Bounded-domain solvability for the planar stationary stochastic Landau-Lifshitz-Gilbert equation

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Stationary stochastic Landau–Lifshitz–Gilbert equation

- Consider the formal stationary stochastic Landau-Lifshitz-Gilbert equation on a bounded domain $\mathbb{D} \subset \mathbb{R}^2$:

$$\alpha \mathbf{u} \times (\mathbf{u} \times \Delta \mathbf{u}) - \beta \mathbf{u} \times \Delta \mathbf{u} = \gamma \mathbf{u} \times \mathbf{O}\mathscr{W}, \qquad |\mathbf{u}| = 1,$$

where $\mathscr{W}$ is a 3D white noise, $\alpha, \beta > 0$, $\alpha^2 + \beta^2 = 1$, and $\mathbf{u}$ takes values in $\mathbb{R}^3$.

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  where $\mathscr{W}$ is a 3D white noise, $\alpha, \beta > 0$, $\alpha^2 + \beta^2 = 1$, and $\mathbf{u}$ takes values in $\mathbb{R}^3$.

- We want to prove the existence of a solution $(\mathbf{u}, \mathbf{O})$, where $\mathbf{u} : \mathbb{D} \to \mathbb{R}^3$, $\mathbf{O} : \mathbb{D} \to SO(3)$, and the third column of $\mathbf{O}$ is $\mathbf{u}$.

The noise in a moving frame

- Let $\mathbf{e} : D \to \mathbb{R}^3$ be such that $\mathbf{e} \perp \mathbf{u}$ and $|\mathbf{u}| = |\mathbf{e}| = 1$. Then $(\mathbf{u}, \mathbf{e}, \mathbf{u} \times \mathbf{e})$ is a moving frame. Formally the white noise $\mathbf{W}$ is

$$\mathbf{W} = w_1(\mathbf{e} \times \mathbf{u}) + w_2\mathbf{e} + w_3\mathbf{u},$$

$$w_1 = (\mathbf{W}, \mathbf{e} \times \mathbf{u}), \qquad w_2 = (\mathbf{W}, \mathbf{e}), \qquad w_3 = (\mathbf{W}, \mathbf{u}).$$

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- In the physics literature w_1, w_2, w_3 are regarded as white noises. Mathematically this is not justified; the moving frame contains $\mathbf{u}$.
- We reverse the argument. In the equation, instead of $\mathbf{W}$, we introduce $\mathbf{O}\mathscr{W}$ with $\mathbf{O} := (\mathbf{e} \times \mathbf{u}, \mathbf{e}, \mathbf{u}) \in SO(3)$, so that

$$\mathbf{O}\mathscr{W} = \mathscr{W}_1(\mathbf{e} \times \mathbf{u}) + \mathscr{W}_2\mathbf{e} + \mathscr{W}_3\mathbf{u}, \qquad \mathbf{u} \times \mathbf{O}\mathscr{W} = \mathscr{W}_1\mathbf{e} + \mathscr{W}_2(\mathbf{u} \times \mathbf{e}).$$

Main result

Theorem

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathbb{D}$ be a bounded domain. Let $\mathscr{W}$ be a 3D white noise, $\mathscr{W}^\varepsilon = \rho_\varepsilon * \mathscr{W}$, $\varepsilon > 0$, be its mollification with a standard mollifier, and let $\kappa \in (0, \frac{1}{2})$. For any $\delta > 0$, there exist a set $\Omega_\delta \subset \Omega$, $\mathbb{P}(\Omega_\delta) > 1 - \delta$, and $\gamma_0 > 0$ such that for $\gamma \in (0, \gamma_0)$ and $\varepsilon > 0$, there exist functions

$$\mathbf{u}^\varepsilon : \Omega_\delta \to \mathrm{C}^{1-\kappa}(\mathbb{D}, \mathbb{S}^2), \qquad \mathbf{O}^\varepsilon : \Omega_\delta \to \mathrm{C}^{1-\kappa}(\mathbb{D}, SO(3)),$$

such that the third column of $\mathbf{O}^\varepsilon$ is $\mathbf{u}^\varepsilon$ and, for every $\omega \in \Omega_\delta$, the pair $(\mathbf{u}^\varepsilon, \mathbf{O}^\varepsilon)$ is a solution to

$$\alpha \mathbf{u}^\varepsilon \times (\mathbf{u}^\varepsilon \times \Delta \mathbf{u}^\varepsilon) - \beta \mathbf{u}^\varepsilon \times \Delta \mathbf{u}^\varepsilon = \gamma \mathbf{u}^\varepsilon \times \mathbf{O}^\varepsilon \mathscr{W}^\varepsilon, \quad |\mathbf{u}^\varepsilon| = 1, \qquad \text{(sLLG)}$$

on $\mathbb{D}$, where $\alpha, \beta > 0$ and $\alpha^2 + \beta^2 = 1$.

Main result

Theorem (Continuation)

Moreover, there exist functions

$$\mathbf{u} : \Omega_\delta \to \mathrm{C}^{1-\kappa}(\overline{\mathbb{D}}, \mathbb{S}^2) \quad \textit{and} \quad \mathbf{O} : \Omega_\delta \to \mathrm{C}^{1-\kappa}(\overline{\mathbb{D}}, SO(3))$$

such that, uniformly in $\omega \in \Omega_\delta$*,*

$$|\mathbf{u}^\varepsilon - \mathbf{u}|_{\mathrm{C}^{1-\kappa}(\overline{\mathbb{D}})} \lesssim \varepsilon^{\frac{\kappa}{8}} \quad \textit{and} \quad |\mathbf{O}^\varepsilon - \mathbf{O}|_{\mathrm{C}^{1-\kappa}(\overline{\mathbb{D}})} \lesssim \varepsilon^{\frac{\kappa}{8}}.$$

Cauchy-Riemann system and moving-frames

Theorem

Let $\mathscr{W}^\varepsilon := (\mathscr{W}_1{}^\varepsilon, \mathscr{W}_2{}^\varepsilon, \mathscr{W}_3{}^\varepsilon)^\top$ *be a mollification of a 3D white noise* $\mathscr{W} := (\mathscr{W}_1, \mathscr{W}_2, \mathscr{W}_3)$ *on* $\mathbb{R}^2$*, and let* $(r_1^\varepsilon, r_2^\varepsilon, r_3^\varepsilon)$ *be a complex-valued solution to the system*

$$\begin{cases} -2\partial_{\bar z} r_1^\varepsilon = -r_2^\varepsilon \overline{r_3^\varepsilon} + \gamma(\alpha \mathscr{W}_1{}^\varepsilon + \beta \mathscr{W}_2{}^\varepsilon), \\ -2\partial_{\bar z} r_2^\varepsilon = r_1^\varepsilon \overline{r_3^\varepsilon} - \gamma(\beta \mathscr{W}_1{}^\varepsilon - \alpha \mathscr{W}_2{}^\varepsilon), \\ 4\partial_z r_3^\varepsilon = r_1^\varepsilon \overline{r_2^\varepsilon} - \overline{r_1^\varepsilon} r_2^\varepsilon. \end{cases} \tag{CR}$$

in $\mathbb{D}$*. Further, let for* $k = 1, 2, 3$,

$$p_k^\varepsilon := \operatorname{Re} r_k^\varepsilon, \qquad q_k^\varepsilon := -\operatorname{Im} r_k^\varepsilon, \quad \textit{i.e.} \quad r_k^\varepsilon = p_k^\varepsilon - i q_k^\varepsilon$$

Cauchy-Riemann system and moving-frames

THEOREM (CONTINUATION)

$$\mathscr{P}^{\varepsilon} := \begin{pmatrix} 0 & p_1^{\varepsilon} & p_2^{\varepsilon} \\ -p_1^{\varepsilon} & 0 & p_3^{\varepsilon} \\ -p_2^{\varepsilon} & -p_3^{\varepsilon} & 0 \end{pmatrix} \quad \textit{and} \quad \mathscr{Q}^{\varepsilon} := \begin{pmatrix} 0 & q_1^{\varepsilon} & q_2^{\varepsilon} \\ -q_1^{\varepsilon} & 0 & q_3^{\varepsilon} \\ -q_2^{\varepsilon} & -q_3^{\varepsilon} & 0 \end{pmatrix}.$$

Assume that

$$\mathbf{v}^{\varepsilon} := (\mathbf{u}^{\varepsilon}, \mathbf{e}^{\varepsilon}, \mathbf{u}^{\varepsilon} \times \mathbf{e}^{\varepsilon})^{\top}$$

is a solution to the system

$$\begin{cases} \partial_x \mathbf{v}^{\varepsilon} = \mathscr{P}^{\varepsilon}\mathbf{v}^{\varepsilon}, & \\ \partial_y \mathbf{v}^{\varepsilon} = \mathscr{Q}^{\varepsilon}\mathbf{v}^{\varepsilon}, & (x,y) \in \mathbb{D}. \\ |\mathbf{u}^{\varepsilon}| = 1, \qquad |\mathbf{e}^{\varepsilon}| = 1, \qquad (\mathbf{u}^{\varepsilon}, \mathbf{e}^{\varepsilon}) = 0, & \end{cases}$$

Cauchy-Riemann system and moving-frames

Theorem (Continuation)

Then, the pair $(\mathbf{u}^\varepsilon, \mathbf{O}^\varepsilon)$ *with*

$$\mathbf{O}^\varepsilon := (-\mathbf{u}^\varepsilon \times \mathbf{e}^\varepsilon, \mathbf{e}^\varepsilon, \mathbf{u}^\varepsilon).$$

solves equation (sLLG).

Lemma (Compatibility condition)

Let $(r_1^\varepsilon, r_2^\varepsilon, r_3^\varepsilon)$ *be a complex-valued solution to the system* (CR) *and let* $\mathscr{P}^\varepsilon$ *and* $\mathscr{Q}^\varepsilon$ *be the matrices defined above. Then, in* $\mathbb{D}$,

$$\partial_y \mathscr{P}^\varepsilon - \partial_x \mathscr{Q}^\varepsilon + [\mathscr{P}^\varepsilon, \mathscr{Q}^\varepsilon] = 0,$$

where $[\mathscr{P}^\varepsilon, \mathscr{Q}^\varepsilon] := \mathscr{P}^\varepsilon \mathscr{Q}^\varepsilon - \mathscr{Q}^\varepsilon \mathscr{P}^\varepsilon$.

Solving system (CR)

In view of the previous theorem, in order to find a solution to (sLLG), we have to find a solution to (CR).

- Since the equation is given on $\mathbb{D}$ and no boundary condition is imposed, we may assume without loss of generality that $\mathbb{D}$ is a bounded open rectangle $J_1 \times J_2$.

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In view of the previous theorem, in order to find a solution to (sLLG), we have to find a solution to (CR).

- Since the equation is given on $\mathbb{D}$ and no boundary condition is imposed, we may assume without loss of generality that $\mathbb{D}$ is a bounded open rectangle $J_1 \times J_2$.
- Indeed, otherwise we carry out the construction on the rectangle $J_1 \times J_2$ containing $\overline{\mathbb{D}}$ and subsequently restrict the resulting solution to $\mathbb{D}$.

Representations for system (CR)

To simplify system (CR), introduce the notation

$$r^\varepsilon := \begin{pmatrix} r_1^\varepsilon \\ r_2^\varepsilon \end{pmatrix}, \qquad r^{\varepsilon\perp} := \begin{pmatrix} -r_2^\varepsilon \\ r_1^\varepsilon \end{pmatrix}, \qquad s^\varepsilon := \overline{r_3^\varepsilon}, \qquad \mathscr{W}^\varepsilon := \begin{pmatrix} \mathscr{W}_1{}^\varepsilon \\ \mathscr{W}_2{}^\varepsilon \end{pmatrix}.$$

With this notation, system (CR) becomes

$$\begin{cases} -2\partial_{\bar z} r^\varepsilon = r^{\varepsilon\perp} \cdot s^\varepsilon + \gamma \mathscr{W}^\varepsilon, \\ \quad 2\partial_{\bar z} s^\varepsilon = \dfrac{1}{2}\langle r^{\varepsilon\perp}, r^\varepsilon \rangle. \end{cases} \tag{CR$_1$}$$

where $\langle a, b\rangle = a_1 \overline{b}_1 + a_2 \overline{b}_2$ is the scalar product in $\mathbb{C}^2$.

Regularizing kernel for a domain

Lemma (Regularizing kernel for a domain)

Let $\mathbb{D} \subset \mathbb{D}_2 := \mathbb{D} + B_2(0)$ *be open bounded domains of* $\mathbb{R}^2$*. Further let* $G(z) = \frac{1}{2\pi z}$, $z \in \mathbb{C}$, *be the Green function for the Cauchy-Riemann operator* $\partial_{\bar{z}}$.

Then there exist $\rho > 0$ *and a smooth cutoff function* χ_ρ *for the open ball* $B_\rho(0)$ *such that*

$$\chi_\rho(x - y) = 1 \qquad \text{for } x \in \mathbb{D},\ y \in \mathbb{D}_2$$

and

$$K(z) := G(z)\chi_\rho(z)$$

is a 1*-regularizing kernel. Moreover,*

$$G(x - y) = K(x - y) \qquad \forall\, x \in \mathbb{D},\ y \in \mathbb{D}_2.$$

Representations for system (CR)

Introduce the new functions

$$R^\varepsilon = r^\varepsilon - \gamma\xi^\varepsilon \qquad \text{and} \qquad S^\varepsilon = s^\varepsilon, \qquad \text{where} \quad \xi^\varepsilon = K * (\chi_{\mathbb{D},\mathbb{D}_2}\mathscr{W}^\varepsilon)$$

and $\chi_{\mathbb{D},\mathbb{D}_2} \in \mathrm{C}_c^\infty(\mathbb{R}^2)$, $\chi_{\mathbb{D},\mathbb{D}_2} \equiv 1$ on $\mathbb{D}$, $\quad \operatorname{supp}\chi_{\mathbb{D},\mathbb{D}_2} \subset \mathbb{D}_2 := \mathbb{D} + B_2(0)$.

In particular, $-2\partial_{\bar z}\xi^\varepsilon = \mathscr{W}^\varepsilon$ on $\mathbb{D}$.

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and $\chi_{\mathbb{D},\mathbb{D}_2} \in \mathrm{C}_c^\infty(\mathbb{R}^2)$, $\chi_{\mathbb{D},\mathbb{D}_2} \equiv 1$ on $\mathbb{D}$, $\operatorname{supp}\chi_{\mathbb{D},\mathbb{D}_2} \subset \mathbb{D}_2 := \mathbb{D} + B_2(0)$.

In particular, $-2\partial_{\bar z}\xi^\varepsilon = \mathscr{W}^\varepsilon$ on $\mathbb{D}$. Then

$$\begin{cases} -2\partial_{\bar z} R^\varepsilon = \left(R^{\varepsilon\perp} + \gamma\xi^{\varepsilon\perp}\right)\cdot S^\varepsilon, \\ 2\partial_{\bar z} S^\varepsilon = \frac{1}{2}\Big[\langle R^{\varepsilon\perp}, R^\varepsilon\rangle + \gamma\big(\langle\xi^{\varepsilon\perp}, R^\varepsilon\rangle - \langle R^\varepsilon, \xi^{\varepsilon\perp}\rangle\big)\Big] + \gamma^2\vartheta^\varepsilon, \end{cases} \tag{CR$_2$}$$

where

$$R^{\varepsilon\perp} := \begin{pmatrix} -R_2^\varepsilon \\ R_1^\varepsilon \end{pmatrix}, \qquad \xi^{\varepsilon\perp} := \begin{pmatrix} -\xi_2^\varepsilon \\ \xi_1^\varepsilon \end{pmatrix}, \qquad \vartheta^\varepsilon = \frac{1}{2}\langle\xi^{\varepsilon\perp}, \xi^\varepsilon\rangle.$$

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- By the solution to system (CR_2) on $\mathbb{D}$ for $\varepsilon > 0$, we understand a $\mathrm{C}^\infty(\mathbb{D})$-solution.

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- By the solution to system (CR_2) on $\mathbb{D}$ for $\varepsilon > 0$, we understand a $\mathrm{C}^\infty(\mathbb{D})$-solution.
- As it will be clear from the next slides, any solution to (CR_2) on $\mathbb{D}$ for $\varepsilon > 0$ is a C^∞-solution.

The fixed-point argument

For the fixed-point argument we use the space $\mathrm{C}^{1-\kappa}(\mathbb{D})$, $\kappa \in (0,1)$.

For $R_i^j \in \mathrm{C}^\nu(\mathbb{D})$, $i,j = 1,2$, and $S_i \in \mathrm{C}^\nu(\mathbb{D})$, $i = 1,2$, where $\nu \in (0,1)$, write

$$R := \begin{pmatrix} R_1^1 + iR_1^2 \\ R_2^1 + iR_2^2 \end{pmatrix}, \qquad S := S_1 + iS_2.$$

Define

$$\|(R,S)\|_{\mathrm{C}^\nu(\mathbb{D})} := \Big(\sum_{i,j=1}^{2} \|R_i^j\|^2_{\mathrm{C}^\nu(\mathbb{D})} \Big)^{\frac{1}{2}} + \Big(\sum_{i=1}^{2} \|S_i\|^2_{\mathrm{C}^\nu(\mathbb{D})} \Big)^{\frac{1}{2}}.$$

We identify $(R,0)$ with R and $(0,S)$ with S, and set

$$\|R\|^2_{\mathrm{C}^\nu(\mathbb{D})} := \sum_{i,j=1}^{2} |R_i^j|^2_{\mathrm{C}^\nu(\mathbb{D})}, \qquad \|S\|^2_{\mathrm{C}^\nu(\mathbb{D})} := \|S_1\|^2_{\mathrm{C}^\nu(\mathbb{D})} + \|S_2\|^2_{\mathrm{C}^\nu(\mathbb{D})}.$$

The fixed-point argument

Introduce the Banach spaces

$$E_{1-\kappa} = \Big\{(R,S): \ R:\mathbb{D}\to\mathbb{C}^2, \quad S:\mathbb{D}\to\mathbb{C}, \quad |(R,S)|_{\mathrm{C}^{1-\kappa}(\mathbb{D})} < \infty\Big\}.$$

Lemma (Hölder extension)

Let $\mathbb{D}\subset\mathbb{R}^d$ be a bounded domain and $\varkappa\in(0,1)$.

Then there exists a linear continuous extension operator

$$\mathscr{E}_\varkappa : \mathrm{C}^\varkappa(\overline{\mathbb{D}}) \to \mathrm{C}^\varkappa(\mathbb{R}^d), \qquad \mathscr{E}_\varkappa f|_{\overline{\mathbb{D}}} = f \quad \forall f\in \mathrm{C}^\varkappa(\overline{\mathbb{D}}).$$

Moreover,

$$\operatorname{supp}\mathscr{E}_\varkappa f \subset \overline{\mathbb{D}+B_2(0)} \qquad \forall f\in \mathrm{C}^\varkappa(\overline{\mathbb{D}}),$$

and the bound for $\|\mathscr{E}_\varkappa\|_{\mathcal{L}(\mathrm{C}^\varkappa(\overline{\mathbb{D}}),\mathrm{C}^\varkappa(\mathbb{R}^d))}$ does not depend on $\mathbb{D}$.

The fixed-point argument

Let $\chi_{\mathbb{D},\mathbb{D}_2} \in \mathrm{C}_c^\infty(\mathbb{R}^2)$, $\chi_{\mathbb{D},\mathbb{D}_2} \equiv 1$ on $\mathbb{D}$, $\operatorname{supp} \chi_{\mathbb{D},\mathbb{D}_2} \subset \mathbb{D}_2$.

Define

$$\zeta^\varepsilon = K * (\chi_{\mathbb{D},\mathbb{D}_2} \vartheta^\varepsilon).$$

Let

$$\mathscr{E}_{1-\kappa} : \mathrm{C}^{1-\kappa}(\overline{\mathbb{D}}) \to \mathrm{C}^{1-\kappa}(\mathbb{R}^2)$$

be the extension operator defined in the Hölder extension lemma.

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Let $\quad \chi_{\mathbb{D},\mathbb{D}_2} \in \mathrm{C}_c^\infty(\mathbb{R}^2), \quad \chi_{\mathbb{D},\mathbb{D}_2} \equiv 1$ on $\mathbb{D}, \quad \operatorname{supp}\chi_{\mathbb{D},\mathbb{D}_2} \subset \mathbb{D}_2.$

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be the extension operator defined in the Hölder extension lemma.

For each $\varepsilon > 0$, define the map

$$\Gamma_\varepsilon : E_{1-\kappa} \to E_{1-\kappa}, \qquad \Gamma_\varepsilon(R,S) = \big(\Gamma_\varepsilon^{(1)}(R,S), \Gamma_\varepsilon^{(2)}(R,S)\big).$$

$$\Gamma_\varepsilon^{(1)}(R,S) := K * \Big[\big((\mathscr{E}_{1-\kappa}R)^\perp + \gamma\xi^{\varepsilon\perp}\big)\cdot \mathscr{E}_{1-\kappa}S\Big]\Big|_{\mathbb{D}},$$

$$\begin{aligned}\Gamma_\varepsilon^{(2)}(R,S) := & -\frac{1}{2}K * \Big[\langle(\mathscr{E}_{1-\kappa}R)^\perp, \mathscr{E}_{1-\kappa}R\rangle \\ & + \gamma\big(\langle\xi^{\varepsilon\perp}, \mathscr{E}_{1-\kappa}R\rangle - \langle\mathscr{E}_{1-\kappa}R, \xi^{\varepsilon\perp}\rangle\big)\Big]\Big|_{\mathbb{D}} \\ & - \gamma^2\zeta^\varepsilon\big|_{\mathbb{D}},\end{aligned}$$

The fixed-point argument

Lemma

The map Γ_ε is well-defined for $\varepsilon > 0$.

Lemma

Let $\varepsilon > 0$ and let $(R^\varepsilon, S^\varepsilon) \in E_{1-\kappa}$ be a fixed point of Γ_ε. Then, on $\mathbb{D}$, $(R^\varepsilon, S^\varepsilon)$ is a solution to

$$\begin{cases} -2\partial_{\bar{z}} R^\varepsilon = \left(R^{\varepsilon\perp} + \gamma\xi^{\varepsilon\perp}\right) \cdot S^\varepsilon, \\ \quad 2\partial_{\bar{z}} S^\varepsilon = \frac{1}{2}\left[\langle R^{\varepsilon\perp}, R^\varepsilon\rangle + \gamma\left(\langle \xi^{\varepsilon\perp}, R^\varepsilon\rangle - \langle R^\varepsilon, \xi^{\varepsilon\perp}\rangle\right)\right] + \gamma^2\vartheta^\varepsilon, \end{cases} \qquad (\mathrm{CR}_2)$$

Elements of proof

- For any $f \in \mathrm{C}^{\nu}(\mathbb{R}^d)$, $\nu \in \mathbb{R}$,

$$K * (\chi_{\mathbb{D},\mathbb{D}_2} f) = G * (\chi_{\mathbb{D},\mathbb{D}_2} f) \quad \text{on} \quad \mathbb{D}.$$

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- $K(z) = G(z)\chi_\rho(z)$ and
$K(z_1 - z_2)\chi_{\mathbb{D},\mathbb{D}_2}(z_2) = G(z_1 - z_2)\chi_\rho(z_1 - z_2)\chi_{\mathbb{D},\mathbb{D}_2}(z_2) =$
$G(z_1 - z_2)\chi_{\mathbb{D},\mathbb{D}_2}(z_2), \quad z_1 \in \mathbb{D}, \quad z_2 \in \mathbb{C}$

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  $G(z_1 - z_2)\chi_{\mathbb{D},\mathbb{D}_2}(z_2), \quad z_1 \in \mathbb{D}, \quad z_2 \in \mathbb{C}$
- We also mention that $\mathscr{E}_{1-\kappa} = \chi_{\mathbb{D},\mathbb{D}_2}\mathcal{S}_{1-\kappa}$ by construction, where $\mathcal{S}_{1-\kappa}$ is Stein's extension, i.e., a bounded linear operator

$$\mathcal{S}_{\varkappa} : \mathrm{C}^{\varkappa}(\overline{\mathbb{D}}) \to \mathrm{C}^{\varkappa}(\mathbb{R}^d), \qquad \mathcal{S}_{\varkappa} f|_{\overline{\mathbb{D}}} = f.$$

The fixed-point argument

The fixed-point argument

In $E_{1-\kappa}$, we choose a complete metric subspace

$$M_{1-\kappa,\sigma} := \Big\{(R,S) \in E_{1-\kappa} : \ \|R\|_{\mathrm{C}^{1-\kappa}(\overline{\mathbb{D}})} \leqslant \sigma, \quad \|S\|_{\mathrm{C}^{1-\kappa}(\overline{\mathbb{D}})} \leqslant \sigma\Big\}.$$

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PROPOSITION

For each $\delta \in (0,1)$, there exists a set $\Omega_\delta \subset \Omega$, $\mathbb{P}(\Omega_\delta) > 1-\delta$, and a number $\gamma_0 > 0$ such that one can choose $\sigma \geqslant \gamma_0$ in such a way that for all $\omega \in \Omega_\delta$ and $\gamma \in (0,\gamma_0)$, Γ_ε is a map

$$\Gamma_\varepsilon : M_{1-\kappa,\sigma} \to M_{1-\kappa,\sigma}.$$

Furthermore, Γ_ε has a unique fixed point in $M_{1-\kappa,\sigma}$ and the fixed-point constant is uniform in ε.

Elements of proof

Lemma

Let $\kappa \in (0,1)$ *and let* $\mathcal{K} \subset \mathbb{R}^2$ *be compact. Then, there exists a compact* $\widetilde{\mathcal{K}} \supset \mathcal{K}$ *such that, for all* $\varepsilon \in (0,1]$,

$$\|\xi^\varepsilon\|_{-\kappa,\mathcal{K}} \lesssim \|\mathscr{W}\|_{-1-\kappa,\widetilde{\mathcal{K}}}, \qquad \mathscr{W} = (\mathscr{W}_1, \mathscr{W}_2)^\top$$

Since $\mathscr{E}_{1-\kappa}$ is a bounded linear operator, for $R_1, R_2, R, S_1, S_2, S \in \mathrm{C}^{1-\kappa}(\overline{\mathbb{D}})$,

$$\|\mathscr{E}_{1-\kappa}(R_1 - R_2)\|_{\mathrm{C}^{1-\kappa}(\mathbb{R}^2)} \lesssim \|R_1 - R_2\|_{\mathrm{C}^{1-\kappa}(\overline{\mathbb{D}})},$$

$$\|\mathscr{E}_{1-\kappa}(S_1 - S_2)\|_{\mathrm{C}^{1-\kappa}(\mathbb{R}^2)} \lesssim \|S_1 - S_2\|_{\mathrm{C}^{1-\kappa}(\overline{\mathbb{D}})}.$$

$$\|\mathcal{E}_{1-\kappa} R\|_{\mathrm{C}^{1-\kappa}(\mathbb{R}^2)} \lesssim \|R\|_{\mathrm{C}^{1-\kappa}(\overline{\mathbb{D}})} \lesssim \sigma,$$

$$\|\mathcal{E}_{1-\kappa} S\|_{\mathrm{C}^{1-\kappa}(\mathbb{R}^2)} \lesssim \|S\|_{\mathrm{C}^{1-\kappa}(\overline{\mathbb{D}})} \lesssim \sigma.$$

Hölder-Zygmund spaces

For $\lambda \in (0,1]$ and $x \in \mathbb{R}^d$, define the rescaled centered test function

$$\varphi_x^\lambda(y) = \frac{1}{\lambda^d}\varphi\left(\frac{y-x}{\lambda}\right), \qquad y \in \mathbb{R}^d.$$

For $r \in \mathbb{N}$ and $\alpha \in \mathbb{R}$, define

$$\mathscr{B}^r := \left\{\varphi \in \mathcal{D}(B_1(0)) : \|\partial^k \varphi\|_\infty \leqslant 1 \quad \text{for all } 0 \leqslant |k| \leqslant r\right\},$$

$$\mathscr{B}_\nu := \left\{\varphi \in \mathcal{D}(B_1(0)) : \int_{\mathbb{R}^d} \varphi(z) z^k \, dz = 0 \quad \text{for all } 0 \leqslant |k| \leqslant \nu\right\}.$$

For $\nu \geqslant 0$, put $\mathscr{B}^r_\nu := \mathscr{B}^r \cap \mathscr{B}_\nu$.

Here $k = (k_1, \ldots, k_d) \in \mathbb{N}_0^d$ is a multiindex, $z^k = z_1^{k_1} \cdots z_d^{k_d}$, and $|k| = k_1 + \cdots + k_d$.

Hölder-Zygmund spaces

Let $\nu \in \mathbb{R}$ and $f \in \mathcal{D}'(\mathbb{R}^d)$.

For $\nu \geqslant 0$ and a compact $\mathcal{K} \subset \mathbb{R}^d$, set

$$|f|_{\nu,\mathcal{K}} := \sup_{\substack{x\in\mathcal{K} \\ \psi\in\mathscr{B}^0}} |f(\psi_x)| + \sup_{\substack{x\in\mathcal{K},\ \lambda\in(0,1] \\ \varphi\in\mathscr{B}^0_\nu}} \frac{|f(\varphi_x^\lambda)|}{\lambda^\nu}.$$

For $\nu < 0$, let $r = [-\nu + 1]$, the smallest positive integer $r > -\nu$, and set

$$|f|_{\nu,\mathcal{K}} := \sup_{\substack{x\in\mathcal{K},\ \lambda\in(0,1] \\ \varphi\in\mathscr{B}^r}} \frac{|f(\varphi_x^\lambda)|}{\lambda^\nu}.$$

Hölder-Zygmund spaces

Definition

The Hölder-Zygmund space $\mathscr{Z}^\nu$ *is the space of all distributions* $f \in \mathcal{D}'(\mathbb{R}^d)$ *such that*

$$|f|_{\nu,\mathcal{K}} < \infty$$

for every compact $\mathcal{K} \subset \mathbb{R}^d$.

If f is complex- or vector-valued, we use

$$\|f\|_{\nu,\mathcal{K}}$$

instead of $|f|_{\nu,\mathcal{K}}$.

Little Hölder-Zygmund spaces

Definition

For $\nu \in \mathbb{R}$, define the little Hölder-Zygmund space

$$\mathscr{z}^{\nu}(\mathbb{R}^d) := \overline{C^{\infty}(\mathbb{R}^d) \cap \mathscr{Z}^{\nu}(\mathbb{R}^d)}^{\mathscr{Z}^{\nu}(\mathbb{R}^d)}.$$

We denote by $|\cdot|_{\nu}$ the restriction to $\mathscr{z}^{\nu}(\mathbb{R}^d)$ of the Hölder-Zygmund norm defined above, for both $\nu \geqslant 0$ and $\nu < 0$.

Inequalities used in the fixed-point argument

Lemma

For any δ-regularising kernel K, the map $\zeta \mapsto K * \zeta$ is continuous from $\mathscr{Z}^\nu$ to $\mathscr{Z}^{\delta+\nu}$ for every $\nu \in \mathbb{R}$. Moreover,

$$|K * \zeta|_{\mathcal{K},\delta+\nu} \lesssim |\zeta|_{\mathcal{K}',\nu},$$

where $\mathcal{K}' \supset \mathcal{K}$ is a compact which depends only on the compact $\mathcal{K}$ and the kernel K.

Lemma

For $\nu \notin \mathbb{N}$, $\mathscr{Z}^\nu(\mathbb{R}^d) = \mathrm{C}^\nu(\mathbb{R}^d)$. In particular, for each compact $\mathcal{K}$, there exists a compact $\mathcal{K}'$ such that

$$\|f\|_{\mathrm{C}^\nu(\mathcal{K})} \lesssim |f|_{\nu,\mathcal{K}'} \qquad \text{and} \qquad |f|_{\nu,\mathcal{K}} \lesssim \|f\|_{\mathrm{C}^\nu(\mathcal{K}_1)},$$

where $\mathcal{K}_1 := \overline{\mathcal{K} + B_1(0)}$.

Inequalities used in the fixed-point argument

Lemma (Young product)

Fix $\varkappa > 0$, $\varkappa \notin \mathbb{N}$, *and* $\nu \leqslant 0$ *such that* $\varkappa + \nu > 0$. *Then there exists a unique bilinear continuous map*

$$\mathscr{Z}^{\varkappa}(\mathbb{R}^d) \times \mathscr{Z}^{\nu}(\mathbb{R}^d) \ni (f, g) \mapsto f \cdot g \in \mathscr{Z}^{\nu}(\mathbb{R}^d),$$

which extends the usual product

$\mathscr{Z}^{\varkappa}(\mathbb{R}^d) \times \mathrm{C}^{\infty}(\mathbb{R}^d) \ni (f, g) \mapsto f \cdot g \in \mathrm{C}^{\varkappa}(\mathbb{R}^d)$.

Moreover, for all $f \in \mathscr{Z}^{\varkappa}(\mathbb{R}^d)$ *and* $g \in \mathscr{Z}^{\nu}(\mathbb{R}^d)$,

$$|f \cdot g|_{\nu,\mathcal{K}} \lesssim |f|_{\varkappa,\mathcal{K}_4} |g|_{\nu,\mathcal{K}_4},$$

where $\mathcal{K} \subset \mathbb{R}^d$ *is compact and* $\mathcal{K}_4 := \overline{\mathcal{K} + B_4(0)}$.

Inequalities used in the fixed-point argument

Lemma

Fix $0 < \varkappa < 1$, $\nu \leqslant 0$, such that $\varkappa + \nu > 0$. Let $\mathbb{D} \subset \mathbb{R}^d$ be a bounded open domain and let $\mathscr{E}_\varkappa f$ be the $\varkappa$-Hölder extension of f to $\mathbb{R}^d$. Then

$$|\mathscr{E}_\varkappa f \cdot g|_{\nu,\overline{\mathbb{D}}} \lesssim |g|_{\nu,\overline{\mathbb{D}_4}} \, \|f\|_{\mathrm{C}^\varkappa(\overline{\mathbb{D}})}, \qquad \forall\, g \in \mathscr{Z}^\nu(\mathbb{R}^d), \ \forall\, f \in \mathrm{C}^\varkappa(\overline{\mathbb{D}}),$$

where $\mathbb{D}_4 := \mathbb{D} + B_4(0)$.

In the fixed-point argument, this estimate is applied to products of the form

$$(\mathscr{E}_{1-\kappa} R)^\perp \cdot \mathscr{E}_{1-\kappa} S \qquad \text{and} \qquad \big\langle (\mathscr{E}_{1-\kappa} R)^\perp, \mathscr{E}_{1-\kappa} R \big\rangle.$$

The term $\vartheta^\varepsilon = \frac{1}{2}\langle \xi^{\varepsilon\perp}, \xi^\varepsilon \rangle$.

Lemma

It holds that

$$\vartheta^\varepsilon = \partial_z[\eta_1^\varepsilon \overline{\xi_2^\varepsilon}] - \partial_{\bar z}[\eta_1^\varepsilon \xi_2^\varepsilon] \quad \text{on } \mathbb{D},$$

$$\text{where} \quad \eta_1^\varepsilon := \Phi * \left(\chi_{\mathbb{D},\mathbb{D}_2} \mathscr{W}_1{}^\varepsilon\right), \qquad \eta_2^\varepsilon := \Phi * \left(\chi_{\mathbb{D},\mathbb{D}_2} \mathscr{W}_2{}^\varepsilon\right).$$

Moreover, for every $x \in \mathbb{D}$,

$$\vartheta^\varepsilon = \partial_z[(\eta_1^\varepsilon - \eta_1^\varepsilon(x))\overline{\xi_2^\varepsilon}] - \partial_{\bar z}[(\eta_1^\varepsilon - \eta_1^\varepsilon(x))\xi_2^\varepsilon] \qquad \text{on } \mathbb{D}.$$

Furthermore, for every $\kappa \in (0, \frac{1}{2})$, there exists a limit

$$\vartheta := \lim_{\varepsilon\to 0} \vartheta^\varepsilon = \partial_z[\eta_1 \overline{\xi_2}] - \partial_{\bar z}[\eta_1 \xi_2] = \partial_z[(\eta_1 - \eta_1(x))\overline{\xi_2}] - \partial_{\bar z}[(\eta_1 - \eta_1(x))\xi_2],$$

where the products in the square brackets are understood as Young products.

Elements of Proof

We will prove that for every constant $A \in \mathbb{R}$,

$$\vartheta^{\varepsilon} = \partial_z\big[(\eta_1^{\varepsilon} - A)\overline{\xi_2^{\varepsilon}}\big] - \partial_{\bar z}\big[(\eta_1^{\varepsilon} - A)\xi_2^{\varepsilon}\big] \qquad \text{on } \mathbb{D}.$$

In particular, taking $A = 0$ or $A = \eta_1^{\varepsilon}(x)$, $x \in \mathbb{D}$, respectively, we obtain the statement.

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Indeed,

$$\begin{aligned} &\partial_z\left[(\eta_1^\varepsilon - A)\overline{\xi_2^\varepsilon}\right] - \partial_{\bar z}\left[(\eta_1^\varepsilon - A)\xi_2^\varepsilon\right] \\ &\quad = \underbrace{\frac{1}{2}\xi_1^\varepsilon\overline{\xi_2^\varepsilon} - \frac{1}{2}\overline{\xi_1^\varepsilon}\xi_2^\varepsilon}_{\vartheta^\varepsilon} + (\eta_1^\varepsilon - A)\underbrace{\left(\partial_z\overline{\xi_2^\varepsilon} - \partial_{\bar z}\xi_2^\varepsilon\right)}_{0} = \vartheta^\varepsilon. \end{aligned}$$

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The last term vanishes since

$$\partial_z\overline{\xi_2^\varepsilon} = 2\partial_z\partial_{\bar z}\eta_2^\varepsilon = 2\partial_{\bar z}\partial_z\eta_2^\varepsilon = \partial_{\bar z}\xi_2^\varepsilon,$$

The term $\vartheta^\varepsilon = \frac{1}{2}\langle \xi^{\varepsilon\perp}, \xi^\varepsilon \rangle$.

Computation of regularities

$$\mathrm{Reg}(\mathscr{W}_j) = -1-\kappa, \qquad \mathrm{Reg}(\xi_j) = -\kappa, \qquad \mathrm{Reg}(\eta_j) = 1-\kappa, \quad j = 1,2.$$

$$\mathrm{Reg}\,\big((\eta_1 - \eta_1(x))\xi_2\big) = -\kappa, \qquad (1-\kappa) + (-\kappa) = 1 - 2\kappa > 0.$$

$$\mathrm{Reg}\,\big[K * \partial_z\big((\eta_1 - \eta_1(x))\xi_2\big)\big] = -\kappa.$$

.

The term $\vartheta^\varepsilon = \frac{1}{2}\langle \xi^{\varepsilon\perp}, \xi^\varepsilon\rangle$.

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This is not good enought for fixed point argument in $\mathrm{C}^{1-\kappa}$.

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$$\mathrm{Reg}\left[K * \partial_z\left((\eta_1 - \eta_1(x))\xi_2\right)\right] = -\kappa.$$

This is not good enought for fixed point argument in $\mathrm{C}^{1-\kappa}$.

A key analytic point: a careful regularity analysis of ϑ^ε and ζ^ε is needed to close the fixed-point argument.

The limit ζ and its regularity

Recall that

$$\zeta^{\varepsilon} := K * (\chi_{\mathbb{D},\mathbb{D}_2} \vartheta^{\varepsilon}), \qquad \zeta := \lim_{\varepsilon \to 0} \zeta^{\varepsilon},$$

where, for every $\kappa \in (0, \frac{1}{2})$, the limit exists in $\mathscr{L}^{-\kappa}(\mathbb{R}^2)$.

Proposition

Let $\kappa \in (0, \frac{1}{2})$ and let $\mathcal{K} \subset \mathbb{R}^2$ be compact. Then there exists a compact $\mathcal{K}' \supset \mathcal{K}$ such that, for all $\varepsilon \in [0, 1]$,

$$|\zeta^{\varepsilon}|_{1-\kappa,\mathcal{K}} \lesssim |\mathscr{W}_1|_{-1-\frac{\kappa}{4},\mathcal{K}'} |\mathscr{W}_2|_{-1-\frac{\kappa}{4},\mathcal{K}'}.$$

Here $\zeta^0 := \zeta$. In particular,

$$\zeta \in \mathrm{C}^{1-\kappa}(\mathbb{R}^2).$$

Rate of convergence

Lemma

Let $\kappa \in (0, \frac{1}{2})$ *and let* $\mathcal{K} \subset \mathbb{R}^2$ *be compact. Then, there exists a compact* $\mathcal{K}' \supset \mathcal{K}$ *such that, for all* $\varepsilon \in (0, 1)$,

$$|\xi^\varepsilon - \xi|_{-\kappa,\mathcal{K}} \lesssim \varepsilon^{\frac{\kappa}{2}} \|\mathscr{W}\|_{-1-\frac{\kappa}{2},\mathcal{K}'}, \qquad \mathscr{W} = (\mathscr{W}_1, \mathscr{W}_2).$$

Moreover,

$$\left|\vartheta^\varepsilon - \vartheta\right|_{-\kappa,\mathcal{K}} \lesssim \varepsilon^{\frac{\kappa}{8}} \Big(|\mathscr{W}_1|_{-1-\frac{\kappa}{8},\mathcal{K}'} |\mathscr{W}_2|_{-1-\frac{\kappa}{4},\mathcal{K}'} + |\mathscr{W}_2|_{-1-\frac{\kappa}{8},\mathcal{K}'} |\mathscr{W}_1|_{-1-\frac{\kappa}{4},\mathcal{K}'} \Big),$$

$$|\zeta^\varepsilon - \zeta|_{1-\kappa,\mathcal{K}} \lesssim \varepsilon^{\frac{\kappa}{8}} \Big(|\mathscr{W}_1|_{-1-\frac{\kappa}{8},\mathcal{K}'} |\mathscr{W}_2|_{-1-\frac{\kappa}{4},\mathcal{K}'} + |\mathscr{W}_2|_{-1-\frac{\kappa}{8},\mathcal{K}'} |\mathscr{W}_1|_{-1-\frac{\kappa}{4},\mathcal{K}'} \Big).$$

Rate of convergence

Lemma

Let $(R^\varepsilon, S^\varepsilon)$ and (R, S) be the fixed points of Γ_ε and Γ, respectively, in $M_{1-\kappa,\sigma}$. Then, $|(R,S)-(R^\varepsilon,S^\varepsilon)|_{\mathrm{C}^{1-\kappa}(\overline{\mathbb{D}})} \lesssim \varepsilon^{\frac{\kappa}{8}}$ on Ω_δ.

Proposition

Let the objects $\delta \in (0,1)$, $\kappa \in (0, \frac{1}{2})$, $\Omega_\delta \subset \Omega$, $\sigma > 0$, $(R^\varepsilon, S^\varepsilon)$, and (R, S) be as in the previous lemma. Then, for every $\varepsilon > 0$ and $\omega \in \Omega_\delta$, the pair $(r^\varepsilon, s^\varepsilon)$ solves (RC_1) on $\mathbb{D}$. Moreover, there exists a limit

$$(r,s) := \lim_{\varepsilon\to 0}(r^\varepsilon, s^\varepsilon)$$

in $\mathrm{C}^{-\kappa}(\overline{\mathbb{D}}) \times \mathrm{C}^{1-\kappa}(\overline{\mathbb{D}})$ and a constant $C_\delta > 0$, depending only on δ, such that, for all $\omega \in \Omega_\delta$,

$$\|r^\varepsilon - r\|_{\mathrm{C}^{-\kappa}(\overline{\mathbb{D}})} + \|s^\varepsilon - s\|_{\mathrm{C}^{1-\kappa}(\overline{\mathbb{D}})} \leqslant C_\delta \varepsilon^{\frac{\kappa}{8}}.$$

Back to the original problem

By the construction above, the matrices

$$\mathscr{P}^\varepsilon := \begin{pmatrix} 0 & p_1^\varepsilon & p_2^\varepsilon \\ -p_1^\varepsilon & 0 & p_3^\varepsilon \\ -p_2^\varepsilon & -p_3^\varepsilon & 0 \end{pmatrix} \quad \text{and} \quad \mathscr{Q}^\varepsilon := \begin{pmatrix} 0 & q_1^\varepsilon & q_2^\varepsilon \\ -q_1^\varepsilon & 0 & q_3^\varepsilon \\ -q_2^\varepsilon & -q_3^\varepsilon & 0 \end{pmatrix}.$$

are defined on an open coordinate rectangle $J_1 \times J_2$ containing $\mathbb{D}$.

Lemma

Let $J_1 \times J_2$ be an open coordinate rectangle containing $\mathbb{D}$ and $(r_1^\varepsilon, r_2^\varepsilon, r_3^\varepsilon)$ be a solution to system (CR) constructed for this rectangle. Then, the system

$$\partial_x \mathbf{v}^\varepsilon = \mathscr{P}^\varepsilon \mathbf{v}^\varepsilon, \quad \partial_y \mathbf{v}^\varepsilon = \mathscr{Q}^\varepsilon \mathbf{v}^\varepsilon, \quad \mathbf{v}^\varepsilon(x_0, y_0) = I_{3\times 3}, \quad (x_0, y_0) \in J_1 \times J_2,$$

has a unique solution $\mathbf{v}^\varepsilon := (\mathbf{u}^\varepsilon, \mathbf{e}^\varepsilon, \mathbf{h}^\varepsilon)^\top$.

Proof

First, solve

$$\partial_x \mathbf{v}^\varepsilon(x, y_0) = \mathscr{P}^\varepsilon(x, y_0)\mathbf{v}^\varepsilon(x, y_0), \qquad \mathbf{v}^\varepsilon(x_0, y_0) = I_{3\times 3},$$

with respect to x. Equivalently,

$$\mathbf{v}^\varepsilon(x, y_0) = I_{3\times 3} + \int_{x_0}^{x} \mathscr{P}^\varepsilon(s, y_0)\mathbf{v}^\varepsilon(s, y_0)\, ds.$$

Since $\mathscr{P}^\varepsilon$ is smooth for $\varepsilon > 0$, Picard iterations give existence and uniqueness.

Proof

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Since $\mathscr{P}^\varepsilon$ is smooth for $\varepsilon > 0$, Picard iterations give existence and uniqueness.

Likewise, for every $x \in J_1$, solve

$$\partial_y \mathbf{v}^\varepsilon(x, y) = \mathscr{Q}^\varepsilon(x, y)\mathbf{v}^\varepsilon(x, y), \qquad \mathbf{v}^\varepsilon(x, y_0) \text{ given above.}$$

Equivalently,

$$\mathbf{v}^\varepsilon(x, y) = \mathbf{v}^\varepsilon(x, y_0) + \int_{y_0}^{y} \mathscr{Q}^\varepsilon(x, s)\mathbf{v}^\varepsilon(x, s)\, ds.$$

Proof

Set $\mathbf{F}^\varepsilon(x,y) := \partial_x \mathbf{v}^\varepsilon(x,y) - \mathscr{P}^\varepsilon \mathbf{v}^\varepsilon(x,y)$. We know $\mathbf{F}^\varepsilon(x,y_0) = 0$.

Then

$$\begin{aligned}\partial_y \mathbf{F}^\varepsilon &= \partial_x(\mathscr{Q}^\varepsilon \mathbf{v}^\varepsilon) - \partial_y(\mathscr{P}^\varepsilon \mathbf{v}^\varepsilon) \\ &= \mathscr{Q}^\varepsilon \mathbf{F}^\varepsilon + \big(\partial_x \mathscr{Q}^\varepsilon - \partial_y \mathscr{P}^\varepsilon + \mathscr{Q}^\varepsilon \mathscr{P}^\varepsilon - \mathscr{P}^\varepsilon \mathscr{Q}^\varepsilon\big)\mathbf{v}^\varepsilon.\end{aligned}$$

By the compatibility condition,

$$\partial_y \mathscr{P}^\varepsilon - \partial_x \mathscr{Q}^\varepsilon + [\mathscr{P}^\varepsilon, \mathscr{Q}^\varepsilon] = 0.$$

Hence

$$\partial_y \mathbf{F}^\varepsilon = \mathscr{Q}^\varepsilon \mathbf{F}^\varepsilon, \qquad \mathbf{F}^\varepsilon(x,y_0) = 0.$$

By uniqueness, $\mathbf{F}^\varepsilon \equiv 0$. Therefore,

$$\partial_x \mathbf{v}^\varepsilon = \mathscr{P}^\varepsilon \mathbf{v}^\varepsilon, \qquad \partial_y \mathbf{v}^\varepsilon = \mathscr{Q}^\varepsilon \mathbf{v}^\varepsilon.$$

Back to the original problem

Lemma

Let

$$\mathbf{v}^\varepsilon := (\mathbf{u}^\varepsilon, \mathbf{e}^\varepsilon, \mathbf{h}^\varepsilon)^\top$$

satisfy the system from the previous slide. Then the function

$$\mathbf{g}^\varepsilon := \left(|\mathbf{u}^\varepsilon|^2, |\mathbf{e}^\varepsilon|^2, |\mathbf{h}^\varepsilon|^2, (\mathbf{u}^\varepsilon, \mathbf{e}^\varepsilon), (\mathbf{u}^\varepsilon, \mathbf{h}^\varepsilon), (\mathbf{e}^\varepsilon, \mathbf{h}^\varepsilon)\right)^\top$$

is a solution to

$$\begin{cases} \partial_x \mathbf{g}^\varepsilon = \mathcal{R}_p^\varepsilon \mathbf{g}^\varepsilon, \\ \partial_y \mathbf{g}^\varepsilon = \mathcal{R}_q^\varepsilon \mathbf{g}^\varepsilon, \end{cases}$$

where

$$\mathcal{R}_p^\varepsilon := \mathcal{R}(p_1^\varepsilon, p_2^\varepsilon, p_3^\varepsilon), \qquad \mathcal{R}_q^\varepsilon := \mathcal{R}(q_1^\varepsilon, q_2^\varepsilon, q_3^\varepsilon).$$

Back to the original problem

LEMMA (CONTINUATION)

and

$$\mathcal{R}(a_1, a_2, a_3) := \begin{pmatrix} 0 & 0 & 0 & 2a_1 & 2a_2 & 0 \\ 0 & 0 & 0 & -2a_1 & 0 & 2a_3 \\ 0 & 0 & 0 & 0 & -2a_2 & -2a_3 \\ -a_1 & a_1 & 0 & 0 & a_3 & a_2 \\ -a_2 & 0 & a_2 & -a_3 & 0 & a_1 \\ 0 & -a_3 & a_3 & -a_2 & -a_1 & 0 \end{pmatrix},$$

$a_1, a_2, a_3 \in \mathbb{R}$

Back to the original problem

Proposition

Let $\mathbf{v}^\varepsilon := (\mathbf{u}^\varepsilon, \mathbf{e}^\varepsilon, \mathbf{h}^\varepsilon)^\top$, $\varepsilon > 0$, *be the unique solution to system containing* $\mathscr{P}^\varepsilon$ *and* $\mathscr{Q}^\varepsilon$, *constructed above. Then,*

$$|\mathbf{u}^\varepsilon| = |\mathbf{e}^\varepsilon| = |\mathbf{h}^\varepsilon| = 1, \qquad (\mathbf{u}^\varepsilon, \mathbf{e}^\varepsilon) = (\mathbf{u}^\varepsilon, \mathbf{h}^\varepsilon) = (\mathbf{e}^\varepsilon, \mathbf{h}^\varepsilon) = 0.$$

Lemma

Let $\varepsilon > 0$, *and* $\mathbf{g}_0 \in \mathbb{R}^6$. *Further let* $\mathcal{R}_p^\varepsilon$ *and* $\mathcal{R}_q^\varepsilon$ *be as above and let the compatibility condition hold for* $\mathscr{P}^\varepsilon$ *and* $\mathscr{Q}^\varepsilon$ *Then, the system*

$$\begin{cases} \partial_x \mathbf{g}^\varepsilon = \mathcal{R}_p^\varepsilon \mathbf{g}^\varepsilon, \\ \partial_y \mathbf{g}^\varepsilon = \mathcal{R}_q^\varepsilon \mathbf{g}^\varepsilon, \\ \mathbf{g}^\varepsilon(x_0, y_0) = \mathbf{g}_0, \qquad (x_0, y_0) \in J_1 \times J_2, \end{cases}$$

has a unique solution $\mathbf{g}^\varepsilon \in C^\infty(J_1 \times J_2; \mathbb{R}^6)$.

Proof of proposition

Let $\mathbf{g}^\varepsilon := \big(|\mathbf{u}^\varepsilon|^2, |\mathbf{e}^\varepsilon|^2, |\mathbf{h}^\varepsilon|^2, (\mathbf{u}^\varepsilon, \mathbf{e}^\varepsilon), (\mathbf{u}^\varepsilon, \mathbf{h}^\varepsilon), (\mathbf{e}^\varepsilon, \mathbf{h}^\varepsilon)\big)^\top$, as above. By the previous lemmas, $\mathbf{g}^\varepsilon$ is the unique solution to the system from the previous slide, subject to the initial condition

$$\mathbf{g}^\varepsilon(x_0, y_0) = \mathbf{g}_0, \qquad \mathbf{g}_0 := (1,1,1,0,0,0)^\top.$$

However, the constant function $\mathbf{g}_0$ is also a solution to this system with the same initial condition. By uniqueness,

$$\mathbf{g}^\varepsilon = \mathbf{g}_0.$$

Proof of proposition

Let $\mathbf{g}^\varepsilon := \big(|\mathbf{u}^\varepsilon|^2, |\mathbf{e}^\varepsilon|^2, |\mathbf{h}^\varepsilon|^2, (\mathbf{u}^\varepsilon, \mathbf{e}^\varepsilon), (\mathbf{u}^\varepsilon, \mathbf{h}^\varepsilon), (\mathbf{e}^\varepsilon, \mathbf{h}^\varepsilon)\big)^\top$, as above. By the previous lemmas, $\mathbf{g}^\varepsilon$ is the unique solution to the system from the previous slide, subject to the initial condition

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Lemma

For each $\varepsilon > 0$, let $\mathbf{v}^\varepsilon := (\mathbf{u}^\varepsilon, \mathbf{e}^\varepsilon, \mathbf{h}^\varepsilon)^\top$ be the unique solution to system
$\partial_x \mathbf{v}^\varepsilon = \mathscr{P}^\varepsilon \mathbf{v}^\varepsilon, \quad \partial_y \mathbf{v}^\varepsilon = \mathscr{Q}^\varepsilon \mathbf{v}^\varepsilon, \quad \mathbf{v}^\varepsilon(x_0, y_0) = I_{3\times 3},$ *constructed above.*
Then, $$\mathbf{h}^\varepsilon = \mathbf{u}^\varepsilon \times \mathbf{e}^\varepsilon \qquad \text{on} \qquad J_1 \times J_2.$$

Main result

Theorem

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathbb{D}$ be a bounded domain. Let $\mathscr{W}$ be a 3D white noise, $\mathscr{W}^\varepsilon = \rho_\varepsilon * \mathscr{W}$, $\varepsilon > 0$, be its mollification with a standard mollifier, and let $\kappa \in (0, \frac{1}{2})$. For any $\delta > 0$, there exist a set $\Omega_\delta \subset \Omega$, $\mathbb{P}(\Omega_\delta) > 1 - \delta$, and $\gamma_0 > 0$ such that for $\gamma \in (0, \gamma_0)$ and $\varepsilon > 0$, there exist functions

$$\mathbf{u}^\varepsilon : \Omega_\delta \to \mathrm{C}^{1-\kappa}(\mathbb{D}, \mathbb{S}^2), \qquad \mathbf{O}^\varepsilon : \Omega_\delta \to \mathrm{C}^{1-\kappa}(\mathbb{D}, SO(3)),$$

such that the third column of $\mathbf{O}^\varepsilon$ is $\mathbf{u}^\varepsilon$ and, for every $\omega \in \Omega_\delta$, the pair $(\mathbf{u}^\varepsilon, \mathbf{O}^\varepsilon)$ is a solution to

$$\alpha \mathbf{u}^\varepsilon \times (\mathbf{u}^\varepsilon \times \Delta \mathbf{u}^\varepsilon) - \beta \mathbf{u}^\varepsilon \times \Delta \mathbf{u}^\varepsilon = \gamma \mathbf{u}^\varepsilon \times \mathbf{O}^\varepsilon \mathscr{W}^\varepsilon, \quad |\mathbf{u}^\varepsilon| = 1, \qquad \text{(sLLG)}$$

on $\mathbb{D}$, where $\alpha, \beta > 0$ and $\alpha^2 + \beta^2 = 1$.

Main result

Theorem (Continuation)

Moreover, there exist functions

$$\mathbf{u} : \Omega_\delta \to \mathrm{C}^{1-\kappa}(\overline{\mathbb{D}}, \mathbb{S}^2) \quad \textit{and} \quad \mathbf{O} : \Omega_\delta \to \mathrm{C}^{1-\kappa}(\overline{\mathbb{D}}, SO(3))$$

such that, uniformly in $\omega \in \Omega_\delta$*,*

$$|\mathbf{u}^\varepsilon - \mathbf{u}|_{\mathrm{C}^{1-\kappa}(\overline{\mathbb{D}})} \lesssim \varepsilon^{\frac{\kappa}{8}} \quad \textit{and} \quad |\mathbf{O}^\varepsilon - \mathbf{O}|_{\mathrm{C}^{1-\kappa}(\overline{\mathbb{D}})} \lesssim \varepsilon^{\frac{\kappa}{8}}.$$

Thank you!