Cross Diffusion: The generalised Keller Segel model under random perturbations

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Scope

1. What is chemotaxis?
2. Some deterministic results
3. Our results
4. The idea of the proof
5. Further directions

What is Cross-Diffusion

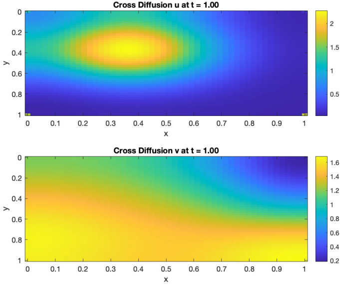
A typical cross-diffusion system;

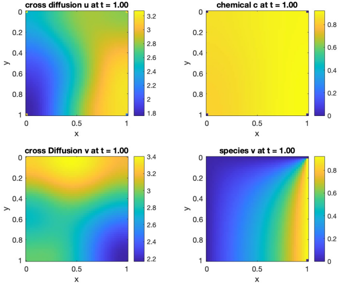
Let $\mathbf{u} := \{u_i : i = 1, \dots, N\}$ denote the density of the i-th species. A typical form is

$$\partial_t u_i = \operatorname{div}\Big(\sum_{j=1}^N a_{ij}(u)\nabla u_j\Big) \qquad i = 1, \dots, N,$$

where the matrix $A(u) := (a_{ij}(u))_{i,j=1}^N$ describes the coupled diffusion mechanism. Often, nonflux boundary conditions are imposed, like

$$\sum_{j=1}^N A_{l,j}(\mathbf{u}(t))\,(\nabla u_j(t)) \cdot \nu = 0 \quad \text{on} \quad \partial\mathcal{O}, \quad t > 0$$





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Applications

- **Population Dynamics**: Models species interactions, including competition and coexistence.
- **Chemical Systems**: Accounts for interactions between different chemical species.
- **Material sience**: Description of atmospheric aerosol[a].

[a] Fowler et all. (2018) *Maxwell–Stefan diffusion: a framework for predicting condensed phase diffusion and phase separation in atmospheric aerosol.*

Cross-Diffusion versus Diffusion

New features

- **Pattern Formation**: Cross-diffusion can lead to complex spatial patterns not observed with standard diffusion[a].

  $\Longrightarrow$ **segregation or clustering of species**

- **Uphill Diffusion:** Unlike standard diffusion, where substances move from higher to lower concentrations, cross-diffusion can cause *uphill* diffusion.
- **Enhanced Stability and Instability:** The inclusion of cross-diffusion terms can either stabilize or destabilize a system, cross diffusion may trigger instabilities.
- **Analytical Challenges**: The inclusion of cross-diffusion terms complicates the analysis of such systems.

[a] Vanag and Epstein (2009) *Cross-diffusion and pattern formation in reaction–diffusion systems.*

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Applications: population dynamics and material science

Example 1: Shigesada–Kawasaki–Teramoto (SKT) system[a]

- Models interacting populations with cross-diffusion effects.
- Typical phenomena: pattern formation, spatial segregation, coexistence, extinction.
- Applications: ecology, mathematical biology, social sciences.

Example 2: Maxwell–Stefan system Introduced by JAMES CLERK MAXWELL [b] (1866) and JOSEF STEFAN[c] (1871) to describe diffusion in multicomponent systems.

- Describes multicomponent diffusion in gases or mixtures.
- Goes beyond Fickian diffusion by incorporating interactions between species.
- Explains effects such as uphill diffusion and osmotic diffusion - see BOTHE [d]
- Applications in material science and multicomponent transport.

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[a] SHIGESADA, KAWASAKI: Spatial segregation of interacting species. J Theor Biol., 79:83–99, 1979, for more details Chen, Jüngle, Wang: *The Shigesada–Kawasaki–Teramoto cross-diffusion system beyond detailed balance* JDE 2023, Chen, Daus, Jüngel, *Global existence analysis of cross-diffusion population systems for multiple species, 2019.*

[b] MAXWELL. On the dynamical theory of gases. Philosophical Transactions of the Royal Society of London, 1866.

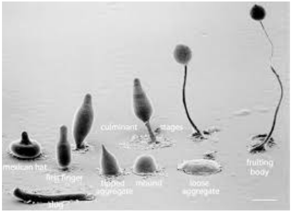
[c] STEFAN. Über das Gleichgewicht und die Bewegung insbesondere die Diffusion von Gasgemengen. Sitzungs-erichte der Kaiserlichen Akademie der Wissenschaften Wien, 1871.

[d] BOTHE, D. (2011). *On the Maxwell–Stefan Approach to Multicomponent Diffusion.* In *Parabolic Problems*, Springer.

Chemotaxis - the Keller Segel model

Chemotaxis

Chemotaxis is defined as the oriented movement of cells (or an organism) in response to a chemical gradient. Many sorts of motile cells undergo chemotaxis. For example, bacteria and many amoeboid cells can move in the direction of a food source. In our bodies, immune cells like macrophages and neutrophils can move towards invading cells. Other cells, connected with the immune response and wound healing, are attracted to areas of inflammation by chemical signals.

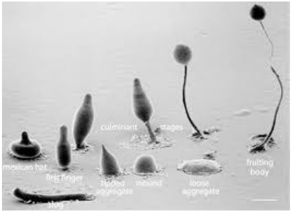


Dictyostelium discoideum is a cellular slime mold that serves as an important model organism in a variety of fields. They exist as separate amoebae, but after consuming all the bacteria in their area they proceed to stream together to form a multicellular organism.

Chemotaxis - the Keller Segel model

First References

- C. S. Patlak. *Random walk with persistence and external bias.* Bull. of Mathematical Biophysics 15:311-338, 1953.
- E. Keller and L. Segel. *Initiation of slime mould aggregation viewed as an instability.* Journal of theoretical Biology, 26:399-415, 1970.
- W. Alt. *Biased random walk models for chemotaxis and related diffusion approximations.* Journal of Mathematical Biology, 9:147–177, (1980).
- A. Stevens: *The Derivation of Chemotaxis Equations as Limit Dynamics of Moderately Interacting Stochastic Many-Particle Systems.* SIAM Journal on applied analysis, (2000).



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Chemotaxis: the mathematical models

The simple Keller-Segel model

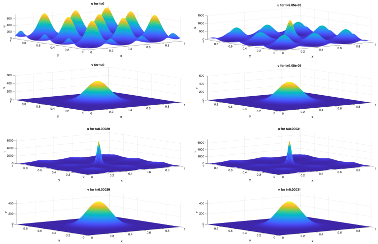
- $u(t, x)$ density of cells at time $t \in [0, T]$ and position $x \in \mathcal{O}$;
- $v(t, x)$ concentration of chemoattractant;
- the parameter χ is the sensitivity of cells to the chemoattractant;
- D_u and D_v are the diffusivity; α some damping constant.

In a collective model, the chemoattractant is emitted by the cells that react according to a biased random walk.

$$\begin{cases} \dot{u}(t) &= D_u \Delta u(t) - \chi \operatorname{div}\big(u(t)\nabla v(t)\big), \\ \tau \dot{v}(t) &= D_v \Delta v(t) + \beta u(t) - \alpha v(t) \quad t \in \mathbb{R}_0^+, \end{cases}$$

with initial conditions $u(0, x) = u_0(x)$ and $v(0, x) = v_0(x)$, where $x \in \mathcal{O}$ and $\mathcal{O} = [0, 1]$ is a bounded interval and Neumann boundaries, $\tau \in \{0, 1\}$.

The Two-Dimensional Dichotomy



The equation: $u(x,0) = u_0(x) \geq 0,\ x \in \mathbb{R}^2$

$$\begin{cases} \partial_t u = \Delta u - \nabla \cdot (u \nabla v), & x \in \mathbb{R}^2,\ t > 0, \\ \tau \partial_t v = \Delta v + u, & x \in \mathbb{R}^2,\ t > 0, \end{cases},$$

Assuming integrable, nonnegative initial data $u_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, the total mass is conserved:

$$M := \int_{\mathbb{R}^2} u(x,t)\, dx = \int_{\mathbb{R}^2} u_0(x)\, dx.$$

Blow-up vs. Global Existence: the parabolic-elliptic system

- If $M < 8\pi$, then the solution exists globally in time and remains bounded.
- If $M > 8\pi$, then solutions can blow up in finite time; that is, there exists $T < \infty$ such that $\limsup_{t \nearrow T} \|u(\cdot,t)\|_{L^\infty} = \infty$.
- The case $M = 8\pi$ is *critical*: there exist both global and blow-up solutions depending on the initial profile. The system is then at the threshold between diffusion and aggregation.

Keller–Segel model: global existence and blow-up

We consider the classical deterministic Keller–Segel system

$$\begin{cases} \partial_t u = \Delta u - \nabla \cdot (u \nabla v), \\ \tau \partial_t v = \Delta v + u - v, \end{cases}$$

where u denotes the cell density and v the chemoattractant.

- In dimension $d = 1$, solutions exist globally and no blow-up occurs.
- In dimension $d = 2$, the model is mass-critical (see last slide)

$$M := \int_{\mathcal{O}} u_0(x)\, dx, \qquad M_c = 8\pi.$$

  - If $M < 8\pi$, then solutions are global.
  - If $M > 8\pi$, finite-time blow-up may occur.
  - At the critical mass $M = 8\pi$, solutions are global, but aggregation may occur a
- In dimensions $d \geq 3$, blow-up can occur even more robustly.

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References for the deterministic Keller–Segel model

- **Jäger–Luckhaus (1992) - Angela Stevenson (limit of a particle system)** global existence for small mass in two space dimensions.
- **Nagai–Senba–Yoshida (1997)** global existence in the two-dimensional case for subcritical mass. Their analysis uses two-dimensional functional inequalities, in particular Trudinger–Moser type estimates.
- **Nagai (2001)** studied global existence versus blow-up phenomena for Keller–Segel type systems.
- **Dolbeault–Perthame (2004)** Gave an entropy-based analysis of the two-dimensional Keller–Segel model and identified the critical mass threshold 8π.
- **Herrero–Medina–Velázquez (1996/1997)** fundamental reference for finite-time aggregation and blow-up, Constructed blow-up solutions for Keller–Segel type systems.
- **Blanchet–Carrillo–Masmoudi (2008)** Studied the critical mass case in the two-dimensional parabolic–elliptic Keller–Segel system.
- **Blanchet–Carlen–Carrillo (2012)** Developed a functional-inequality and entropy approach to the critical mass problem,
- **Carlen–Figalli (2013)** Studied stability and critical behaviour connected with the sharp functional inequalities at the critical mass 8π.

Chemotaxis with random perturbation

The Keller-Segel model with proliferation in dimension 2

- $u(t,x)$ density of cells at time $t \in [0,T]$ and position $x \in \mathcal{O}$;
- $v(t,x)$ concentration of chemoattractant;
- the parameter χ is the sensitivity of cells to the chemoattractant;
- D_u and D_v are the diffusivity;

In a collective model, the chemoattractant is emitted by the cells that react according to a biased random walk.

$$(*) \quad \left\{ \begin{array}{rcl} du(t) & = & \big(D_u \Delta u(t) - \chi \mathrm{div}\big(u(t)\nabla v(t)\big)\big)\, dt \\ & & -|u|^{q-1}(t)u(t) + u(t)dW_1(t) \\ dv(t) & = & \big(D_v \Delta v(t) + \beta u(t)\big)dt + v(t)dW_2(t), \quad t \in \mathbb{R}_0^+, \end{array} \right.$$

with initial conditions $u(0,x) = u_0(x)$ and $v(0,x) = v_0(x)$, where $x \in \mathcal{O}$ and $\mathcal{O} \subset \mathbb{R}^2$ is a bounded interval and Neumann boundaries.

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Chemotaxis with random perturbation

The Keller-Segel model with porous media term in dimension 2

- $u(t,x)$ density of cells at time $t \in [0,T]$ and position $x \in \mathcal{O}$;
- $v(t,x)$ concentration of chemoattractant;
- the parameter χ is the sensitivity of cells to the chemoattractant;
- D_u and D_v are the diffusivity;

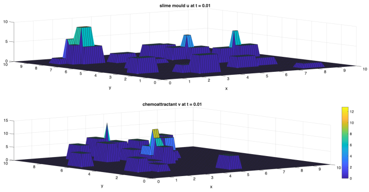
In a collective model, the chemoattractant is emitted by the cells that react according to a biased random walk.

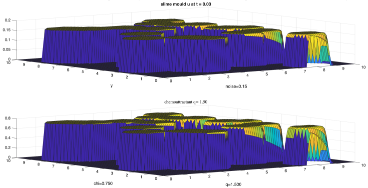
$$(*) \quad \begin{cases} du(t) &= \Big(D_u \Delta \Phi(u(t)) - \chi \operatorname{div}\big(u(t)\nabla v(t)\big)\Big)\, dt \\ & \quad + u(t) dW_1(t) \\ dv(t) &= \Big(D_v \Delta v(t) + \beta u(t)\Big) dt + v(t) dW_2(t)\,, \quad t \in \mathbb{R}_0^+, \end{cases}$$

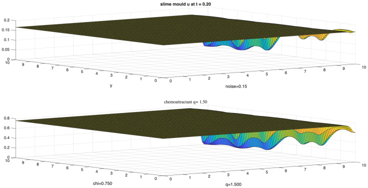
where $\Phi(x) = |x|^{q-1}x$ and with initial conditions $u(0,x) = u_0(x)$ and $v(0,x) = v_0(x)$, where $x \in \mathcal{O}$ and $\mathcal{O} \subset \mathbb{R}^2$ is a bounded interval and Neumann boundaries.

Chemotaxis with $-|u|^{q-1}u$ - some deterministic results

- Osaki, Tohru Tsujikawa, Atsushi Yagi, Masayasu Mimura: Exponential attractor for a chemotaxis-growth system of equations, Nonlinear Analysis: Theory, Methods and Applications, 2000.
- Hai-Yang Jin , Tian Xiang: Chemotaxis effect vs. logistic damping on boundedness in the 2-D minimal Keller–Segel model, C. R. Acad. Sci. Paris, Ser. I, 2018.
- Xiang, Tian: How strong a logistic damping can prevent blow-up for the minimal Keller-Segel chemotaxis system? J. Math. Anal. Appl. 459 (2018).
- Xie Li: On a fully parabolic chemotaxis system with nonlinear signal secretion. Nonlinear Analysis: Real World Applications Volume 49, 2019.
- Fujie, Kentarou; Winkler, Michael; Yokota, Tomomi: Blow-up prevention by logistic sources in a parabolic-elliptic Keller-Segel system with singular sensitivity, AIMS, 2014.







Chemotaxis with $\Delta\Phi_q(u)$ - some deterministic results

Critical diffusion exponent: $q_c = 2 - \frac{2}{d} \qquad (d \geq 2)$.

- $q < q_c$: diffusion too weak $\Rightarrow$ concentration or blow-up mechanisms may occur.
- $q = q_c$: *critical regime* (often critical mass / threshold phenomena).
- $q > q_c$: stronger diffusion $\Rightarrow$ global existence/boundedness expected in many settings.

Articles:

- Blanchet-Carrillo-Laurencot (2009): critical mass for the Patlak-Keller-Segel system with degenerate diffusion in higher dimensions.

Chemotaxis with $\Delta\Phi_q(u)$ - some deterministic results

Critical diffusion exponent: $q_c = 2 - \frac{2}{d} \qquad (d \geq 2)$.

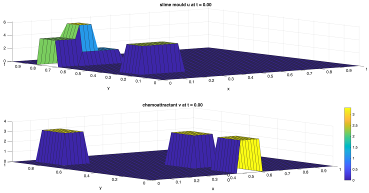
Supercritical diffusion $q > q_c$:

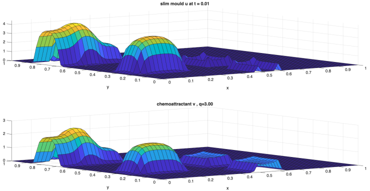
Global existence / no blow-up: When the diffusion is strong enough (supercritical porous-medium regime), global weak solutions and various boundedness/regularity properties are expected/available under standard assumptions; precise statements depend on the parabolic–elliptic vs parabolic–parabolic coupling and domain.

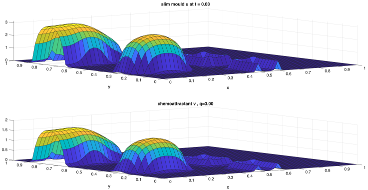
- Bian–Liu (2013) discuss global existence vs blow-up/spreading regimes in the Keller–Segel system with nonlinear diffusion and identify parameter ranges tied to m_c.
- For boundedness techniques and modern unified estimates in Keller-Segel-type systems (including nonlinear diffusion): see Winkler (2022).

Chemotaxis with $\Delta\Phi_q(u)$ - related topics

- Luckhaus–Sugiyama (2006): large-time behavior in *super-critical* cases for degenerate Keller–Segel systems.
- Sugiyama (2009): ε-regularity and asymptotic behavior tools for Keller-Segel systems with degenerate diffusion.
- Tao and Winkler (2012): global existence and boundedness in Keller–Segel–Stokes with arbitrary porous-medium diffusion (shows the strength of the porous diffusion mechanism in coupled settings with fluid)
- Carrillo-Hittmeir-Jüngel (2010/2011) (cross-diffusion and nonlinear diffusion) show mechanisms preventing blow-up via entropy structures.







Chemotaxis under random perturbation

Assumptions on the initial conditions:

Let $v_0, u_0 \in L^2(\mathcal{O})$ be two random variables over $\mathfrak{A}$ such that

1. $u_0 \geq 0$ and $v_0 \geq 0$;
2. (u_0, v_0) is $\mathcal{F}_0$–measurable;
3. $\mathbb{E}|u_0|^2_{L^2} < \infty$ and $\mathbb{E}|\nabla v_0|^2_{L^2} < \infty$.

Chemotaxis with random perturbation

The Entropy Functional

A typical entropy functional in cross-diffusion systems is: The following Lyapunov type functional is given:

$$\mathcal{F}(u,v) := \int (\ln(u(\xi)) - 1)\, u(\xi)\, d\xi + c_1 |v|_{L^2}^2 + c_2 |\nabla v|_{L^2}^2 .$$

where $\mathcal{O}$ is the spatial domain, and u and v are the concentrations of the substances or populations.

$$\begin{aligned} \frac{\partial}{\partial u}\left((\ln(u) - 1)\, u\right) &= \ln(u), \\ \frac{\partial^2}{\partial u^2}\left((\ln(u) - 1)\, u\right) &= \frac{1}{u}. \end{aligned}$$

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Definition of the probabilistic strong solution

Definition

Given a filtered probability space $\mathfrak{A} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, two independent Wiener processes W_1 and W_2, defined over $\mathfrak{A}$. Then, we call the tuple

$$(u, v) : \Omega \times [0, T] \to H_2^{-1}(\mathcal{O}) \times L^2(\mathcal{O})$$

unique strong solution (u, v), to the system $(*)$, along with the initial and boundary conditions, if

$$\mathbb{P}\left(u, v \in L^2(0, T; H_2^{-1}(\mathcal{O}) \cap L^2(0, T; L^2(\mathcal{O}))\right) = 1,$$

and $\mathbb{P}$ a.s. we have for all $\phi_1 \in L^2(0, T; H_0^1(\mathcal{O})$ and $\phi_2 \in L^2(0, T; L^2(\mathcal{O}))$

$$\begin{aligned}
\langle u(t), \phi_1 \rangle &= r_u \int_0^t \langle \Delta u(s), \phi_1 \rangle \, ds - \int_0^t u^{[q]}(s), \phi_1 \rangle \, ds \\
&\quad - \chi \int_0^t \langle \operatorname{div}\big(u(s) \nabla v(s)\big), \phi_1 \rangle + \sum_{k \in \mathbb{I}} \int_0^t \langle u(s) \psi_k^1, \phi_1 \rangle \circ d\beta_k^{(1)}(s)
\end{aligned}$$

and

$$\begin{aligned}
\langle v(t), \phi_2 \rangle &= r_v \int_0^t \langle \Delta v(s), \phi_2 \rangle \, ds + \beta \int_0^t \langle u(s), \phi_2 \rangle \, ds \\
&\quad - \alpha \int_0^t \langle v(s), \phi_2 \rangle \, ds + \sum_{k \in \mathbb{I}} \int_0^t \langle u(s) \psi_k^2, \phi_2 \rangle \circ d\beta_k^{(2)}(s).
\end{aligned}$$

Assumptions

Assumption on the Wiener process

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be isomorphic to Bessel potential spaces[a]. Let us assume that the embedding $\mathcal{H}_1$ in $H_2^{\frac{d}{2}}(\mathcal{O})$ and the embedding $\mathcal{H}_1$ in $H_2^2(\mathcal{O})$ are Hilbert-Schmidt operators.

[a] Two simple examples are $\mathcal{H}_1 = H_2^{\delta_1}(\mathcal{O})$ for $\delta_1 > 1$, and $\mathcal{H}_2 = H_2^{\delta_2}(\mathcal{O})$ for $\delta_2 > 1$.

Assumption on the initial conditions

Let

$$m_0 = \begin{cases} \frac{7q+696}{(q-2)(q+18)} & \text{if } q \in (2, 2.5] \\ \frac{54-5q}{4(q-2)(q+6)} & \text{if } q \in (2.5, 3) \end{cases}$$

Let $\mathbb{E}|\mathcal{F}(u_0, v_0)|^{m_0} \leq \infty$ and $\mathbb{E}|u_0|_{L^2}^2 + \mathbb{E}|v_0|_{L^4}^4 + \mathbb{E}|\nabla v_0|_{L^4}^4 < \infty$.

Chemotaxis with random perturbation

Theorem

Let the Assumption stated before be satisfied and $q \in (2,3)$. Then, for all initial data $(u_0, v_0) \in L^2(\mathcal{O}) \times H^1_4(\mathcal{O})$ satisfying Assumption on the initial conditions, there exists a unique global mild solution (u, v) to the system $(*)$. In addition, there exist positive constants C_0 and a_1 such that the following inequality holds:

$$\frac{1}{4}\mathbb{E}\Big[\sup_{0\le s\le T}|u(s)|^2_{L^2}\Big] + \frac{r_u}{2}\mathbb{E}\Big[\int_0^T |\nabla u(s)|^2_{L^2}\,ds\Big] + \delta\,\mathbb{E}\Big[\int_0^T |u(s)|^q_{L^{q+1}}\,ds\Big]$$

$$+\alpha\,\mathbb{E}\Big[\int_0^T |\nabla v(s)|^4_{L^4}\,ds\Big] + \frac{1}{4}\mathbb{E}\Big[\sup_{0\le s\le T}|v(s)|^4_{H^1_4}\Big] + \frac{r_v}{2}\mathbb{E}\Big[\int_0^T |\nabla v(s)|^4_{H^1_4}\,ds\Big]$$

$$\le \quad C(u_0, v_0)e^{Ta_1}.$$

Idea of the proof

Let $\frac{4-q}{2-q} > \epsilon \geq 2$ and $p \geq 2$ be chosen so that

$$(\circ) \qquad \frac{1}{2} \leq \frac{1}{p} + \frac{1}{1+\epsilon} \leq \frac{5}{4}$$

holds.

Let us introduce a cut-off function $\phi \in C_c^\infty(\mathbb{R})$, let $\phi_n(x) := \phi(x/n)$, $x \in \mathbb{R}$, $n \in \mathbb{N}$. For any $n \in \mathbb{N}$, progressively measurable process ξ over $\mathfrak{A}$, let us denote for $t \geq 0$

$$h_1^n(\xi, t) := \phi_n\Big(\|\xi\|_{L^p([0,t];L^{1+\epsilon})} \Big), \quad h_2^n(\xi, t) := \phi_n\Big(\sup_{0\leq s\leq t} |\nabla \xi(s)|_{L^4} \Big).$$

We consider the truncated system for $n \in \mathbb{N}$, which is given by

$$(*)_n \quad \begin{cases} du_{n_1,n_2}(t) & = \Big(r_u \Delta u_{n_1,n_2}(t) - \chi h_1^n(u_{n_1,n_2}, t) h_2^n(v_{n_1,n_2}, t) \operatorname{div}(u_{n_1,n_2}(t) \nabla v_{n_1,n_2}(t)) \Big) dt \\ & \qquad - \delta |u^n(t)|^{q-1} u^n(t) dt + u_{n_1,n_2}(t) dW_1(t), \\ dv_{n_1,n_2}(t) & = \Big(r_v \Delta v_{n_1,n_2}(t) + \beta h_1^n(u_{n_1,n_2}, t) u_{n_1,n_2}(t) - \alpha v_{n_1,n_2}(t) \Big) dt + v_{n_1,n_2}(t) dW_2(t). \end{cases}$$

Idea of the proof

For all $n \in \mathbb{N}$, existence and uniquness of $(*)_n$ is shown by a Banach fix point Theorem:

Definition of the integral operator: For any $n \in \mathbb{N}$, we define an operator[1]

$$\Phi_n : \mathcal{M}^p_{\mathfrak{A},\lambda}([0,\infty); L^{1+\epsilon}(\mathcal{O})) \to \mathcal{M}^p_{\mathfrak{A},\lambda}([0,\infty); L^{1+\epsilon}(\mathcal{O})),$$

where $\Phi_n(\xi)$ is given for $t \geq 0$ by $\Phi_n(\xi) = u^n_\xi$ and the couple (u^n_ξ, v^n_ξ) is a solution to

$$(\circ)_u \qquad du(t) := \Big(r_u \Delta u(t) + \chi h^n_1(\xi, t) h^n_2(v, t) \operatorname{div}\big(\xi(t) \nabla v(\xi)(t)\big) - \delta |u(t)|^{q-1} u(t)\Big) dt + u(t) dW_1(t),$$

with initial condition $u^n_\xi(0) = u_0$, and where $v(\xi) := v$ is given by

$$(\circ)_v \qquad v(t) := e^{(r_v \Delta - \alpha I)t} v_0 + \beta \int_0^t h^n_1(\xi, s) e^{(r_v \Delta - \alpha I)(t-s)} \xi(s) ds.$$

[1] $\mathcal{M}^p_{\mathfrak{A},\lambda}([0,\infty); E) := \{\xi : [0,\infty) \times \Omega \to \mathbb{R}$ where ξ is progressively measurable such that , $\int_0^\infty e^{-\lambda s} |\xi(s)|^p_E \, ds < \infty\}$.

Idea of the proof

Important: The norm is weighted by a factor λ:

$$\|\xi\|^p_{\mathcal{M}_{\mathfrak{A},\lambda}} = \mathbb{E}\int_0^\infty e^{-\lambda s}|\xi(s)|^p_{L^{1+\epsilon}}\, ds$$

1. Well posedness of the operator $\Rightarrow$ Standard literature.
   - $(\circ)^n_v$: Semigroup approach gives existence, uniqueness, and exact regularity of the process v (see Da Prato and Zabczyk)
   - $(\circ)^n_u$: Variational Ansatz (see Wei and Röckner)

$$A_\xi(u,t) := r_u\Delta u + \chi h_1^n(\xi,t) h_2^n(v,t)\mathrm{div}\big(\xi(t)\nabla v(t)\big) - \delta|u|^{q-1}u$$

2. for all $n\in\mathbb{N}$ there exists a $\lambda_0\geq 0$ (depending on n), such that for all $\lambda\geq\lambda_0$, Φ_n is a strict contraction

Globalising the local solution by a uniform bound

Step II:

Extending the solution by linearisation - showing non-negativity

Step III: Verifying a uniform bound

For all $q \in (2,3)$, $n \in \mathbb{N}$ with $n \geq 3$, there exist positive constants a_1, a_2 and some number $m > 1$ such that for all initial data (u_0, v_0) satisfying Assumption beforewe have the following inequality for all $t \in [0, \tau_n]$

$$\begin{aligned}
&\frac{1}{4}\mathbb{E}\Big[\sup_{0\le s\le t\wedge\tau_n} |\bar{u}^{n_1,n_2}(s)|^2_{L^2}\Big] + \frac{r_u}{2}\mathbb{E}\Big[\int_0^{t\wedge\tau_n} |\nabla \bar{u}^{n_1,n_2}(s)|^2_{L^2}\, ds\Big] \\
&\quad + \alpha\,\mathbb{E}\Big[\int_0^{t\wedge\tau_n} |\nabla \bar{v}^{n_1,n_2}(s)|^4_{L^4}\, ds\Big] + \frac{1}{4}\mathbb{E}\Big[\sup_{0\le s\le t\wedge\tau_n} |\bar{v}^{n_1,n_2}(s)|^4_{H^1_4}\Big] \\
&\quad + \frac{r_v}{2}\mathbb{E}\Big[\int_0^{t\wedge\tau_n} |\nabla \bar{v}^{n_1,n_2}(s)|^4_{H^1_4}\, ds\Big] + \frac{r_v}{2}\mathbb{E}\Big[\int_0^{t\wedge\tau_n} |\bar{u}^{n_1,n_2} s)|^{q+1}_{L^{q+1}}\, ds\Big] \\
&\le \quad a_1\, e^{a_2\mathbb{E}[t\wedge\tau_n]}\left\{\frac{1}{2}\mathbb{E}|u_0|^2_{L^2} + \frac{1}{4}\mathbb{E}|\nabla v_0|^4_{L^4}\right\} + \mathbb{E}|u(t\wedge\tau_n)|^m_{L^1}
\end{aligned}$$

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&\quad + \alpha\,\mathbb{E}\Big[\int_0^{t\wedge\tau_n} |\nabla \bar{v}^{n_1,n_2}(s)|_{L^4}^4\, ds\Big] + \frac{1}{4}\mathbb{E}\Big[\sup_{0\le s\le t\wedge\tau_n} |\bar{v}^{n_1,n_2}(s)|_{H_4^1}^4\Big] \\
&\quad + \frac{r_v}{2}\mathbb{E}\Big[\int_0^{t\wedge\tau_n} |\nabla \bar{v}^{n_1,n_2}(s)|_{H_4^1}^4\, ds\Big] + \frac{r_v}{2}\mathbb{E}\Big[\int_0^{t\wedge\tau_n} |\bar{u}^{n_1,n_2}s)|_{L^{q+1}}^{q+1}\, ds\Big] \\
\le\ & a_1\, e^{a_2 \mathbb{E}[t\wedge\tau_n]} \left\{ \frac{1}{2}\mathbb{E}|u_0|_{L^2}^2 + \frac{1}{4}\mathbb{E}|\nabla v_0|_{L^4}^4 \right\} + \mathbb{E}|u(t\wedge\tau_n)|_{L^1}^m
\end{aligned}$$

Third Step - uniform bounds

The important terms are:

$$\begin{aligned}
&\frac{1}{2}|u(s)|_{L^2}^2 - \frac{1}{2}|u_0|_{L^2}^2 + r_u \int_0^s |\nabla u(r)|_{L^2}^2 dr + \delta \int_0^s |u(r)|_{L^{q+1}}^{q+1} dr \\
\leq\quad & \chi\epsilon \int_0^s |\nabla u(r)|_{L^2}^2 dr + \frac{\chi}{4\epsilon} \int_0^s |u(r)\nabla v(r)|_{L^2}^2 dr + \frac{\gamma_1^2 + \gamma_1}{2} \int_0^s |u(r)|_{L^2}^2 dr \\
& + \left| \int_0^s \langle u(r), u(r) dW_1(r) \rangle \right|,
\end{aligned}$$

and the term

$$\begin{aligned}
&\frac{1}{4}|\nabla v(t)|_{L^4}^4 - \frac{1}{4}|\nabla v_0|_{L^4}^4 + \frac{r_v}{4} \int_0^t \int_{\mathcal{O}} |\nabla |\nabla v(s,x)|^2|^2 \, dx\, ds \\
&\qquad + \alpha \int_0^t |\nabla v(s)|_{L^4}^4 \, ds + \frac{r_v}{2} \int_0^t \int_{\mathcal{O}} |\nabla v(s,x)|^2 |D^2 v(s,x)|^2 \, dx\, ds \\
=\quad & \cdots \leq c_2 \int_0^t |\nabla v(s)|_{L^4}^4 \, ds + c_5 \int_0^t |u(s)\nabla v(s)|_{L^2}^2 \, ds.
\end{aligned}$$

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$$
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\leq \quad & \chi\epsilon \int_0^s |\nabla u(r)|_{L^2}^2 dr + \frac{\chi}{4\epsilon} \int_0^s |u(r)\nabla v(r)|_{L^2}^2 dr + \frac{\gamma_1^2+\gamma_1}{2} \int_0^s |u(r)|_{L^2}^2 dr \\
& + \left| \int_0^s \langle u(r), u(r) dW_1(r) \rangle \right|,
\end{aligned}
$$

and the term

$$
\begin{aligned}
&\frac{1}{4}|\nabla v(t)|_{L^4}^4 - \frac{1}{4}|\nabla v_0|_{L^4}^4 + \frac{r_v}{4} \int_0^t \int_{\mathcal{O}} |\nabla|\nabla v(s,x)|^2|^2 \, dx\, ds \\
&\qquad + \alpha \int_0^t |\nabla v(s)|_{L^4}^4 \, ds + \frac{r_v}{2} \int_0^t \int_{\mathcal{O}} |\nabla v(s,x)|^2 |D^2 v(s,x)|^2 \, dx\, ds \\
= \quad & \cdots \leq c_2 \int_0^t |\nabla v(s)|_{L^4}^4 \, ds + \underbrace{c_5 \int_0^t |u(s)\nabla v(s)|_{L^2}^2 \, ds.}_{\text{gives the problems}}
\end{aligned}
$$

Third Step - uniform bounds the proof

From the Gagliardo-Nirenberg inequality we know that, there exists constants $C_1, C_2, C_3 > 0$ such that

$$\begin{aligned} |u\nabla v|_{L^2}^2 &\leq C(\epsilon_2)\,|u|_{L^s}^\rho + \epsilon_2 c_6 |\nabla|\nabla v|^2|_{L^2}^\alpha |\nabla v|_{L^4}^{2(1-\alpha)} + \epsilon_2 c_7 ||\nabla v|^2|_{L^\sigma} \\ &\leq C(\epsilon_2)\,|u|_{L^s}^\rho + \epsilon_3 |\nabla|\nabla v|^2|_{L^2} + C(\epsilon_3)|\nabla v|_{L^4}^2 + \epsilon_2 c_7 ||\nabla v|^2|_{L^\sigma} \end{aligned}$$

Taken into account the time integral, and interpolation gives

$$\int_0^t |u(s)|_{L^s}^\rho \, ds \leq C(\epsilon) \sup_{0\leq s\leq t} |u(s)|_{L^1}^m + \epsilon \int_0^t |u(s)|_{L^{q+1}}^{q+1} \, ds + \epsilon \int_0^t |u(s)|_{L^r}^2 \, ds.$$

Uniform bounds

Consider the entropy functional

$$\mathcal{F}(u,v) = \int_{\mathcal{O}} (u \ln u - u)\,dx + |\nabla v|_{L^2}^2 := \phi(u) + \psi(v),$$

with $u > 0$.

Second uniform bound

For all $q \in (2,3)$, $n \in \mathbb{N}$ with $n \geq 3$, and any $m \geq 1$, there exists some constant $C > 0$ we have

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0\le s\le t} \mathcal{F}(u(s),v(s)) + \int_0^t |\nabla u^{\frac{1}{2}}(s)|_{L^2}^2\,ds + c_q \int_0^t |u(s)|_{L^2}^2\,ds \\
&\qquad + \alpha \int_0^t |\nabla v(s)|_{L^2}^2\,ds + \alpha \int_0^t |\Delta v(s)|_{L^2}^2\,ds \Big]^m \\
\le\ & C\mathbb{E}\Big[(t\mathrm{Leb}(\mathcal{O})\,a + C(\epsilon)) + \phi(u(0),v(0)) \\
&\qquad + (C(\beta,\epsilon)+\chi) \int_0^t |\Delta v(s)|_{L^2}^2\,ds + (\epsilon+\chi) \int_0^t |u(s)|_{L^2}^2\,ds \Big]^m.
\end{aligned}$$

Step IV: Globalising

Let $\Omega_1 := \{$There exists a solution to system $(*)\}$. We will show $\mathbb{P}(\Omega_1) = 1$.

Uniform bounds

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Step IV: Globalising

Let $\Omega_1 := \{\text{There exists a solution to system } (*)\}$. We will show $\mathbb{P}(\Omega_1) = 1$.

Definition of the probabilistic weak solution

Definition

A *probabilistic weak solution* to the problem $(*)$ is a system

$$(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}, (W_1, W_2), (u, v)) \tag{1}$$

such that

- $\mathfrak{A} := (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a complete filtered probability space with a filtration $\mathbb{F} = \{\mathcal{F}_t : t \in [0, T]\}$ satisfying the usual conditions,
- W_1 and W_2 are cylindrical Wiener processes on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, over the probability space $\mathfrak{A}$;
- $u : [0, T] \times \Omega \to H_2^{-1}(\mathcal{O})$ and $v : [0, T] \times \Omega \to L^2(\mathcal{O})$ are two $\mathbb{F}$-progressively measurable processes such that the couple (u, v) is a probabilistic strong solution to the system $(*)$ over the probability space $\mathfrak{A}$.

Second equation - what happens with the porous media term

Improvement by a porous media term

Let $q > 1$, let $\Phi_q(x) = |x|^{q-1}x$ and check

$$(*) \quad \begin{cases} du(t) &= \Big(D_u \Delta \Phi(u(t)) - \chi \mathrm{div}\big(u(t)\nabla v(t)\big)\Big)\, dt \\ & \quad + u(t) dW_1(t) \\ dv(t) &= \big(\Delta v(t) + \beta u(t)\big) dt + v(t) dW_2(t) , \quad t \in \mathbb{R}_0^+, \end{cases}$$

Here, one gets integrability and pays regularity. For which q, does one can show existence of a solution ($q > 3$ is straightforward).

Theorem (Tölle and E.H.), (Ankit Kumar, Tölle, and E.H., under second revision)

Fix a filtered probability space $\mathfrak{A} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with filtration $\mathbb{F} = (\mathcal{F}(t))_{t\geq 0}$, let W be a Wiener process defined on $\mathfrak{A}$. Let $\mathbb{X}$ be a function space over $[0,T]$ and let $\mathcal{X}(\mathfrak{A})$ be a weakly convex nonempty subset of $\mathcal{M}^p_{\mathfrak{A}}(\mathbb{X})$. Let $\mathcal{V}_{\mathfrak{A},W} : \mathcal{M}^m_{\mathfrak{A}}(\mathbb{X}) \to \mathcal{M}^m_{\mathfrak{A}}(\mathbb{X})$ be a continuous operator such that

1. the operator $\mathcal{V}_{\mathfrak{A},W}$ maps $\mathcal{X}(\mathfrak{A})$ into $\mathcal{X}(\mathfrak{A})$;
2. the operator $\mathcal{V}_{\mathfrak{A},W} : \mathcal{X}(\mathfrak{A}) \to \mathcal{M}^{\phi}_{\mathfrak{A}}(\mathbb{X})$ is continuous;
3. there exists a space $\mathbb{X}'$ and numbers $r, R > 0$ such that $\mathbb{X}' \hookrightarrow \mathbb{X}$ compactly and $\mathbb{E}|\mathcal{V}_{\mathfrak{A},W}(v)|^r_{\mathbb{X}'} \leq R, \quad \forall v \in \mathcal{X}$;
4. there exists a Banach space E_1 with $\mathcal{V}_{\mathfrak{A},W} : \mathcal{X}(\mathfrak{A}) \to \mathbb{D}(0,T;E_1)$.

Then there exists a complete filtered probability space $\tilde{\mathfrak{A}} := (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}})$ with a filtration $\tilde{\mathbb{F}} = \{\tilde{\mathcal{F}}_t : t \in [0,T]\}$ satisfying the usual condition, a martingale $\tilde{M}$ defined over the probability space $\tilde{\mathfrak{A}}$ and a n element $\tilde{v} \in \mathcal{M}^m_{\tilde{\mathfrak{A}}}(\mathbb{X})$ being progressively measurable such that we have for all $t \in [0,T]$, $\tilde{\mathbb{P}}$–a.s.

$$\mathcal{V}_{\tilde{\mathfrak{A}},\tilde{W}}(\tilde{\mathfrak{A}})(\tilde{v})(t) = \tilde{v}(t).$$

Underlying spaces

Let us fix throughout this work the following parameters:

$$r = \frac{4(1+q)}{3+q} \in (2, 1+q), \qquad m = 2(1+q) \in (q+1, \infty), \qquad \delta = \frac{1}{2}.$$

Let us put

$$r_\epsilon = \frac{4(1+q-\epsilon/2)}{q+3-\epsilon}, \quad m_\epsilon := 2(1+q) - \epsilon, \quad \text{where} \quad 0 < \epsilon \, \text{small enough.}$$

Remark

Note, the space $L^m(0,T; H_r^{-\frac{1}{2}})$ is the complex interpolation space of $L^\infty(0,T; H_2^{-1}(\mathcal{O}))$ and $L^{q+1}(0,T; L^{q+1}(\mathcal{O}))$, i.e.,

$$\left[L^\infty(0,T; H_2^{-1}(\mathcal{O})), L^{q+1}(0,T; L^{q+1}(\mathcal{O})\right]_{\frac{1}{2}} = L^m(0,T; H_r^{-1/2}).$$

Motivation of the proof

The integral operator:

Let $\mathcal{M}^{m_\epsilon}_{\mathfrak{A}}(0,T) := \left\{ \xi \in L^{m_\epsilon}(0,T; H^{-\frac{1}{2}}_{r_\epsilon}) : \mathbb{E}\int_0^T |\xi(s)|^{m\epsilon}_{H^{-\frac{1}{2}}_{r_\epsilon}}\, ds < \infty \right\}$
and define $\mathcal{T}(\eta) = u$ on $\mathcal{M}^{m_\epsilon}_{\mathfrak{A}}(0,T)$, where u solves

$$du(t) \quad = \quad \Big(r_u \Delta u^{[q]}(t) - \chi \mathrm{div}\big(\eta(t) \nabla v(t)\big)\Big)\, dt + \sigma_u u(t)\, dW_1(t),$$

and v solves

$$dv(t) \quad = \quad \Big(r_v \Delta v(t) + \beta\eta(t) - \alpha v(t)\Big)\, dt + \sigma_v v(t)\, dW_2(t), \qquad t \in [0,T].$$

Problem

However, there is a small problem. There is no solution to the systems. The nonlinearity is not controllable. (Do not forget, $\mathcal{X}$ has to be chosen to be invariant).

Solution

Cut Off function ψ. I.e., a function ψ such that $\psi \in C_c^\infty(\mathbb{R})$ and satisfy

$$\psi(x) := \begin{cases} 0 & \text{if } |x| \geq 2, \\ \in [0,1] & \text{if } |x| \in (1,2), \\ 1 & \text{if } |x| \leq 1, \end{cases}$$

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The Cut Off

Let ψ be a cut-off function such that $\psi \in C_c^\infty(\mathbb{R})$ and satisfy

$$\psi(x) := \begin{cases} 0 & \text{if } |x| \geq 2, \\ \in [0,1] & \text{if } |x| \in (1,2), \\ 1 & \text{if } |x| \leq 1, \end{cases}$$

and $\psi_\kappa(x) := \psi(x/\kappa)$, $x \in \mathbb{R}$, $\kappa \in \mathbb{N}$. In addition, for any η of progressively measurable process with $\eta \in \mathcal{M}_{\mathfrak{A}}^{m_\epsilon}(0,T; H_{r_\epsilon}^{-1/2}(\mathcal{O}))$ and $\xi \in \mathcal{M}_{\mathfrak{A}}^{m_\epsilon}(0,T; H_{r_\epsilon}^{1/2}(\mathcal{O}))$ let us define

$$h^1(\eta,t) := \|\mathbb{1}_{[0,t)}(\cdot)\eta\|_{L^{m_\epsilon}(0,T;H_{r_\epsilon}^{-1/2})} \quad \text{and} \quad h^2(\xi,t) := \|\mathbb{1}_{[0,t)}(\cdot)\xi\|_{L^{m_\epsilon}(0,T;H_{r_\epsilon}^{1-1/2})}.$$

Underlying spaces

The underlying space

$$\mathcal{M}_{\mathfrak{A}}^{m_\epsilon}(0,T) \quad := \quad \left\{ \xi \in L^{m_\epsilon}(0,T; H_{r_\epsilon}^{-\frac{1}{2}}) : \mathbb{E}\int_0^T |\xi(s)|^{m\epsilon}_{H_{r_\epsilon}^{-\frac{1}{2}}}\, ds < \infty \right\}$$

The subspace where the operator is defined

$$\mathcal{X}_{\mathfrak{A}}(R_1,R_2) := \Big\{ \xi \in \mathcal{M}_{\mathfrak{A}}^{m_\epsilon}(0,T) : \mathbb{E}\Big(\sup_{0\le s\le T} |\eta(s)|^2_{H_2^{-1}} + \int_0^T |\eta(s)|^{q+1}_{L^{q+1}}\, ds \Big) \le R_1$$
$$\mathbb{E} \sup_{0\le s\le T} \big(\ln(\eta(s)) - 1)\eta(s) \big)\, ds + \int_0^T |\eta(s)|^q_{H_q^\theta}\, ds \le R_2 \Big\}.$$

Definition of the Operator:

Given $n_1, n_2 \in \mathbb{N}$. Let us define the operator $\mathcal{T}_{n_1,n_2}$ acting on $\mathcal{X}_{\mathfrak{A}}(R_1,R_2)$ as follows. For $\eta \in \mathcal{X}$, let $\mathcal{T}_{n_1,n_2}\eta := u_{n_1,n_2}$, where $(u_{n_1,n_2}, v_{n_1,n_2})$ is the solution of the coupled system

$$du(t) \quad = \quad \Big(r_u \Delta u^{[q]}(t) - \chi h^1_{n_1}(\eta,t) h^2_{n_2}(\nabla v,t)\, \mathrm{div}\big(\eta(t)\nabla v(t)\big)\Big)\, dt + \sigma_u u(t)\, dW_1(t),$$

and v solves

$$dv(t) \quad = \quad \Big(r_v \Delta v(t) + \beta h_2^{n_2}(\nabla v,t)\eta(t)) - \alpha v(t)\Big)\, dt + \sigma_v v(t)\, dW_2(t), \qquad t \in [0,T].$$

Verification of the Conditions for the Schauder-Tychonoff type fix point Theorem

In this step, we verify all the conditions required to apply Theorem Schauder Fix Theorem. In particular, we will show:

1. well-posedness of the operator $\mathcal{T}_{n_1,n_2}$ on $\mathcal{X}_{\mathfrak{A}}(R_1, R_2)$ for all $R_1, R_2 > 0$;
2. existence of $R_1^{n_1,n_2}, R_2^{n_1,n_2} > 0$ such that the set $\mathcal{X}_{\mathfrak{A}}(R_1, R_2)$ is invariant under $\mathcal{T}_{n_1,}$ all $R_1 \geq R_1^{n_1,n_2}$ and $R_2 \geq R_2^{n_1,n_2}$;
3. continuity of the operator $\mathcal{T}_{n_1,n_2}$, restricted to $\mathcal{X}_{\mathfrak{A}}(R_1, R_2)$, in the topology of $\mathcal{M}^{m_\epsilon}(0, T; H_{r_\epsilon}^{-1/2}(\mathcal{O}))$ for all $R_1, R_2 > 0$;
4. precompactness in $\mathcal{M}^{m_\epsilon}(0, T; H_{r_\epsilon}^{-1/2}(\mathcal{O}))$ of the image of $\mathcal{X}_{\mathfrak{A},W}(R_1, R_2)$ under th operator $\mathcal{T}_{n_1,n_2}$ for all $R_1, R_2 > 0$;
5. the existence of $m_0 > m$ such that

$$\mathbb{E} \left\| \mathcal{T}_{n_1,n_2} \eta \right\|^{m_0}_{L^{m_\epsilon}(0,T;H_{r_\epsilon}^{-\frac{1}{2}})} \leq C$$

for all $\eta \in \mathcal{X}_{\mathfrak{A}}(R_1, R_2)$.

Verification of the Conditions for the Schauder-Tychonoff type fix point Theorem

1. Well posedness of the operator ⇒ Standard literature.
   $(\circ)^n_v$: Semigroup approach gives existence, uniqueness, and exact regularity of the process v (see Da Prato and Zabczyk)
   $(\circ)^n_u$: Variational Ansatz (see Wei and Röckner)

$$A(u,t) := r_u \Delta |u|^{q-1} u + \chi h_1^n(\xi,t) h_2^n(v,t) \mathrm{div}\big(\xi(t)\nabla v(t)\big) + \text{noise}$$

2. Continuity: Straightforward
3. Compactness: Choice of the interpolation space! $m \mapsto m+\epsilon$ increased θ_ϵ, however, we chosed $\frac{1}{2}$;

Preperation for the globalising step

Reduction of one cut-off parameter

Since, we can show that there exists a constant $C > 0$ such that

$$\left\| \nabla v_{n_1,n_2} \right\|_{L^{m_\epsilon}(0,T;H^{1-1/2}_{r_\epsilon})} \leq C \left\| u_{n_1,n_2} \right\|_{L^{m_\epsilon}(0,T;H^{-1/2}_{r_\epsilon})}.$$

Glueing together the solutions (technical)

This is not that simple, since we do not have uniqness, however one can manage it.

Non-negativity

- for porous media operator: see book of Wei Röckner, or Barbu, Da Prato, and Röckner
- for the chemotaxis term: See Winkler.

Uniform bound

Here, we will show, that there exists a constant $C > 0$ such that for all $n \in \mathbb{N}$ we have

$$\frac{1}{4}\,\mathbb{E}\Big[\sup_{0\le s\le t}|\bar{u}_n(s)|^2_{L^2}\Big] + r_u\int_0^s |\bar{u}_n^{q-1}(\nabla \bar{u}_n(r))^2|_{L^1}\,ds + \alpha\,\mathbb{E}\Big[\int_0^t |\nabla \bar{v}^{n_1,n_2}(s)|^4_{L^4}\,ds\Big]$$
$$+\frac{1}{4}\,\mathbb{E}\Big[\sup_{0\le s\le t}|\bar{v}^{n_1,n_2}(s)|^4_{H^1_4}\Big] + \frac{r_v}{2}\mathbb{E}\Big[\int_0^t |\nabla \bar{v}^{n_1,n_2}(s)|^4_{H^1_4}\,ds\Big] \le C.$$

$\Rightarrow$ The proof is similar to the proof before

Stroock-Varuopoulos type inequality

For any $\gamma > 1$, $\theta \in (0, \frac{1}{\gamma})$, there exists a constant $C > 0$ such that

$$\|\eta\|^{2\gamma}_{L^{2\gamma}(0,T;H^\theta_{2\gamma})} \le C\int_0^T |\eta(s)^{[\gamma-1]}\nabla\eta(s)|^2_{L^2}\,ds,$$

where $z^{[\gamma-1]} := |z|^{\gamma-2}z$ for $z \in \mathbb{R}$.

Uniform bound

Here, we will show, that there exists a constant $C > 0$ such that for all $n \in \mathbb{N}$ we have

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where $z^{[\gamma-1]} := |z|^{\gamma-2}z$ for $z \in \mathbb{R}$.

Globalising the solution (standard)

Remind: The inner part of the cut-off function is given by

$$h(u_n, v_n, u, t) := \|u_n\|_{L^{m_\epsilon}(0,T;H_{r_\epsilon}^{-\frac{1}{2}})}$$

Due to the uniform bounds on u_n and v_n, there exists a constant $C > 0$ such that

$$\mathbb{E}\|u_n\|^{m_\epsilon}_{L^{m_\epsilon}(0,T;H_{r_\epsilon}^{-1/2})} + \mathbb{E}\|\nabla v\|^{m_\epsilon}_{L^{m_\epsilon}(0,T;H_{r_\epsilon}^{1/2})} \leq C$$

Fix $n \in \mathbb{N}$ and let $A_n \subset \Omega$ be given by

$$A_n := \left\{ \|\eta\|_{L^{m_\epsilon}(0,T;H_{r_\epsilon}^{-\frac{1}{2}})}, \|\nabla v\|_{L^{m_\epsilon}(0,T;H_{r_\epsilon}^{\frac{1}{2}})} \leq n \right\}.$$

Globalising the solution

Fix $n \in \mathbb{N}$ and let $A_n \subset \Omega$ be given by

$$A_n := \left\{ \|\eta\|_{L^{m_\epsilon}(0,T;H^{-\frac{1}{2}}_{r_\epsilon})}, \|\nabla v\|_{L^{m_\epsilon}(0,T;H^{\frac{1}{2}}_{r_\epsilon})} \le n \right\}.$$

It is straightforward that there exists a progressively measurable process $(\bar{u}_n, \bar{v}_n)$ over $\mathfrak{A} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P})$ such that $(\bar{u}_n, \bar{v}_n)$ solves $\mathbb{P}$–a.s. the stochastic Klausmaier system up to time τ^*_n. In addition $(\bar{u}_n, \bar{v}_n = (\bar{u}_{n+1}, \bar{v}_{n+1})$ on $[0, \tau^*_n)$ and we have for the conditioned probability

$$\mathbb{P}\left(\{\text{a solution } (u,v) \text{ to the stochastic chemotaxis system exists}\} \mid A_n\right) = 1.$$

Set $\bar{\Omega}_0 = \bigcup_{n=1}^\infty A_n$. We have by the Markov inequality and the definition of $\bar{\tau}^*_n$

$$\mathbb{P}\left(\bar{\Omega}_0 \setminus A_n\right) \le \mathbb{P}\left(\bar{\tau}^*_n \ge T\right) \le \frac{C(T)}{n} \to 0, \qquad \text{as } n \to \infty.$$

Thus the solution process is well defined on $\bar{\Omega} = \bigcup_{n=1}^\infty A_n$ with $\mathbb{P}(\bar{\Omega}) = 1$. This finishes the proof. □

Further Plans

- Its numerical modelling (adapted schemes - nonlinearity and chemotaxis term needs different treatments)
- Long Term Asymptotic's: Is it possible that for some choices of parameters a global solution exists ? Is there any blow up, or concentration on one measure?
- Invariant measure, uniqueness of the invariant measure $\Rightarrow$ (proliferation) in deterministic converges to a flat surface
- Possible equilibrium or stationary solution.
- Jump processes $\rightarrow$ Marcus integral

The End

Thank you for the attention !