On local well-posedness of the stochastic incompressible density-dependent Euler equations

Workshop on Stochastic Analysis

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Plan of the talk

Introduction

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Euler equations: incompressible fluids.

The classical 3-dimensional Euler equations are given by the following system of PDEs

$$\begin{cases} \dfrac{\partial v}{\partial t} + (v \cdot \nabla) v + \dfrac{\nabla \pi}{\rho} = f \\ \operatorname{div} \, v = 0 \end{cases}$$

where $v : [0, T] \times D \to \mathbb{R}^3$ is the velocity field of the fluid, ρ is the fluid density, $\pi : [0, T] \times D \to \mathbb{R}$ is the pressure field and f represents external forces.

This system describes a non viscous and incompressible fluid which occupies a region $D \subset \mathbb{R}^3$, and whose density only varies spatially.

Some previous results

- **Chemin, J.-Y. (1998).** *Perfect Incompressible Fluids*. Oxford Lecture Series in Mathematics and Its Applications. The Clarendon Press Oxford Univ. Press, New York.
- **Majda, A. J. and Bertozzi, A. L. (2002).** *Vorticity and Incompressible Flow*. Cambridge Texts in Applied Mathematics 27. Cambridge Univ. Press, Cambridge.
- **Bardos, K. and Titi, È. S. (2007).** *Euler equations for an ideal incompressible fluid*. Uspekhi Mat. Nauk 62 5–46.
- **Constantin, P. (2007).** *On the Euler equations of incompressible fluids*. Bull. Amer. Math. Soc. (N.S.) 44 603–621.

Euler equations: density dependent case.

When dealing with incompressible fluids whose **density may vary in time** due to differences in composition, temperature, or other factors, we turn to the deterministic density-dependent Euler equations:

$$\begin{cases} \rho_t + v \cdot \nabla \rho = 0, \\ \dfrac{\partial v}{\partial t} + (v \cdot \nabla) v + \dfrac{\nabla \pi}{\rho} = f, \\ \operatorname{div} \, v = 0 \,. \end{cases}$$

Some previous results

- **H. Beirão da Veiga, A. Valli (1980)** - *On the Euler equations for nonhomogeneous fluids II*: local well-posedness results, in bounded domains, by considering initial data of class C^∞.
- **S. Itoh (1994)**, *Cauchy problem for the Euler equations of a nonhomogeneous ideal incompressible fluid*: local existence of solution with initial condition in H^s in whole the space $\mathbb{R}^3$.
- **S. Itoh, A. Tani, (1999)** - *Solvability of nonstationary problems for nonhomogeneous incompressible fluids and the convergence with vanishing viscosity*: local well-posedness in the Sobolev space $W^{2,p}(U)$ with U being either bounded or unbounded smooth domain in $\mathbb{R}^3$.

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Density-dependent Euler equations: multiplicative and additive noise

We are concerned with the following model of incompressible inviscid fluid with variable density:

$$\begin{cases} d\rho + v \cdot \nabla \rho \, dt = 0, \\ dv + (v \cdot \nabla) v \, dt + \dfrac{\nabla \pi}{\rho} \, dt = -v \circ d\mathcal{W}, \qquad (\text{or } d\mathcal{W}^Q), \\ \operatorname{div} v = 0. \end{cases} \tag{1}$$

with initial conditions

$$\rho(0, x) = \rho_0(x), \qquad v(0, x) = v_0(x),$$

in $\Omega \times [0, T] \times \mathbb{R}^3$, where ρ, v and π denote the density, velocity and pressure, respectively. We work on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, where $\mathcal{W}$ is a standard scalar Brownian motion and $\mathcal{W}^Q$ is a Q-cylindrical Brownian motion on $W^{k,2}(\mathbb{R}^3)$, with $k > \frac{5}{2}$.

Our goal

Remark: The literature on the stochastic homogeneous incompressible Euler equations consists of several works of the authors **Bessaih, Brzeźniak, Flandoli, Glatt-Holtz, Kim**. They show, among other things, global well-posedness for the 2D Euler equations with additive and multiplicative noise and local well-posedness for $d = 3$.

Objetive: To show the local existence and uniqueness of solutions of the stochastic inhomogeneous Euler equation (1) in the whole space $\mathbb{R}^3$.

Main theorem

Local pathwise solutions

Definition

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a stochastic basis with a complete right-continuous filtration, and $\mathcal{W}$ a Brownian process relative to the filtration.
Assuming the conditions $3 < p < \infty$ and $(\rho_0, v_0) \in C(\mathbb{R}^3) \times W^{2,p}(\mathbb{R}^3)$, we call $(\rho, \nabla\pi, v, \tau)$ a **local pathwise solution** to the system (1) provided

(i) τ is an a.s. strictly positive $(\mathcal{F}_t)$-stopping time;

(ii) the density ρ is a $C(\mathbb{R}^3)$- progressively measurable process satisfying

$$0 < m \leq \rho(\cdot \wedge \tau) \leq M, \quad \nabla\rho(\cdot \wedge \tau) \in C\left([0, T]; W^{1,p}(\mathbb{R}^3)\right) \qquad \mathbb{P} - a.s$$

for some constants m and M;

(iii) the velocity v and $\nabla\pi$ are $W^{2,p}(\mathbb{R}^3)$- progressively measurable process satisfying

$$v(\cdot \wedge \tau), \nabla\pi(\cdot \wedge \tau) \in C\left([0,T]; W^{2,p}(\mathbb{R}^3)\right) \qquad \mathbb{P} - a.s;$$

(iv) there holds $\mathbb{P}$-a.s. in the weak sense

$$\rho(t \wedge \tau) = \rho_0 - \int_0^{t\wedge\tau} v \cdot \nabla\rho \, ds,$$

$$v(t \wedge \tau) = v_0 - \int_0^{t\wedge\tau} v \cdot \nabla v \, ds - \int_0^{t\wedge\tau} \frac{\nabla\pi}{\rho} \, ds - \int_0^{t\wedge\tau} v \circ d\mathcal{W}_s,$$

$$\operatorname{div} v = 0,$$

for every $t \in [0,T]$.

Uniqueness of solutions

Definition

We say that local pathwise solutions are unique if, given any pair $(\rho^1, \pi^1, v^1, \tau^1)$, $(\rho^2, \pi^2, v^2, \tau^2)$ of local pathwise solutions with the same initial condition (ρ_0, v_0) we get

$$\mathbb{P}\left(\left(\rho^1, \pi^1, v^1\right)(t) = \left(\rho^2, \pi^2, v^2\right)(t); \forall t \in [0, \tau^1 \wedge \tau^2]\right) = 1.$$

Our result: Local well-posedness

Theorem (E., Olivera, Mollinedo)

Suppose that for $3 < p < \infty$*, the function* ρ_0 *satisfies*

$$\rho_0 \in C(\mathbb{R}^3), \quad 0 < m \leq \rho_0 \leq M, \quad \nabla\rho_0 \in W^{1,p}(\mathbb{R}^3), \tag{2}$$

and the vector field v_0 *verifies*

$$v_0 \in W^{2,p}(\mathbb{R}^3), \quad div\ v_0 = 0\,. \tag{3}$$

Then, there exists a unique local pathwise solution $(\rho, \nabla\pi, v, \tau)$ *for Problem* (1).

Sketch of the proof

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Change of variables

$$\boxed{\tilde{v} = e^{\mathcal{W}(t)} v}$$

Using the well-known chain rule for the Stratonovich integral,

$$dv + (v \cdot \nabla) v \, dt + \frac{\nabla \pi}{\rho} \, dt = -v \circ d\mathcal{W}_t$$

$$\Downarrow$$

$$d\tilde{v} = -z^{-1}(\tilde{v} \cdot \nabla)\tilde{v} \, dt - \frac{z}{\rho} \nabla \pi \, dt,$$

where $z(t) = e^{\mathcal{W}(t)}$.

Auxiliary problems

- **Transport equation**

$$\begin{cases} \rho_t + z^{-1}\tilde{v} \cdot \nabla \rho = 0, \\ \rho\big|_{t=0} = \rho_0. \end{cases}$$

- **Elliptic equation**

$$\operatorname{div}(\rho^{-1}\nabla \pi) = -z^{-2} \sum_{i,j=1}^{3} \tilde{v}^i_{x_j} \tilde{v}^j_{x_i}.$$

- **Linear Euler equation**

$$\begin{cases} u_t + z^{-1}(\tilde{v} \cdot \nabla) u + z \dfrac{\nabla \pi}{\rho} = 0, \\ u\big|_{t=0} = v_0. \end{cases}$$

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A priori estimates

Assume

$$\tilde{v} \in C([0,T]; W^{2,p}(\mathbb{R}^3)), \ \operatorname{div} \tilde{v} = 0$$

then there exist unique solutions satisfying

- $\rho \in C^{1,1}([0,T] \times \mathbb{R}^3)$:
$$m \le \rho \le M, \qquad \|\nabla \rho(t)\|_{W^{1,p}} \le \|\nabla \rho_0\|_{W^{1,p}} e^{c_1 \int_0^t z^{-1}(s)\|\tilde{v}(s)\|_{W^{2,p}}\, ds}.$$
- $\nabla \pi \in C([0,T]; W^{2,p}(\mathbb{R}^3))$:
$$\|\nabla \pi(t)\|_{W^{2,p}} \le z^{-2}(t) K(\|\nabla \rho(t)\|_{W^{1,p}}) \|\tilde{v}(t)\|^2_{W^{2,p}}.$$
- $u \in C([0,T]; W^{2,p}(\mathbb{R}^3))$:
$$\|u(t)\|_{W^{2,p}} \le e^{c_2 \int_0^t z^{-1}(s)\|\tilde{v}(s)\|_{W^{2,p}}\, ds} \left[\|v_0\|_{W^{2,p}} + \int_0^t z^{-1}(s) L(\|\nabla \rho(s)\|_{W^{1,p}}) \|\tilde{v}(s)\|^2_{W^{2,p}}\, ds \right].$$

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Iterative construction

Let $\tilde{v}^{(0)} = 0$, and for $k \in \mathbb{N}$, we iteratively define $\rho^{(k)}$, $\pi^{(k)}$ and $u^{(k)}$ as the corresponding solutions of the following problems

$$\begin{cases} \rho_t^{(k)} + z^{-1}\tilde{v}^{(k-1)} \cdot \nabla \rho^{(k)} = 0, \\ \rho^{(k)}|_{t=0} = \rho_0(x), \end{cases} \tag{4}$$

$$\left\{ \operatorname{div}\left(\frac{1}{\rho^{(k)}} \nabla \pi^{(k)}\right) = -z^{-2} \sum_{i,j=1}^{3} \tilde{v}_{x_j}^{(k-1),i} \tilde{v}_{x_i}^{(k-1),j}, \right. \tag{5}$$

$$\begin{cases} u_t^{(k)} + z^{-1}\tilde{v}^{(k-1)} \cdot \nabla u^{(k)} + \frac{z}{\rho^{(k)}} \nabla \pi^{(k)} = 0 \\ u^{(k)}|_{t=0} = v_0(x), \end{cases} \tag{6}$$

Finally, we define

$$\tilde{v}^{(k)} = u^{(k)} - \nabla \phi^{(k)} \tag{7}$$

where $\phi^{(k)}$is the solution of

$$\Delta \phi^{(k)} = \operatorname{div}\ u^{(k)}. \tag{8}$$

Boundness of the aproximations

Lemma

If we define the stopping time

$$\tau = \inf\{t \geq 0 : \int_0^t z^{-1}(s)\, ds \geq A^{-2}\} \wedge T,$$

with $A \geq \max\{c_3 exp(c_2) \left[\|v_0\|_{2,p} + L\left(\|\nabla \rho_0\|_{1,p} exp(c_1)\right)\right], 1\}$.

Then, the sequence $\left\{\tilde{v}^{(k)}\right\}_k$ *is bounded in* $C([0,\tau]; W^{2,p}(\mathbb{R}^3))$*:*

$$\sup_{0 \leq t \leq \tau} \|\tilde{v}^{(k)}(t)\|_{2,p} \leq A.$$

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Sequences of differences

We consider the sequences of differences given by: $\sigma^{(k)} = \rho^{(k)} - \rho^{(k-1)}$, $h^{(k)} = u^{(k)} - u^{(k-1)}$, $q^{(k)} = \pi^{(k)} - \pi^{(k-1)}$, $w^{(k)} = v^{(k)} - v^{(k-1)}$ and $\tilde{w}^{(k)} = \tilde{v}^{(k)} - \tilde{v}^{(k-1)}$. Subtracting the equations in each of the problems (4), (5) and (6) corresponding to the steps (k) and $(k-1)$, we obtain

$$\begin{cases} \sigma_t^{(k)} + z^{-1}\tilde{v}^{(k-1)} \cdot \nabla\sigma^{(k)} = -z^{-1}\nabla\rho^{(k-1)} \cdot \tilde{w}^{(k-1)}, \\ \sigma^{(k)}|_{t=0} = 0, \end{cases} \tag{9}$$

$$\begin{cases} \operatorname{div}\left(\frac{1}{\rho^{(k)}} z\nabla q^{(k)}\right) & = \operatorname{div}\left(\frac{\sigma^{(k)}}{\rho^{(k)}\rho^{(k-1)}} z\nabla\pi^{(k-1)}\right) - z^{-1}\sum_{i,j=1}^{3} \tilde{w}_{x_j}^{(k-1),i}\,\tilde{v}_{x_i}^{(k-1),j} \\ & \quad -z^{-1}\sum_{i,j=1}^{3} \tilde{v}_{x_j}^{(k-2),i}\,\tilde{w}_{x_i}^{(k-1),j}, \end{cases} \tag{10}$$

$$\begin{cases} h_t^{(k)} + z^{-1}\tilde{v}^{(k-1)} \cdot \nabla h^{(k)} + z\frac{\nabla q^{(k)}}{\rho^{(k)}} = z\frac{\sigma^{(k)}}{\rho^{(k-1)}\rho^{(k)}}\nabla\pi^{(k-1)} - z^{-1}\tilde{w}^{(k-1)}\nabla u^{(k-1)}, \\ h^{(k)}|_{t=0} = 0. \end{cases} \tag{11}$$

Estimates

Lemma

Let $\sigma^{(k)}$, $q^{(k)}$ and $h^{(k)}$ be the solution of the system (9), (10) and (11), respectively, then

$$\|\sigma^{(k)}(t)\|_{1,p} \leq L_5 \int_0^t z^{-1}(s)\|\tilde{w}^{(k-1)}(s)\|_{1,p}\, ds,$$

$$\|\nabla q^{(k)}(t)\|_{1,p} \leq L_6 \left(\|\sigma^{(k)}(t)\|_{1,p} + z^{-2}(t)\|\tilde{w}^{(k-1)}(t)\|_{1,p}\right),$$

$$\|h^{(k)}(t)\|_{1,p} \leq L_9 \int_0^t \left(z(s)\|\nabla q^{(k)}(s)\|_{1,p} + z^{-1}(s)\|\tilde{w}^{(k-1)}(s)\|_{1,p} + z(s)\,\|\sigma^{(k)}(s)\|_{1,p}\right) ds,$$

for all $t \in [0, \tau]$.

Existence

The sequence $\{\tilde{v}^{(k)}\}_k$ converge in $C([0,\tau]; W^{1,p}(\mathbb{R}^3))$ to a random variable $\tilde{v} \in C([0,\tau]; W^{2,p}(\mathbb{R}^3))$.

- From (8) we have: $\|\tilde{w}^{(k)}\|_{1,p} \leq \|h^{(k)}\|_{1,p} + \|\nabla(\phi^{(k)} - \phi^{(k-1)})\|_{1,p} \leq c_9 \|h^{(k)}\|_{1,p}$.
- Using the previous lemmas

$$\|\tilde{w}^{(k)}(t)\|_{1,p} \leq L_{10} \int_0^t z^{-1}(s) \|\tilde{w}^{(k-1)}(s)\|_{1,p} \, ds,$$

- $\sup_{0\leq t\leq\tau} \|\tilde{w}^{(k)}(t)\|_{1,p} \leq A\, L_{10}^{k-1} \sup_{0\leq t\leq\tau} (z^{-1}(s))^{k-1} \frac{\tau^{k-1}}{(k-1)!}$.
- Finally, we get:

$$\sum_{k=1}^{\infty} \|\tilde{w}^{(k)}\|_{C([0,\tau];W^{1,p}(\mathbb{R}^3))} < \infty.$$

Existence

This implies that $(\rho^{(k)}, \nabla\pi^{(k)}, u^{(k)}, \tilde{v}^{(k)}) \to (\rho, \nabla\pi, u, \tilde{v})$, as $k \to \infty$ (in the respective spaces according the definition of local pathwise solution), which satisfies the desired equations:

$$\begin{cases} \rho_t + z^{-1}\tilde{v} \cdot \nabla\rho = 0 \\ \rho|_{t=0} = \rho_0(x) \end{cases}$$

$$\mathrm{div}(\rho^{-1}(t)\nabla\pi(t)) = -z^{-2}(t)\mathrm{div}(\tilde{v}(t) \cdot \nabla\tilde{v}(t)),$$

$$\begin{cases} u_t + z^{-1}\tilde{v} \cdot \nabla u + z\dfrac{\nabla\pi}{\rho} = 0, \\ u|_{t=0} = v_0(x). \end{cases}$$

$$\tilde{v} = u - \nabla\phi,$$

with ϕ being a weak solution of the elliptic equation

$$\Delta\phi = \mathrm{div}\ u.$$

Existence

Proposition

If $v = z^{-1}\tilde{v}$, then $(\rho, \nabla\pi, v, \tau)$ determines a local pathwise solution for Problem 1.

The key is to show that $u = \tilde{v}$.

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Uniqueness of local solutions

Proposition

The solution for Problem (1) is pathwise unique.

Proof of uniqueness

Let $(\rho^1, \pi^1, v^1, \tau_1)$ and $(\rho^2, \pi^2, v^2, \tau_2)$ be two solutions with the same initial data, and let $\tau = \tau_1 \wedge \tau_2$.

Define

$$\sigma = \rho^2 - \rho^1, \quad q = \pi^2 - \pi^1, \quad \tilde{w} = \tilde{v}^2 - \tilde{v}^1.$$

Subtracting the equations gives a system for $(\sigma, q, \tilde{w})$. Using the same estimates as before,

$$\frac{d}{dt}\|\tilde{w}(t)\|_{W^{1,p}} \leq L_{12}\|\tilde{w}(t)\|_{W^{1,p}}.$$

Hence, by Gronwall's lemma,

$$(\rho^1, \nabla\pi^1, v^1) = (\rho^2, \nabla\pi^2, v^2) \quad \text{on } [0, \tau], \text{ a.s.}$$

End: Thank you!

D. Breit, E. Feireisl, M. Hofmanová. Local strong solutions to the stochastic compressible Navier–Stokes system. Communications in Partial Differential Equations, 43(2), 313-345, 2018.

C. Espitia, D. A. C. Mollinedo, and C. H. Olivera. *On local well-posedness of the stochastic incompressible density-dependent Euler equations.* Indiana University Mathematics Journal, to appear.

S. Itoh, A. Tani, Solvability of nonstationary problems for nonhomogeneous incompressible fluids and the convergence with vanishing viscosity. Tokyo Journal of Mathematics, 22, 1999.

S. Itoh, Cauchy problem for the Euler equations of a nonhomogeneous ideal incompressible fluid. J. Korean Math. Soc. 31 , 1994.

J.U. Kim, On the stochastic Euler equations in a two-dimensional domain, SIAM J. Math. Anal. 33, 2002.