A stochastic collisional picture of the Wave Kinetic Equation

Joaquin Fontbona
work in progress with
R. Cortez (UNAB), A. Bernou (U. Lyon)
& G. Krstulovic (U. Côte d'Azur)

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1. The Wave Kinetic Equation

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- Cubic integro-differential equation that **describes evolution of the energy spectrum in a weakly interacting nonlinear wave system**. Fundamental effective model in **wave turbulence theory**, with important applications in plasma physics, oceanography, condensed matter, nonlinear optics...

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  particles in kinetic theory $\longleftrightarrow$ "phonons" or linear waves

Example: Schrödinger

Schrödinger equation

$$i\partial_t \psi = \Delta \psi \quad \text{on } \mathbb{R}^d$$

admits (individual) plane waves $\psi(t,x) = e^{i(k\cdot x - \omega t)}$ in $\mathbb{R}^d$ as solutions if the "dispersion relation" $\omega = |k|^2$ holds.

Physically admissible solutions are "continuous superposition" of plane waves:

$$\psi(t,x) = \frac{1}{2\pi} \int_{\mathbb{R}^d} \hat{\psi}(0,k) e^{i(k\cdot x - t|k|^2)}\, dk,$$

with $\hat{\psi}(0,\cdot)$ an L^2 function.

The 4-wave WKE:

The **WKE** in $\mathbb{R}^d$ (typically $d = 2$ or $d = 3$) with "dispersion relation" $\omega = |k|^2$ is given by

$$\frac{d}{dt}n(k) = \int_{(\mathbb{R}^d)^3} \delta(k + k_1 - k_2 - k_3)\delta(|k|^2 + |k_1|^2 - |k_2|^2 - |k_3|^2) \\ \times n\, n_1 n_2 n_3 \left(\frac{1}{n} + \frac{1}{n_1} - \frac{1}{n_2} - \frac{1}{n_3}\right) dk_1\, dk_2\, dk_3,$$

with

- $n = n_t(k)$ **spectrum density** at time $t \geq 0$ and wave-vector $k \in \mathbb{R}^d$, and n_i, $i = 1, 2, 3$ short-hand for $n_i = n_t(k_i)$
- δ Dirac mass at 0, either on $\mathbb{R}^d$ or $\mathbb{R}$.
- $\delta(k + k_1 - k_2 - k_3)$ and $\delta(|k|^2 + |k_1|^2 - |k_2|^2 - |k_3|^2)$ restrict interactions to **resonant manifold**: quadruples (k, k_1, k_2, k_3) satisfying conservation of momentum and energy: $k + k_1 = k_2 + k_3$ and $|k|^2 + |k_1|^2 = |k_2|^2 + |k_3|^2$

The 4-wave WKE

- **WKE preserves mass, momentum and energy**: for $\phi(k) = 1, k, |k|^2$ one has

$$\frac{d}{dt}\int_{\mathbb{R}^d} \phi(k)n(k)dk = 0.$$

Rich phenomenology, for instance

- expected to exhibit *Bose–Einstein condensation* i.e. **finite-time mass concentration at zero energy**, characterized by emergence of Dirac mass at the origin (blow up)
- admits exact **scale-invariant power law solutions**: *Kolmogorov–Zakharov solutions*, associated with cascade phenomena and constant fluxes (non-equilibrium steady states) of conserved quantities across scales. However: infinite mass

The 4-wave WKE: some mathematical result

- Best general well-posedness result: Germain, Ionescu, Tran (JFA2020), only local in time.
- Isotropic case Escobedo, Velasquez (MemoirsAMS2015).
- Recent ground breaking work of **Deng and Hani** Invent. Math.'23: first full-range **mathematical derivation of the WKE from the nonlinear Schrödinger equation** as fundamental nonlinear dispersive system, when size of finite domain grows and strength of the nonlinearity vanishes in suitable joint scaling.

Deng and Hani's result: Cubic nonlinear Schrödinger equation on the square torus:

$$\begin{cases} (i\partial_t - \Delta)u + \alpha|u|^2 u = 0, \quad x \in \mathbb{T}_L^d = [0, L]^d, \\ u(0, x) = u_0(x) \end{cases}$$

with $\alpha > 0$ strength of the nonlinearity, and random initial condition:

$$u_0(x) = \frac{1}{L^{d/2}} \sum_{k \in \mathbb{Z}_L^d} \widehat{u}_0(k) e^{2\pi i k \cdot x}, \quad \widehat{u}_0(k) = \sqrt{\Psi_0(k)} G_k,$$

with $\Psi : \mathbb{R}^d \to \mathbb{R}_+$ smooth and $(G_k)_k$ i.i.d. e.g. standard Gaussians.

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with $\Psi : \mathbb{R}^d \to \mathbb{R}_+$ smooth and $(G_k)_k$ i.i.d. e.g. standard Gaussians. Then, as $L \to \infty$ with $\alpha = L^{-\gamma}$, for $\gamma \in (0, 1]$,

$$\mathbb{E}(|\widehat{u}(t, k)|^2) \to n_t(k)$$

2. The Spatially Homogeneous Boltzmann Equation

Spatially homogenous Boltzmann equation (HBE)

Describes evolution of velocities of infinitely many physical (gas) particles undergoing pairwise ellastic "collisions":

$$\frac{df}{dt}(v) = Q(f,f)(v) := \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} \left(f' f'_* - f f_*\right) B(|v - v_*|, \sigma)\, d\sigma\, dv_*$$

with $f = f_t(v)$, $f_* = f_t(v_*)$, $f' = f_t(v')$, $f'_* = f_t(v'_*)$ and:

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \qquad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma \tag{1}$$

post-collisional velocities (see drawing) satisfying :

$$v + v_* = v' + v'_*, \qquad |v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2$$

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- Equation preserves mass and mean momentum and energy.
- Introduced by **Kac** in the 50's as simplified version of the true Boltzmann equation $\frac{d}{dt} f(x,v) = Q(f,f)(x,v) + v \cdot \nabla_x f(x,v)$.

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- HBE extensively studied last decades, especially regarding long-time convergence and **propagation of chaos** (see below)

Collision kernel

$B(|v - v_*|, \sigma)$ expresses collision rate dependance on physical forces between particles. It has the general form:

$$B(|v - v_*|, \sigma) = |v - v_*|^\gamma b(\sigma),$$

with $\gamma \in [-3, 2]$ and b some function of σ. Relevant cases:

- Maxwell molecules: $\gamma = 0$, hence $B(|v - v_*|, \sigma)$ does not depend on $|v - v_*|$.
- Hard spheres:
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- Hard potentials: $\gamma \in (1,2)$
- Soft potentials: $\gamma \in (-3,0)$, Coulomb interactions: $\gamma = -3$

2.1. Nonlinear process

Probabilistically: Tanaka *nonlinear* process ('78)

Stochastic jump process $(V_t)_{t\geq 0}$ on $\mathbb{R}^d$, representing the trajectory of typical particle in infinite system. For *hard spheres* it is given by the jump SDE:

$$dV_t = \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}}\int_{\mathbb{R}_+} (v' - V_{t^-})\,\mathbb{1}\{\xi \leq |V_{t^-} - v_*|\}\mathcal{P}(dt, dv_*, d\sigma, d\xi),$$

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- Atom at time $t \geq 0$ samples $v_* \sim \mathrm{Law}_{V_t}$ and $\sigma \sim U(\mathbb{S}^{d-1})$, and jump is proposed from $v = V_{t\text{-}}$ to v' obtained from v,v_* as in (1).
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- $\xi > 0$ accepts/rejects jumps, so that inst. jump rate is $|V_{t\text{-}} - v_*|$.
- One can show $\mathrm{Law}_{V_t} = f_t$ **is solution to HBE** (Itô calculus).

Non linear process and weak solutions: idea

- Using compensation formula for Poisson point measures, for φ test function $\varphi : \mathbb{R}^d \to \mathbb{R}$ we have

$$\varphi(V_t)-\varphi(V_0) = \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} (\varphi(V_t')-\varphi(V_t))|V_t-v_*|d\sigma \mathsf{Law}_{V_t}(dv_*)dt + \mathsf{Martingale}_t$$

- Taking $\mathbb{E}$ we get $g_t = \mathsf{Law}_{V_t}$ solves

$$\int \varphi(v)g_t(dv) - \int \varphi(v)g_0(dv) = \int_0^t \int_{(\mathbb{R}^d)^2} \int_{\mathbb{S}^{d-1}} (\varphi(v')-\varphi(v))\, |v-v_*| d\sigma\, g_t(dv_*) g_t(dv) dt$$

which is weak form of $\mathbf{HBE}$.

2.2. Kac's particle systems

N- Particle system (Kac'56)

(version by Cortez-F. 2018)

$$dX_t^i = \sum_{\substack{j=1 \\ j\neq i}}^{N} \int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \mathbb{1}\{\xi \leq |V_{t^-}^i - V_{t^-}^j|\} \qquad i = 1, \dots, N$$

$$\times \left(v' - V_{t^-}^i\right) \left[\mathcal{P}^{ij}(dt, d\sigma, d\xi) + \mathcal{P}^{ji}(dt, -d\sigma, d\xi)\right] \text{ with}$$

- $\mathcal{P}^{ij}(dt, d\sigma, d\xi)$, $i, j \in \{1, \dots, N\}$, $i \neq j$ Poisson point measure on $\mathbb{R}_+ \times \mathbb{S}^{d-1} \times \mathbb{R}_+$ with intensity $\frac{1}{2(N-1)} dt\, d\sigma\, d\xi\, d\zeta$ and collection $(\mathcal{P}^{ij})_{i\neq j}$ are independent
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- v' computed from $v = V_{t^-}^i$, $v_* = V_{t^-}^j$ and σ as in (1)
- momentum & energy exactly preserved at each collision
- **easy to simulate**: at rate $\sim 2Nm$, for $m > 0$ large, sample pair $(V_{t^-}^i, V_{t^-}^j)$ uniformly, perform (accept) collision if $\xi \leq |V_{t^-}^i - V_{t^-}^j|$ (and reject otherwise), for $\xi \sim U([0, m])$, repeat independently...

2.3. Propagation of chaos

Propagation of chaos for HBE

Theorem: Assume particles are i.i.d. and f_0 distributed at $t = 0$. Then, under certain conditions (on initial condition, collision kernel), for all $t \geq 0$,

$$\lim_{N\to\infty} \text{Law}(V_t^1, \dots, V_t^k) = f_t^{\otimes k} \quad \text{weakly for each } k \in \mathbb{N},$$

i.e. independence propagates to later times $t > 0$, as $N \to \infty$. Equivalently (cf. exchangeability), the empirical measure $\frac{1}{N}\sum_{i=1}^N \delta_{V_t^i}$ converges in law to f_t solution of BHE. (Refinements possible too).

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(True BE:short-time:Lanford70's,Bodinau-S.Raymond-Gallagher$\sim$15,
Long-time: Deng-Hani-Ma,'24, Fields Medal 2026 Deng)

Toy simulation of Kac's system:

Kac's system for 2D Maxwell-like HBE
(with angular dependence dropped):

$$dV_t^i = \sum_{\substack{j=1 \\ j\neq i}}^{N} \int_{\mathbb{R}_+} \int_{\mathbb{S}^{d-1}} \left(v' - V_{t^-}^i\right) \left[\mathcal{P}^{ij}(dt, d\sigma) + \mathcal{P}^{ji}(dt, -d\sigma)\right]$$

- Long-time, large N limit: centered isotropic 2D Gaussian (Maxwellian)
- $N = 5000$
- empirical measure regularized for visualization with $\rho_\varepsilon(x) := \rho(x/\varepsilon)\varepsilon^{-2}$ with ρ isotropic 2D Gaussian and $\varepsilon = 0.2$.
- $\sim$ 300.000 collisions per (nominal) minute

3. From Boltzmann to WKE: post-collisional acceptance/rejection

Key insight: weak forms of HBE and WKE:

For $\sigma \in \mathbb{S}^{d-1}$ and "**pre-collisional**" $z, z_* \in \mathbb{R}^d$, recall the notation

$$z' = \frac{z + z_*}{2} + \frac{|z - z_*|}{2}\sigma, \qquad z'_* = \frac{z + z_*}{2} - \frac{|z - z_*|}{2}\sigma$$

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Integrating HBE against test function φ (and dropping angular dependance), r.h.s. reads:

$$\int_{(\mathbb{R}^d)^2} \int_{\mathbb{S}^{d-1}} \Big(\varphi(v') - \varphi(v)\Big)|v - v_*|^\gamma f(v) f(v_*) \, d\sigma \, dv_* \, dv$$

for e.g. $\gamma = 0$ (Maxwell molecules) or $\gamma = 1$ (Hard spheres).

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$$\int_{(\mathbb{R}^d)^2} \int_{\mathbb{S}^{d-1}} \Big(\varphi(k') - \varphi(k)\Big) |k - k_*|^{d-2} [n(k') + n(k'_*)] n(k) n(k_*) \, d\sigma \, dk_* \, dk$$

3.1. Nonlinear process for WKE

Nonlinear process for WKE

Theorem (Bernou, Cortez, F. '26+) Let $(n_t)_{t\in[0,T]}$ be (locally bounded) solution to **WKE** (+ assumptions on n_0). There exists a unique **non-linear process** $(K_t)_{t\in[0,T]}$ on $\mathbb{R}^d$ solution to jump SDE

$$dK_t = \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}}\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} (k' - K_{t^-})\,\mathbb{1}\{\xi \le |K_{t^-} - k_*|^{d-2}\} \times \mathbb{1}\{\zeta \le \mathsf{Law}_{K_t}(k') + \mathsf{Law}_{K_t}(k'_*)\}\mathcal{P}(dt, dk_*, d\sigma, d\xi, d\zeta).$$

with $\mathcal{P}(dt, dk_*, d\sigma, d\xi, d\zeta)$ a Poisson point measure on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{S}^{d-1} \times \mathbb{R}_+ \times \mathbb{R}_+$ with intensity $dt\,\mathsf{Law}_{K_t}(dk_*)d\sigma\,d\xi\,d\zeta$

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$$dK_t = \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}}\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} (k' - K_{t^-})\,\mathbb{1}\{\xi \le |K_{t^-} - k_*|^{d-2}\} \\ \times \mathbb{1}\{\zeta \le \mathrm{Law}_{K_t}(k') + \mathrm{Law}_{K_t}(k'_*)\}\mathcal{P}(dt, dk_*, d\sigma, d\xi, d\zeta).$$

with $\mathcal{P}(dt, dk_*, d\sigma, d\xi, d\zeta)$ a Poisson point measure on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{S}^{d-1} \times \mathbb{R}_+ \times \mathbb{R}_+$ with intensity $dt\,\mathrm{Law}_{K_t}(dk_*)d\sigma\, d\xi\, d\zeta$
Note that:

- Atom at time $t \ge 0$ samples $k_* \sim \mathrm{Law}_{K_t}$ and $\sigma \sim U(\mathbb{S}^{d-1})$; jump is then proposed from K_{t^-} to $K_t = k'$ defined as usual
- Random $\xi > 0$ and $\zeta > 0$ accept/reject jumps, so that instantaneous jump rate is $|K_{t^-} - k_*| \times [\mathrm{Law}_{K_t}(k') + \mathrm{Law}_{K_t}(k'_*)]$.

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- One can show that $\mathrm{Law}_{K_t} = n_t$ **solution of KWE**.

3.2. Regularized particle system for WKE

Regularized particle system for WKE

KWE process requires evaluating at k', k'_* the density n_t, **not defined** for particle system. Let $\varepsilon > 0$ and $(K_t^{1,\varepsilon}, \dots, K_t^{N,\varepsilon})$ given by

$$dK_t^{i,\varepsilon} = \sum_{\substack{j=1 \\ j\neq i}}^{N} \int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \left(k' - K_{t^-}^{i,,\varepsilon}\right) \mathbb{1}\{\xi \leq |K_{t^-}^{i,\varepsilon} - K_{t^-}^{j,\varepsilon}|^{d-2}\}$$

$$\mathbb{1}\{\zeta \leq \rho_\varepsilon * \bar{n}_{t^-}^{N,\varepsilon}(k') + \rho_\varepsilon * \bar{n}_{t^-}^{N,\varepsilon}(k'_*)\}$$

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- k' and k'_* computed from $k = K_{t^-}^{i,\varepsilon}$, $k_* = K_{t^-}^{j,\varepsilon}$ and σ as usual.
- $\bar{n}_t^{N,\varepsilon} = \frac{1}{N} \sum_{i=1}^{N} \delta_{K_t^{i,\varepsilon}}$ empirical measure and ρ_ε mollifier.
- Energy & momentum: exactly preserved at each collision

Regularized nonlinear process and regularized WKE

For fixed ε, **propagation of chaos** expected to hold for regularized particle system, towards the limit given by the **reguralized non-linear process** $(K_t^\varepsilon)_{t\in[0,T]}$ on $\mathbb{R}^d$ solution to the jump SDE:

$$dK_t^\varepsilon = \int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}}\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} (k' - K_{t\text{-}}^\varepsilon)\,\mathbb{1}\{\xi \le |K_{t\text{-}}^\varepsilon - k_*|^{d-2}\}\times \mathbb{1}\{\zeta \le \rho_\varepsilon * \mathsf{Law}_{K_t^\varepsilon}(k') + \rho_\varepsilon * \mathsf{Law}_{K_t^\varepsilon}(k'_*)\}\mathcal{P}(dt, dk_*, d\sigma, d\xi, d\zeta).$$

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$$\frac{d}{dt}\int_{\mathbb{R}^d}\varphi(k)n_t^\varepsilon(k)dk = \int_{(\mathbb{R}^d)^2}\int_{\mathbb{S}^{d-1}}\Big(\varphi(k')-\varphi(k)\Big)|k-k_*|^{d-2} \times n_t^\varepsilon(k)n_t^\varepsilon(k_*)[\rho_\varepsilon * n_t^\varepsilon(k') + \rho_\varepsilon * n_t^\varepsilon(k'_*)]d\sigma dk_* dk$$

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New, simple (random) regularization of resonant manifold constraint!

Toy simulation of regularized KWE system:

Regularized Kac's system for regularized 2D KWE :

$$dK_t^{i,\varepsilon} = \sum_{\substack{j=1 \\ j\neq i}}^{N} \int_{\mathbb{S}^{d-1}} \int_{\mathbb{R}_+} \left(k' - K_{t^-}^{i,\varepsilon}\right) \mathbb{1}\{\zeta \leq \rho_\varepsilon * \bar{n}_{t^-}^{N,\varepsilon}(k') + \rho_\varepsilon * \bar{n}_{t^-}^{N,\varepsilon}(k'_*)\}$$

$$\times \left[\mathcal{P}^{ij}(dt, d\sigma, d\zeta) + \mathcal{P}^{ji}(dt, -d\sigma, d\zeta)\right] \text{ with}$$

- Long-time, large N limit ?
- $N = 5000$
- empirical measure regularized for visualization with the same mollifier $\rho_\varepsilon(x) := \rho(x/\varepsilon)\varepsilon^{-2}$ as for post-collisional filtering.
- $\sim$ 300.000 proposed collisions per (nominal) minute.
- Decreasing acceptance rate: $\sim 20\% \searrow 10\%$ in this movie
- "Pseudo-condensate" seems to emerge.

4. Partial results and agenda

Some partial mathematical results

- Easy (conditional result): we can construct **nonlinear process** (non-regularized) associated with any (locally bounded) solution $(n_t)_{t\in[0,T]}$, $T>0$ of **WKE** in any dimension $d \geq 2$

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- More concretely: **local well posedness for nonlinear process, regularized and non-regularized**. Based on Germain et al. 2020 locall well possedness result for WKE, extended to hold "**uniformely w.r.t.** $\varepsilon>0$" (mollifier well behaved!)

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- **Moreover: global well-posedness of regularized model** (equation and nonlinear process) for $d=2$ and ε fixed by probabilistic argument (construction à la Tanaka: fixed point in space of probability measures)
- **Propagation of chaos of regularized particle system,** $\varepsilon > 0$ **fixed, in** $d=2$ (non quantitative yet).

Agenda and open questions:

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- For regularized particle system in $d = 2$ or for isotropic $d = 3$ version : **can we prove something on "condensation"? How is this compatible with energy preservation?**
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- Apply **particle method** to relevant/real-scale $(d = 2, 3)$ cases

5. Conclusions

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- Potential tool to study (mathematically, numerically) pertinent physical questions: KZ solutions, out of equilibrium steady-states, spontaneous stochasticity (and others)

Thank you!