$

Some ingredients: well-posedness

In weak sense, including multiplicative case:
σ div free, C^2 and s.t. $\sum_{ij} \|\sigma^{ij}\|_{C^2} \leq C$.

Theorem [K., Olivera, Souza (JDE'26)]

Assume $0 \leq \rho_0 \in L^1 \cap H_q^{1-\frac{2}{q}}(\mathbb{T}^d)$. There exists a $T > 0$ depending only on $d, \|\rho_0\|_{L^q}$ s.t. the above SPDE admits a unique nonnegative solution $\rho_t(\omega)$ with

$$\left(\mathbb{E}\left[\int_0^T \|\rho_t\|^q_{H^1_q(\mathbb{T}^d)}\, dt\right]\right)^{1/q} < \infty, \quad \left\| \sup_{t\in[0,T]} \|\rho_t\|^q_{L^q(\mathbb{T}^d)} \right\|_{L^\infty(\Omega)} < \infty$$

Proof mimicks *Huang, Qiu - J. Nonlin. Sci.'21.*

Krylov-Ito formula for the L^p-norm of stochastic H^1_q-valued processes

$$\|\rho_t - \rho_t^N\|_{L^q}^q = \|\rho_0 - \rho_0^N\|_{L^q}^q + \text{ Mart.1} + \text{Mart.2} + \text{Q.var.1} + \text{Q.var.2} + \ldots$$

The comparison in the nonlinearity manifests as

$$\rho_s(x) K * \rho_s(x) - \left\langle S_s^N, V^N(x - \cdot) \cdot K * \rho_s^N(\cdot) \right\rangle$$

Which is dealt with commutator estimates.

Trick

$$
\begin{aligned}
V^N(x-\cdot)|K*\rho_s^N(x)-K*\rho_s^N(\cdot)| &\leq \|\rho_s^N\|_{L^q}\, V^N(x-\cdot)|x-\cdot|^{\gamma} \\
&\leq \|\rho_s^N\|_{L^q}\, V^N(x-\cdot)N^{-\frac{\beta}{d}\gamma}.
\end{aligned}
$$

A key uniform estimate

$$
\sup_{N\in\mathbb{N}^*} \mathbb{E}\left[\sup_{t\in[0,T]} \left\|\rho_t^N\right\|_{L^q}^m\right]^{\frac{1}{m}} < \infty
$$

which helps also with the Martingale terms.

Multiplicative common noise

Martingale estimate that fails in L^q.

$$\int_0^t \int_{\mathbb{T}^d} q|\rho_s^N(x)|^{q-2}\rho_s^N(x) \left\langle S_s^N, \sigma^\top(\cdot)\nabla V^N(x-\cdot) \right\rangle dx\, dB_s^0$$

$$\text{commutator} \qquad \pm\left\langle S_s^N, \sigma^\top(x)\nabla V^N(x-\cdot)\right\rangle$$

Tightness via [Simon '87] $\mathcal{D}_0 \doteq L^2(0,T;X) \cap \mathbb{W}^{\gamma,m}(0,T;Y)$ is relatively compact in

$$\mathcal{D} \doteq C(0,T;B)$$

where X, Y, B are negative Sobolev spaces.

Tool to compute

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|\rho_t^N\|_{H^{-\lambda}}\right]^q$$

is the semigroup for $\frac{1}{2}\left(\Delta + \sum_k (\sigma^\dagger)^k D^2(\cdot)\sigma^k\right)$.

For now, $\lambda > d/2$ and K falls out of desired class.

Thank you!