Weighted Helmholtz–Hodge decompositions, Lyapunov functions, and invariant measures

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Motivation

Consider a diffusion-type operator on $\mathbb{R}^d$

$$L^{A,\mathbf{G}} f = \frac{1}{2}\,\mathrm{trace}(A\nabla^2 f) + \langle \mathbf{G}, \nabla f\rangle, \qquad A = A^T > 0, \quad f \in C_0^\infty(\mathbb{R}^d).$$

Here $\mathbf{G}$ is a vector field, and $\langle\cdot,\cdot\rangle :=$ Euclidean inner product.

Two questions usually studied separately

- Construct Lyapunov functions for deterministic stability or for stochastic non-explosion and long-time ergodic behavior.
- Identify an infinitesimally invariant or invariant measure for the diffusion semigroup.

The main point is that, in many explicit situations, both questions are linked by a drift decomposition. The deterministic and stochastic Lyapunov roles are logically distinct, but in some of our examples the same potential-derived function serves both.

Guiding principle

Since

$$L^{A,A\nabla\Phi}f = \frac{1}{2}\,\mathrm{trace}(A\nabla^2 f) + \langle A\nabla\Phi, \nabla f\rangle$$

is symmetric with respect to

$$\mu = e^{2\Phi}dx,$$

we look for decompositions of the form

$$\mathbf{G} = A\nabla\Phi + \mathbf{B},$$

where the remainder $\mathbf{B}$ is divergence-free with respect to μ:

$$\mathrm{div}_\mu(\mathbf{B}) = 0, \qquad \text{i.e.} \qquad \int_{\mathbb{R}^d} \langle \mathbf{B}, \nabla f\rangle \, d\mu = 0 \quad \forall f \in C_0^\infty(\mathbb{R}^d).$$

Consequently,

$$\int_{\mathbb{R}^d} L^{A,\mathbf{G}} f \, d\mu = 0, \qquad \forall f \in C_0^\infty(\mathbb{R}^d).$$

Interpretation

The gradient part produces the reference density $e^{2\Phi}$, while the weighted divergence-free part preserves this density.

Analytic framework

Throughout, let $d \geq 2$, let A be a symmetric positive definite matrix, and let

$$\mathbf{G} \in L^2_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}^d).$$

For the vector field $\mathbf{G}$ we consider

$$L^{A,\mathbf{G}} f = \frac{1}{2}\operatorname{trace}(A\nabla^2 f) + \langle \mathbf{G}, \nabla f \rangle, \qquad f \in C_0^\infty(\mathbb{R}^d).$$

Infinitesimal invariance

A positive locally finite measure $\widehat{\mu}$ satisfying $L^{A,\mathbf{G}} f \in L^1(\mathbb{R}^d, \widehat{\mu})$ for all $f \in C_0^\infty(\mathbb{R}^d)$ is infinitesimally invariant for $(L^{A,\mathbf{G}}, C_0^\infty(\mathbb{R}^d))$ if

$$\int_{\mathbb{R}^d} L^{A,\mathbf{G}} f \, d\widehat{\mu} = 0, \qquad \forall f \in C_0^\infty(\mathbb{R}^d).$$

Regularity of densities: if $\mathbf{G} \in L^p_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}^d)$ for some $p > d$, then every infinitesimally invariant measure has a density $\widehat{\mu} = \widehat{\rho} dx$ with

$$\widehat{\rho} \in H^{1,p}_{\mathrm{loc}}(\mathbb{R}^d) \cap C(\mathbb{R}^d), \qquad \widehat{\rho} > 0.$$

Conversely: if one starts with an explicit density from the beginning, the regularity assumptions can be relaxed substantially.

Why weighted rather than classical?

If, in the case $A = id$,

$$\mathbf{G} = \nabla\Phi + \mathbf{B},$$

then the classical Helmholtz–Hodge condition would be

$$\mathrm{div}_{dx}(\mathbf{B}) = 0.$$

This is natural for incompressible fluid dynamics.

For stochastic dynamics, the natural reference measure is

$$\mu = e^{2\Phi} dx,$$

which is finite in the invariant-probability setting, whereas Lebesgue measure is infinite on $\mathbb{R}^d$.

Therefore the relevant condition is

$$\mathrm{div}_{\mu}(\mathbf{B}) = 0,$$

because, as we have seen, it is exactly the condition for infinitesimal invariance of μ for the corresponding diffusion operator.

Three decompositions

Let $\widetilde{\mathbf{G}} \in L^2_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}^d)$.

HHD

There are $\Phi \in H^{1,2}_{\mathrm{loc}}(\mathbb{R}^d) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^d)$ and $\mathbf{B} \in L^2_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}^d)$ such that

$$\widetilde{\mathbf{G}} = \nabla\Phi + \mathbf{B}, \qquad \mathrm{div}_{dx}(\mathbf{B}) = 0.$$

OHHD

In addition to HHD,

$$\langle \nabla\Phi, \mathbf{B} \rangle = 0 \quad \text{a.e.}$$

WHHD

There are Φ and $\mathbf{B}$ as above such that

$$\widetilde{\mathbf{G}} = \nabla\Phi + \mathbf{B}, \qquad \mathrm{div}_{\mu}(\mathbf{B}) = 0, \qquad \mu = e^{2\Phi} dx.$$

If $\widetilde{\mathbf{G}}$, Φ, and $\mathbf{B}$ are continuously differentiable, and the identities

$$\widetilde{\mathbf{G}} = \nabla\Phi + \mathbf{B}, \qquad \operatorname{div}\mathbf{B} = 0, \qquad \langle \nabla\Phi, \mathbf{B}\rangle = 0$$

hold pointwise on $\mathbb{R}^d$, then the OHHD is called a strictly orthogonal Helmholtz–Hodge decomposition (SOHHD).

Geometric content of SOHHDs

- $\nabla\Phi$ is normal to the level sets of Φ.
- $\mathbf{B}$ is tangent to the level sets of Φ.
- $\operatorname{div}\mathbf{B} = 0$ gives the usual incompressibility condition.

Main identity: invariance and weighted divergence

Assume

$$\mathbf{G} = A\nabla\Phi + \mathbf{B}, \qquad \Phi \in H^{1,2}_{\mathrm{loc}}(\mathbb{R}^d) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^d), \qquad \mathbf{B} \in L^2_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}^d), \qquad \mu = e^{2\Phi} dx.$$

Theorem 2.5 (using integration by parts and approximation)

$$\mathrm{div}_\mu(\mathbf{B}) = 0 \iff \mu \text{ is infinitesimally invariant for } (L^{A,\mathbf{G}}, C_0^\infty(\mathbb{R}^d))$$

$$\iff \int_{\mathbb{R}^d} \left(\langle \nabla\Phi, \mathbf{B} \rangle \varphi - \frac{1}{2} \langle \mathbf{B}, \nabla\varphi \rangle \right) dx = 0 \quad \forall \varphi \in C_0^\infty(\mathbb{R}^d).$$

- If $\mathrm{div}_{dx}(\mathbf{B}) \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ exists in the weak sense, this is equivalent to

$$\langle \nabla\Phi, \mathbf{B} \rangle + \frac{1}{2}\,\mathrm{div}_{dx}(\mathbf{B}) = 0.$$

- Moreover, if $\mathrm{div}_\mu(\mathbf{B}) = 0$ (WHHD), then:

$$\mathrm{div}_{dx}(\mathbf{B}) = 0 \iff \langle \nabla\Phi, \mathbf{B} \rangle = 0.$$

Thus,

$$\mathrm{div}_{dx}(\mathbf{B}) = 0 \quad \text{and} \quad \langle \nabla\Phi, \mathbf{B} \rangle = 0 \quad \Longrightarrow \quad \mathrm{div}_{\mu}(\mathbf{B}) = 0, \quad \mu = e^{2\Phi}dx,$$

and therefore

$$\text{OHHD} \Longrightarrow \text{WHHD}.$$

The **converse is not automatic**: for instance, if $\mathrm{div}_{dx}(\mathbf{B}) \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, we may have

$$\langle \nabla\Phi, \mathbf{B} \rangle + \frac{1}{2}\,\mathrm{div}_{dx}(\mathbf{B}) = 0.$$

but

$$\text{neither} \quad \mathrm{div}_{dx}(\mathbf{B}) = 0 \quad \text{nor} \quad \langle \nabla\Phi, \mathbf{B} \rangle = 0$$

(see later).

Lyapunov interpretation of deterministic systems

Suppose

$$\frac{dX_t}{dt} = \widetilde{\mathbf{G}}(X_t), \qquad \widetilde{\mathbf{G}} = \nabla\Phi + \mathbf{B}, \qquad \langle \nabla\Phi, \mathbf{B} \rangle = 0.$$

Then along a trajectory,

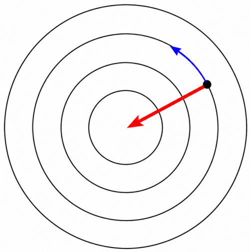
$$\frac{d}{dt}\left[-\Phi(X_t)\right] = -\left\langle \nabla\Phi(X_t), \widetilde{\mathbf{G}}(X_t) \right\rangle = -\|\nabla\Phi(X_t)\|^2 \le 0.$$

Conclusion

$-\Phi$ is a Lyapunov function in the sense that it is non-increasing along the flow. The gradient part $\nabla\Phi$ moves the trajectory from higher to lower level sets of the Lyapunov function $-\Phi$. The field $\mathbf{B}$, by contrast, has no normal component and only produces tangential motion along the level sets of Φ.

This is the deterministic Lyapunov interpretation. It should be distinguished from the generator inequality used later for stochastic non-explosion. In the setting of Remark 3.15 the same potential-derived function serves both roles.

Geometric picture for $\Phi(x) = -\|x\|^2$



For $\Phi(x) = -\|x\|^2$,

$$\nabla\Phi(x) = -2x.$$

Hence $\nabla\Phi$ points inward and is normal to the level sets.

In an OHHD/SOHHD,

$$\widetilde{\mathbf{G}} = \nabla\Phi + \mathbf{B}, \qquad \langle \nabla\Phi, \mathbf{B} \rangle = 0,$$

so $\mathbf{B}$ is tangential to the level sets. Thus $\nabla\Phi$ changes the level of Φ, $\mathbf{B}$ only moves tangentially along the same level set.

Uniqueness of the infinitesimally invariant measure makes the potential canonical

Theorem 2.7

Let $\widetilde{\mathbf{G}} \in L^p_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}^d)$, $p > d$, and write

$$L^{\widetilde{\mathbf{G}}} f := \frac{1}{2}\Delta f + \langle \widetilde{\mathbf{G}}, \nabla f \rangle, \qquad f \in C_0^\infty(\mathbb{R}^d).$$

Suppose that $(L^{\widetilde{\mathbf{G}}}, C_0^\infty(\mathbb{R}^d))$ admits a unique infinitesimally invariant measure, up to multiplication,

$$\mu = e^{2\Phi} dx.$$

Then

$$\widetilde{\mathbf{G}} = \nabla\Phi + \mathbf{B}, \qquad \text{with} \quad \mathbf{B} = \widetilde{\mathbf{G}} - \nabla\Phi,$$

is the unique WHHD of $\widetilde{\mathbf{G}}$. If an OHHD exists, it is unique as well.

When does the unique WHHD become an OHHD?

Under the assumptions of Theorem 2.7, it follows from Theorem 2.5 that the unique WHHD

$$\widetilde{\mathbf{G}} = \nabla\Phi + \mathbf{B}$$

is an OHHD if and only if one, hence both, of the following conditions hold:

$$\mathrm{div}_{dx}(\mathbf{B}) = 0, \qquad \langle \nabla\Phi, \mathbf{B} \rangle = 0.$$

If neither condition holds, then there is no OHHD for $\widetilde{\mathbf{G}}$.

Consequence

If the infinitesimally invariant measure is unique, there is no freedom to change the potential in order to force orthogonality.

Let

$$\Phi(x) = -\frac{1}{2}\|x\|^2, \qquad \mu = e^{-\|x\|^2}dx, \qquad \rho = e^{2\Phi}.$$

Define a skew-symmetric matrix field $C(x) = (c_{ij}(x))$ by

$$c_{ij}(x) = \begin{cases} x_1, & (i,j) = (1,d), \\ -x_1, & (i,j) = (d,1), \\ 0, & \text{otherwise.} \end{cases}$$

Then $C(x)x = (x_1 x_d, 0, \ldots, 0, -x_1^2)$, and we set

$$\mathbf{G}(x) = -x + C(x)x + (0, \ldots, 0, 1/2).$$

Moreover,

$$\langle \mathbf{G}(x), x \rangle = -\|x\|^2 + \frac{1}{2}x_d.$$

Hence the associated semigroup is conservative. Since μ is finite, it is, up to multiplication, the unique infinitesimally invariant measure; after normalization it is also the unique invariant measure, and recurrence holds.

Indeed,

$$\mathbf{G} = \nabla\Phi + \mathbf{B}, \qquad \mathbf{B}(x) = C(x)x + (0, \dots, 0, 1/2),$$

and since

$$\mathbf{B} = \beta^{\rho, C^T} = \frac{1}{2\rho} C^T \nabla\rho + \frac{1}{2} \operatorname{div} C, \qquad \rho = e^{2\Phi},$$

we get

$$\operatorname{div}_\mu(\mathbf{B}) = 0.$$

Thus μ is infinitesimally invariant for $(L^{\mathbf{G}}, C_0^\infty(\mathbb{R}^d))$ and $\nabla\Phi + \mathbf{B}$ is the unique WHHD for $\mathbf{G}$.

However,

$$\langle \nabla\Phi, \mathbf{B} \rangle = -\frac{1}{2} x_d, \qquad \operatorname{div} \mathbf{B} = x_d.$$

Therefore neither condition for an OHHD holds, and no OHHD of $\mathbf{G}$ exists.

Transition to linear vector fields

We saw above:

$$\mathrm{div}_\mu(\mathbf{B}) = 0 \quad \iff \quad \mu \text{ infinitesimally invariant.}$$

Now take matrices $G, A, S \in \mathbb{M}_d(\mathbb{R})$ with $A = A^T > 0$, $S = S^T$ and

$$\mathbf{G}(x) = Gx, \qquad \Phi(x) = \frac{1}{2}\langle x, Sx\rangle.$$

Then

$$\mu = e^{2\Phi}dx = e^{\langle x, Sx\rangle}dx, \qquad \mathbf{B}(x) = (G - AS)x, \qquad A\nabla\Phi(x) = ASx,$$

and

$$L^{A,G}f = \frac{1}{2}\,\mathrm{trace}(A\nabla^2 f) + \langle Gx, \nabla f\rangle = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}\partial_{ij}f + \sum_{i=1}^{d}\Big(\sum_{j=1}^{d} g_{ij}x_j\Big)\partial_i f.$$

The question of infinitesimal invariance becomes algebraic.

Linear computation

Let

$$H = G - AS, \qquad \text{hence} \quad \mathbf{B}(x) = Hx.$$

The condition $\mathrm{div}_\mu(Hx) = 0$ is equivalent to

$$\langle Sx, Hx \rangle + \frac{1}{2}\,\mathrm{trace}(H) = 0, \qquad x \in \mathbb{R}^d.$$

Thus we obtain two algebraic conditions:

$$SH + H^T S = 0, \qquad \mathrm{trace}(H) = 0.$$

Substituting $H = G - AS$ gives the algebraic Riccati equation

$$SG + G^T S = 2SAS$$

with trace condition, see $(*)$ below.

In particular, in the linear case considered here,

$$\text{WHHD} \quad \Longleftrightarrow \quad \text{SOHHD}.$$

We have hence obtained the following result for $G, S \in \mathbb{M}_d(\mathbb{R})$.

Lemma 2.11

Let $S = S^T$ and let $\mathbf{G}(x) = Gx$. Then

$$\mu = e^{\langle x, Sx\rangle} dx$$

is infinitesimally invariant for $(L^{A,G}, C_0^\infty(\mathbb{R}^d))$ if and only if

$$SG + G^T S = 2SAS, \qquad \operatorname{trace}(G - AS) = 0. \tag{*}$$

Moreover, under (*)

$$\widetilde{G}x = Sx + (G - AS)x$$

is a WHHD if and only if it is a SOHHD.

For

$$G = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad A = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}, \qquad \alpha > 0, \quad \alpha\gamma - \beta^2 > 0,$$

Theorem 2.13 gives explicit symmetric solutions S of $(*)$.

Trace zero and singular cases

- If $a + d = 0$, one may choose $S = 0$.
- If $a + d \neq 0$, $ad - bc = 0$, and $b \neq 0$, one may choose
$$S = \frac{a+d}{\alpha a^2 + 2\beta ab + \gamma b^2} \begin{pmatrix} a^2 & ab \\ ab & b^2 \end{pmatrix}.$$
- If $a + d \neq 0$, $ad - bc = 0$, $b = 0$, and $c \neq 0$, one may choose
$$S = \frac{a+d}{\alpha c^2 + 2\beta cd + \gamma d^2} \begin{pmatrix} c^2 & cd \\ cd & d^2 \end{pmatrix}.$$
- If $b = c = 0$, one obtains the diagonal cases $S = \begin{pmatrix} a/\alpha & 0 \\ 0 & 0 \end{pmatrix}$ if $a \neq 0$, and $S = \begin{pmatrix} 0 & 0 \\ 0 & d/\gamma \end{pmatrix}$ if $d \neq 0$.

Dimension two: closed form solution page 2

If $a+d \neq 0$ and $ad-bc \neq 0$, one possible solution is

$$S = \kappa \begin{pmatrix} \gamma a(a+d) + \alpha c^2 - \gamma bc - 2\beta ac & \alpha cd + \gamma ab - 2\beta ad \\ \alpha cd + \gamma ab - 2\beta ad & \alpha d(a+d) + \gamma b^2 - \alpha bc - 2\beta bd \end{pmatrix},$$

$$\kappa = \frac{a+d}{(\alpha\gamma - \beta^2)(a+d)^2 + (\gamma b - \alpha c + \beta(a-d))^2}.$$

Together with the cases on the preceding slide, this gives a closed form solution in dimension two.

Proposition 2.15(i)

For every symmetric positive definite A and every real matrix G, there exists a symmetric real matrix S such that

$$SG + G^T S = 2SAS, \qquad \mathrm{trace}(G - AS) = 0. \tag{*}$$

Therefore an infinitesimally invariant measure of the form

$$\mu = e^{\langle x, Sx \rangle} dx$$

always exists.

A finite invariant measure exists precisely when G is Hurwitz; in that case the negative definite solution is the one described on the next slide.

Hurwitz criterion

Assume G is Hurwitz, i.e. all eigenvalues of G have strictly negative real parts. Set

$$P = \int_0^\infty e^{Gt}(2A)e^{G^T t}\,dt, \qquad S = -P^{-1}.$$

Then S is negative definite and solves the algebraic Riccati equation $(*)$.

Proposition 2.15(ii)–(iii)

- $\mu = e^{\langle x, Sx\rangle}\,dx$ is the unique finite invariant measure of the OU semigroup, up to multiplication.
- Conversely, existence of a finite invariant measure implies that G is Hurwitz.

$$\text{OU finite invariant measure} \quad \Longleftrightarrow \quad G \text{ Hurwitz}.$$

Structural consequence for the linear case

In the Hurwitz case,

$$\widetilde{G}x = Sx + (G - AS)x$$

has the unique SOHHD

$$\widetilde{G}x = \nabla\Phi(x) + (G - AS)x, \qquad \Phi(x) = \frac{1}{2}\langle x, Sx\rangle.$$

The same measure

$$\mu = e^{\langle x, Sx\rangle} dx$$

is the unique finite infinitesimally invariant measure for both

$$(L^{A,G}, C_0^\infty(\mathbb{R}^d)) \quad \text{and} \quad (L^{\widetilde{G}}, C_0^\infty(\mathbb{R}^d)).$$

Structural route in the OU case

From stability to decomposition

$$G \text{ Hurwitz}$$
$$\Downarrow$$
$$P = \int_0^\infty e^{Gt}(2A)e^{G^T t}\,dt$$
$$\Downarrow$$
$$S = -P^{-1} \text{ solves } (*)$$
$$\Downarrow$$
$$\mu = e^{\langle x, Sx\rangle}\,dx$$
$$\Downarrow$$
$$\widetilde{G}(x) = Sx + (G - AS)x \text{ has a unique SOHHD.}$$

Moreover, there are standard algorithms for solving the corresponding continuous-time Lyapunov equation and hence computing S explicitly from G and A.

Nonlinear polynomial perturbations

Start with a potential, often obtained from the linear theory,

$$\Phi(x) = \frac{1}{2}\langle x, Sx\rangle,$$

(or, more generally, with a polynomial potential,) and add nonlinear terms to the drift.

Question

When does the same potential still determine the invariant measure $\mu = e^{2\Phi}dx$?

Answer within the exponential–polynomial class

Let Φ be a non-constant polynomial of even degree $m \geq 2$, let $\mathbf{P}$ be a polynomial vector field, and set

$$\mathbf{B} = \mathbf{P} - A\nabla\Phi.$$

The condition for infinitesimal invariance is $\operatorname{div}_\mu(\mathbf{B}) = 0$, equivalently

$$\langle \nabla\Phi, \mathbf{P} - A\nabla\Phi\rangle + \frac{1}{2}\operatorname{div}(\mathbf{P} - A\nabla\Phi) = 0.$$

Writing $\Phi = \sum_{k=0}^{m} \Phi_k$ and $\mathbf{P} = \sum_{k=0}^{n} \mathbf{P}_k$ in homogeneous parts, this becomes a finite algebraic system by collecting degrees.

Main polynomial theorem

Theorem 3.1

Let Φ be a non-constant polynomial of even degree and let $\mathbf{P}$ be a polynomial vector field. Assume, for some $c \in \mathbb{R}$ and $\alpha > d/2$,

$$\Phi(x) \leq c - \alpha \log \|x\| \qquad \text{for } \|x\| \text{ sufficiently large,}$$

choose $C > 0$ such that $V = -\Phi + C \geq 1$, and assume

$$L^{A,\mathbf{P}} V \leq MV \qquad \text{outside some ball,}$$

together with

$$\langle \nabla \Phi, \mathbf{P} - A\nabla\Phi \rangle + \frac{1}{2} \operatorname{div}(\mathbf{P} - A\nabla\Phi) = 0.$$

Then the associated regularized semigroup is conservative and

$$\mu = e^{2\Phi} dx$$

is, up to multiplication, the unique finite invariant measure and the unique finite infinitesimally invariant measure.
If additionally $\operatorname{div}(\mathbf{P} - A\nabla\Phi) = 0$, then

$$\widetilde{\mathbf{P}} := \nabla\Phi + (\mathbf{P} - A\nabla\Phi)$$

is the unique SOHHD for $\widetilde{\mathbf{P}}$.

A practical sufficient criterion

Let

$$\Phi = \Phi_0 + \Phi_1 + \cdots + \Phi_m$$

be the homogeneous decomposition and suppose $\mathbf{B} = \mathbf{P} - A\nabla\Phi$.

Proposition 3.3

A sufficient condition for the hypotheses on finiteness and Lyapunov-type non-explosion in Theorem 3.1 is:

1. $\deg(\mathbf{B}) \leq 2m - 2$,
2. $-\Phi_m(x) > 0$ for every $x \in S^{d-1}$.

The second condition says that the leading homogeneous part of $-\Phi$ is strictly positive on the sphere.

This is a sufficient leading-order condition; it is not implied by $-\Phi(z) \to \infty$ as $\|z\| \to \infty$. For example,

$$\Phi(x, y) = -(x^2 + y^4), \qquad z = (x, y),$$

has $-\Phi(x, y) \to \infty$ as $\|(x, y)\| \to \infty$, but $\Phi_4(x, y) = -y^4$ vanishes at $(\pm 1, 0) \in S^1$ (Remark 3.4).

Let

$$\Phi(x,y) = -x^{10} - y^{10} + p(x,y), \qquad \deg(p) \le 9,$$

and let $q(x,y)$ be a polynomial with $\deg(q) \le 9$. Define the skew-symmetric matrix

$$C(x,y) = \begin{pmatrix} 0 & q(x,y) \\ -q(x,y) & 0 \end{pmatrix}$$

and

$$\mathbf{P} = \nabla\Phi + C^T \nabla\Phi + \frac{1}{2} \operatorname{div} C.$$

Then the conditions of Proposition 3.3 apply, and

$$\mu = e^{2\Phi} dxdy$$

is finite and the unique infinitesimally invariant and invariant measure for the corresponding semigroup.

Quadratic perturbations in dimension two

$A, G \in \mathbb{M}_2(\mathbb{R})$, $A = A^T > 0$. Then we know that there is an explicit $S \in \mathbb{M}_2(\mathbb{R})$ with $S = S^T$ (solution to the ARE) such that

$$\mu = e^{2\Phi} dx = e^{\langle x, Sx \rangle} dx$$

is infinitesimally invariant for

$$L^{A,G} f = \frac{1}{2} \sum_{i,j=1}^{2} a_{ij} \partial_{ij} f + \sum_{i=1}^{2} \Big(\sum_{j=1}^{2} g_{ij} x_j \Big) \partial_i f, \qquad f \in C_0^\infty(\mathbb{R}^2).$$

Perturb the linear drift by a constant and a homogeneous quadratic part:

$$\mathbf{H}(x,y) = (\alpha_{0,0} + \alpha_{2,0} x^2 + \alpha_{2,1} xy + \alpha_{2,2} y^2, \beta_{0,0} + \beta_{2,0} x^2 + \beta_{2,1} xy + \beta_{2,2} y^2),$$

i.e. set

$$\mathbf{P}(x,y) \;\; = \;\; Gz + \mathbf{H}(z), \qquad z = (x,y).$$

Then μ is an infinitesimally invariant measure for $\big(L^{A,\mathbf{P}}, C_0^\infty(\mathbb{R}^2)\big)$ if and only if

$$\langle \nabla \Phi, \mathbf{H} \rangle + \frac{1}{2} \operatorname{div} \mathbf{H} = 0 \qquad \text{(finite algebraic system in the coefficients)}$$

Suppose from now on that μ is an infinitesimally invariant measure for $\big(L^{A,\mathbf{P}}, C_0^\infty(\mathbb{R}^2)\big)$. Then the decomposition

$$\tilde{\mathbf{P}}(z) := Sz + \mathbf{P}(z) - ASz = \underbrace{\nabla\Phi(x,y) + (G - AS)\begin{pmatrix} x \\ y \end{pmatrix}}_{=:\widetilde{G},\quad \text{SOHHD by construction}} + \mathbf{H}(x,y),$$

is a SOHHD if and only if

$$\operatorname{div}\mathbf{H} = 0 \quad \Longleftrightarrow \quad \alpha_{2,0} = -\frac{\beta_{2,1}}{2} \quad \text{and} \quad \beta_{2,2} = -\frac{\alpha_{2,1}}{2}.$$

Thus the same invariant measure may persist even when orthogonality fails.

Moreover:

- μ is infinitesimally invariant for $(L, C_0^\infty(\mathbb{R}^2))$, $L = L^{A,G}, L^S, L^{\widetilde{G}}, L^{\tilde{\mathbf{P}}}$.
- G is Hurwitz $\Longleftrightarrow$ μ is a finite invariant measure for the semigroup associated with $L^{A,\mathbf{P}}$.
- If $\operatorname{div}\mathbf{H} \neq 0$, then there is no SOHHD for $\widetilde{\mathbf{P}}$.
- If $\operatorname{div}\mathbf{H} = 0$, then (1) is an SOHHD. If, in addition, G is Hurwitz, then (1) is the unique OHHD for $\widetilde{\mathbf{P}}$.

Example 3.8: a concrete quadratic perturbation page 1

Let $z = (x, y) \in \mathbb{R}^2$ and consider

$$G = \begin{pmatrix} 0 & 1 \\ -1 & -3 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

Then G is Hurwitz, A is symmetric and positive definite, and

$$G = AS, \qquad S = \begin{pmatrix} -1 & -1 \\ -1 & -2 \end{pmatrix}$$

is negative definite. Hence, with

$$\Phi(z) = \frac{1}{2}\langle Sz, z\rangle, \qquad \mu = e^{2\Phi(x,y)}dxdy = e^{-x^2-2xy-2y^2}dxdy,$$

we have

$$\mathrm{trace}(G - AS) = 0, \qquad \langle Sz, (G - AS)z\rangle = 0.$$

Since $G - AS = 0$, the transformed linear field is

$$\widetilde{G}z = Sz + (G - AS)z = Sz = \nabla\Phi(z),$$

and this is the unique SOHHD for $\widetilde{G}z$.

Example 3.8: a concrete quadratic perturbation page 2

Now perturb the linear drift by the quadratic vector field

$$\mathbf{H}(x,y) = (3/2 + 2x^2 + xy - 6y^2, 1 - 2x^2 + xy + 3y^2).$$

Set

$$\mathbf{P}(z) = Gz + \mathbf{H}(z) = (3/2 + y + 2x^2 + xy - 6y^2, 1 - x - 3y - 2x^2 + xy + 3y^2).$$

A direct coefficient calculation gives

$$\langle \nabla\Phi, \mathbf{H}\rangle + \frac{1}{2}\operatorname{div}\mathbf{H} = 0.$$

Therefore the same Gaussian measure

$$\mu = e^{-x^2-2xy-2y^2}dxdy$$

is the unique finite infinitesimally invariant and invariant measure for the operator with drift $\mathbf{P}$.

However,

$$\frac{1}{2}\operatorname{div}\mathbf{H} = \frac{5}{2}x + \frac{7}{2}y \neq 0.$$

Thus

$$\widetilde{\mathbf{P}}(z) := Sz + \mathbf{P}(z) - ASz = \nabla\Phi(z) + \mathbf{H}(z)$$

is a WHHD. Since μ is the unique infinitesimally invariant measure for the corresponding transformed operator, Theorem 2.7 and $\operatorname{div}\mathbf{H} \neq 0$ show that there is no OHHD for $\widetilde{\mathbf{P}}$; hence, in particular, no SOHHD exists.

Example 3.8: Lyapunov condition without an OHHD

Set
$$V := -\Phi + 1 = 1 + \frac{1}{2}x^2 + xy + y^2.$$
For the preceding example with $A \neq id$,
$$\mathbf{B} := \mathbf{P} - A\nabla\Phi = \mathbf{H}, \qquad \deg(\mathbf{B}) = 2 = 2\deg(\Phi) - 2,$$
and $-\Phi$ is strictly positive on S^1. Hence Proposition 3.3 applies although no OHHD exists. Directly,
$$L^{A,\mathbf{P}}V = \frac{29}{8} - \left(x + 2y - \frac{5}{4}\right)^2 - \left(y + \frac{3}{4}\right)^2 \leq \frac{29}{8}V.$$
The same phenomenon occurs for $A = id$ if the drift is changed consistently to
$$\mathbf{P}^{id}(z) := Sz + \mathbf{H}(z) = \nabla\Phi(z) + \mathbf{H}(z).$$
Then
$$L^{\mathbf{P}^{id}}V = \frac{37}{16} - 2\left(x + \frac{3}{2}y - \frac{5}{8}\right)^2 - \frac{1}{2}\left(y + \frac{1}{4}\right)^2 \leq \frac{37}{16}V.$$
Thus the stochastic Lyapunov condition, conservativeness, and the invariant-measure conclusions may hold although no OHHD exists.

If the quadratic potential

$$\Phi(z) = \frac{1}{2}\langle Sz, z\rangle$$

is kept fixed, then a highest-order odd perturbation $\mathbf{P}_{2k-1}$ preserving the Gaussian measure must satisfy

$$\langle Sz, \mathbf{P}_{2k-1}(z)\rangle = 0.$$

Thus, at regular points of Φ, this highest-order term is tangent to the level sets of the fixed quadratic potential.
Odd-degree terms need not, however, arise only in the weighted divergence-free part. Let Ψ be a polynomial of degree $2k$ and set

$$\widehat{\mathbf{P}} = A\nabla\Psi + \mathbf{B}, \qquad \widehat{\mu} = e^{2\Psi}dx.$$

Then

$$\widehat{\mu} \text{ is infinitesimally invariant} \quad \Longleftrightarrow \quad \mathrm{div}_{\widehat{\mu}}(\mathbf{B}) = 0.$$

Hence odd-degree polynomial terms can come from the gradient part, from the weighted divergence-free part, or from both. In particular, $A\nabla\Psi$ has degree $2k-1$.

The exponential–polynomial class is not exhaustive

Consider

$$\mathbf{P}(x,y) = (-x^3 + y, -y), \qquad L^{\mathbf{P}} f = \frac{1}{2}\Delta f + \langle \mathbf{P}, \nabla f \rangle.$$

Proposition 3.10

The diffusion is non-explosive and admits a unique invariant probability measure ν. The measure ν is ergodic and has a strictly positive smooth density. However, there is no polynomial Ψ such that

$$e^{2\Psi} \in L^1(\mathbb{R}^2)$$

and $e^{2\Psi}\,dx\,dy$ is infinitesimally invariant for $(L^{\mathbf{P}}, C_0^\infty(\mathbb{R}^2))$.

Sketch of Proof 1) With $W(x,y) = 1 + x^2 + y^2$ one has

$$L^{\mathbf{P}} W \le \frac{7}{2} - W,$$

which supplies the required Lyapunov control and gives a unique finite invariant measure $\nu = \rho\,dx$ with $\rho \in H^{1,p}_{\mathrm{loc}}(\mathbb{R}^2) \cap C(\mathbb{R}^2)$, for some $p > 2$, $\rho > 0$.
2) Since ρ solves the stationary Fokker–Planck equation

$$\frac{1}{2}\Delta\rho - \operatorname{div}(\mathbf{P}\rho) = 0,$$

elliptic regularity implies that ρ is smooth.

3) Suppose that we have

$$\nu = e^{2\Psi}\, dx\, dy$$

where Ψ is a polynomial. Then by Theorem 2.5

$$\langle \nabla\Psi, \mathbf{P} - \nabla\Psi \rangle + \frac{1}{2}\,\mathrm{div}(\mathbf{P} - \nabla\Psi) = 0.$$

But one can show that the latter implies $\deg(\Psi) \neq 2, 4, 6, \ldots$. Thus ν cannot have exponential-polynomial density. $\square$

Similarly in the case $\deg(\mathbf{P}) = 2$ one can show that there is no exponential-polynomial density for

$$\mathbf{P}(x, y) = (-x - xy, -y + x^2).$$

using the Lyapunov function

$$W(x, y) = 1 + x^2 + y^2$$

for which

$$L^{\mathbf{P}} W = 4 - 2W.$$

Anti-symmetric matrix perturbations

Let $A = id$, let $S = S^T$ solve

$$SG + G^T S = 2S^2, \qquad \mathrm{trace}(G - S) = 0,$$

and set

$$\Phi(x) = \frac{1}{2}\langle Sx, x\rangle, \qquad \mu = e^{\langle Sx, x\rangle} dx.$$

Let $C = (c_{ij})$ satisfy $c_{ij} = -c_{ji} \in H^{1,p}_{\mathrm{loc}}(\mathbb{R}^d)$, $p > d$, and write

$$(\mathrm{div}\ C)_i = \sum_{j=1}^{d} \partial_j c_{ji}.$$

Theorem 3.14(i)

Define

$$\mathbf{G}_0(x) = Gx + C(x)^T Sx + \frac{1}{2}\,\mathrm{div}\ C(x).$$

Then

$$\mathbf{G}_0 = \nabla\Phi + \mathbf{B}, \qquad \mathbf{B} = (G - S)x + C^T Sx + \frac{1}{2}\,\mathrm{div}\ C$$

is a WHHD.

The linear part $(G - S)x$ is weighted divergence-free by the Riccati equation, and the skew-symmetric perturbation contributes another weighted divergence-free term.

Assume in addition

$$\langle \operatorname{div} C(x), Sx \rangle = 0 \qquad \text{for a.e. } x \in \mathbb{R}^d.$$

Since $\nabla \Phi(x) = Sx$, Proposition 3.13 identifies this with the equivalent conditions

$$\begin{aligned} \operatorname{div}_\mu(\operatorname{div} C) = 0 &\iff \operatorname{div}_\mu(C^T \nabla \Phi) = 0 \\ &\iff \langle \operatorname{div} C, \nabla \Phi \rangle = 0 \iff \operatorname{div}_{dx}(C^T \nabla \Phi) = 0. \end{aligned}$$

Hence Theorem 3.14(ii) gives, for arbitrary $k_0, k_1, k_2 \in \mathbb{R}$,

$$\mathbf{G}_1(x) = \nabla \Phi(x) + k_0 (G - S)x + k_1 C(x)^T Sx + k_2 \operatorname{div} C(x),$$

and this is an OHHD.

If G is Hurwitz, then S is negative definite and Theorem 3.14(iii) gives uniqueness of this OHHD; moreover $\mu = e^{\langle Sx, x \rangle} dx$ is the unique infinitesimally invariant measure and the unique invariant measure for the corresponding semigroup, up to multiplication.

The same potential in two Lyapunov roles: Remark 3.15

Assume in addition that G is Hurwitz, $\mathbf{G}_1$ is locally Lipschitz, and $\mathbf{G}_1(0) = 0$. Then S is strictly negative definite and

$$V_0(x) := -\Phi(x) = -\frac{1}{2}\langle Sx, x\rangle$$

is positive definite and coercive. Along the deterministic equation

$$\frac{dX_t}{dt} = \mathbf{G}_1(X_t)$$

the OHHD relation gives

$$\frac{d}{dt}V_0(X_t) = -\|SX_t\|^2.$$

Thus V_0 is a strict Lyapunov function for the deterministic dynamics.
For stochastic non-explosion one may use

$$V(x) = 1 + \langle -Sx, x\rangle = 1 + 2V_0(x),$$

for which

$$L^{\mathbf{G}_1}V(x) = -\operatorname{trace}(S) - 2\|Sx\|^2 \le -\operatorname{trace}(S)V(x).$$

Hence, up to multiplication and addition of a constant, the same potential-derived function serves both Lyapunov roles in this class.

Example 3.16: explicit anti-symmetric perturbations

Let K be a constant skew-symmetric matrix and let $\varphi \in C^1(\mathbb{R})$. Then

$$C(x) = \varphi(\langle Sx, x\rangle)K$$

satisfies

$$\langle \operatorname{div} C(x), Sx\rangle = 0.$$

More generally, one may take

$$C(x) = \widehat{K}(\langle Sx, x\rangle),$$

where $\widehat{K} = (\widehat{k}_{ij})$ is a skew-symmetric matrix of C^1-functions of one real variable.

Thus Theorem 3.14 gives explicit nonlinear perturbations preserving the density $e^{\langle Sx,x\rangle}dx$ and, under the orthogonality condition, producing an OHHD. If G is Hurwitz, the density is finite and gives the same Gaussian invariant measure.

Final outlook

Simplified message

- The condition $\mathrm{div}_\mu(\mathbf{B}) = 0$ is exactly the weighted condition which makes $e^{2\Phi}dx$ infinitesimally invariant.
- In the linear case this becomes an algebraic Riccati equation; in the Hurwitz case it yields the finite Gaussian invariant measure and a unique SOHHD for the transformed drift.
- For nonlinear polynomial drifts the same framework produces explicit invariant measures and Lyapunov functions, but orthogonality is not necessary for the stochastic Lyapunov condition, and exponential–polynomial invariant densities do not exhaust the ergodic polynomial diffusions.

Natural directions

- Rough coefficients, in particular possibly degenerate state-dependent diffusion matrices (this can be done if the potential is given explicitly).
- Quantitative consequences for non-reversible perturbations and convergence rates.
- Relation to metastability and non-reversible sampling algorithms.

Thank you for your attention.