Densities of Hermite processes

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Outline

The main object

Main object: the so-called *Hermite process*.

$\Rightarrow$ these are self-similar stochastic processes with stationary increments;

$\Rightarrow$ the class of Hermite processes includes the fractional Brownian motion (fBm) $(B_t^H, t \geq 0)$. This is a centered Gaussian process with covariance function

$$\mathbb{E}\left[B_t^H B_s^H\right] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \quad s, t \geq 0;$$

$\Rightarrow$ except for the fBm, the Hermite processes are non-Gaussian.

Definition

We denote by $(Z_t^{(q,H)})_{t\geq 0}$ the Hermite process of order $q \geq 1$ and with self-similarity index $H \in \left(\frac{1}{2}, 1\right)$.

$\Rightarrow$ it lives in the qth Wiener chaos and it is defined as a multiple Wiener–Itô integral: for every $t \geq 0$,

$$Z_t^{(q,H)} = d(q,H) \int_{\mathbb{R}} dB(y_1) \dots \int_{\mathbb{R}} dB(y_q)$$
$$\left(\int_0^t (s-y_1)_+^{-\left(\frac{1}{2}+\frac{1-H}{q}\right)} \dots (s-y_q)_+^{-\left(\frac{1}{2}+\frac{1-H}{q}\right)} ds \right),$$

where $(B(y))_{y\in\mathbb{R}}$ is a Wiener process indexed by $\mathbb{R}$ and $d(q,H) > 0$ is a normalizing constant chosen so that $\mathbb{E}\left[\left(Z_1^{(q,H)}\right)^2\right] = 1$.

Chaos representation

We can also write

$$Z_t^{(q,H)} = I_q(L_t)$$

with

$$\begin{aligned} & L_t(y_1, \ldots, y_q) \\ = \quad & d(q,H) \left(\int_0^t (s-y_1)_+^{-\left(\frac{1}{2}+\frac{1-H}{q}\right)} \ldots (s-y_q)_+^{-\left(\frac{1}{2}+\frac{1-H}{q}\right)} ds \right), \end{aligned}$$

where I_q is the multiple stochastic integral of order q with respect to B.

The kernel L_t belongs to $L^2(\mathbb{R}^q)$.

So, for every t, the random variable $Z_t^{(q,H)}$ belongs to the qth Wiener chaos.

Properties

Main properties:

- the Hermite process is H-self-similar, i.e. for every $c > 0$,

$$\left(Z_{ct}^{(q,H)}, t \geq 0\right) \stackrel{(\mathrm{d})}{=} \left(c^H Z_t^{(q,H)}, t \geq 0\right);$$

- it has stationary increments, i.e. for every $h > 0$,

$$\left(Z_{t+h}^{(q,H)} - Z_h^{(q,H)}, t \geq 0\right) \stackrel{(\mathrm{d})}{=} \left(Z_t^{(q,H)}, t \geq 0\right);$$

- $Z^{(q,H)}$ has long memory: with $r(n) = \mathbb{E}\left[Z_1^{(q,H)}\left(Z_{n+1}^{(q,H)} - Z_n^{(q,H)}\right)\right]$ we have $r(n) \sim H(2H-1)n^{2H-2}$ and $\sum_{n\geq 1} r(n) = \infty$.

Properties (continued)

- its trajectories are Hölder continuous of any order $\delta \in (0, H)$;
- the covariance of the Hermite process $Z^{(q,H)}$ is the same for all integers $q \geq 1$ and it coincides with the covariance of the fBm, i.e.

$$\mathbb{E}\left[Z_t^{(q,H)} Z_s^{(q,H)} \right] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H} \right), \qquad s, t \geq 0;$$

- but the covariance does *not* determine the law as soon as $q \geq 2$.

Examples

The class of Hermite processes includes

- the fractional Brownian motion, which is obtained for $q = 1$;
- the Rosenblatt process ($q = 2$).

$\Rightarrow$ the fBm is the only Gaussian Hermite process; the Hermite process is non-Gaussian as soon as $q \geq 2$.

Where do Hermite processes come from?

Non-central limit theorem (Taqqu 1979, Dobrushin–Major 1979).

Let $(\xi_k)_{k\geq 1}$ be a centered stationary Gaussian sequence with $\mathbb{E}\left[\xi_0^2\right] = 1$ and

$$\rho(k) = \mathbb{E}\left[\xi_0 \xi_k\right] = k^{-D}\ell(k), \qquad \ell \text{ slowly varying.}$$

Let $g : \mathbb{R} \to \mathbb{R}$ with $\mathbb{E}\left[g(\xi_0)\right] = 0$, $\mathbb{E}\left[g(\xi_0)^2\right] < \infty$ and *Hermite rank* $q \geq 1$.

If $0 < D < \frac{1}{q}$ (long memory), then, in the sense of finite dimensional distributions,

$$\frac{1}{n^H} \sum_{k=1}^{\lfloor nt \rfloor} g(\xi_k) \xrightarrow[n\to\infty]{} c\, Z_t^{(q,H)}, \qquad H = 1 - \frac{qD}{2} \in \left(\tfrac{1}{2}, 1\right).$$

The Rosenblatt case $q = 2$

$\Rightarrow$ for $q = 2$ we have some knowledge about the law of the marginal distribution (for instance, it is determined by its moments).

$\Rightarrow$ indeed, we have the following representation:

$$Z_1^{(2,H)} = \sum_{n\geq 1} \lambda_n \left(N_n^2 - 1\right),$$

with $(N_n)_{n\geq 1}$ i.i.d. $N(0,1)$ and $(\lambda_n)_{n\geq 1}$ the eigenvalues of the Hilbert–Schmidt operator associated with the kernel L_1 (so that $\lambda_n > 0$ and $2\sum_{n\geq 1} \lambda_n^2 = 1$).

Density of the marginal distributions

We know that, for fixed $t > 0$, the random variable $Z_t^{(q,H)}$ admits a density with respect to the Lebesgue measure.

$\Rightarrow$ this is an old result by Shigekawa, which states that a (non-trivial) multiple stochastic integral admits a density;

$\Rightarrow$ no information about the densities of the finite dimensional distributions;

$\Rightarrow$ no information about the smoothness of the densities.

Purpose of the talk

Let $n \geq 2$ and let $t_1, \ldots, t_n$ be distinct times in $(0, \infty)$ with

$$0 < t_1 < \ldots < t_n.$$

Let $Z^{(q,H)} := Z$ be a Hermite process of order $q \geq 1$.

Consider the random vector

$$(Z_{t_1}, \ldots, Z_{t_n}).$$

$\Rightarrow$ **Goal:** analyze the law of this random vector.

The question

Question: is the law of this vector absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^n$?

$\Rightarrow$ we will use Malliavin calculus to analyze this question.

$\Rightarrow$ joint project with L. Loosveldt (Liège), Y. Nachit (Lille) and I. Nourdin (Luxembourg).

Malliavin calculus: notation

Let $D : \mathrm{Dom}(D) = \mathbb{D}^{1,2} \subset L^2(\Omega) \to L^2(T \times \Omega)$ be the Malliavin derivative with respect to an isonormal process

$$W(h), \quad h \in \mathfrak{H} = L^2(T).$$

(For Hermite processes: $T = \mathbb{R}$, $\mathfrak{H} = L^2(\mathbb{R})$ and $W(h) = \int_{\mathbb{R}} h(y)\, dB(y)$.)

We have

$$D_t I_q(f) = q\, I_{q-1}\left(f(\cdot, t)\right)$$

if $f \in L^2_S(T^q)$; in particular $D_s Z_t = q\, I_{q-1}\left(L_t(\cdot, s)\right)$.

Also,

$$D_t g(W(h)) = g'(W(h))\, h(t)$$

if g is differentiable (with the usual integrability conditions).

The Bouleau–Hirsch criterion

A cornerstone of Malliavin calculus is the *Bouleau–Hirsch criterion*.

$\Rightarrow$ it asserts that a random vector $(Z_1, \dots, Z_n)$ with Malliavin differentiable components has an absolutely continuous distribution with respect to the Lebesgue measure on $\mathbb{R}^n$ as soon as its *Malliavin matrix*

$$\Gamma_Z = \Big(\langle DZ_i, DZ_j \rangle_{\mathfrak{H}} \Big)_{1 \le i,j \le n}$$

is almost surely non-singular.

Comments on the criterion

$\Rightarrow$ if Z is a Gaussian vector, the Malliavin matrix reduces to the covariance matrix.

$\Rightarrow$ this extends the Gaussian situation, where absolute continuity is equivalent to the non-degeneracy of the covariance matrix, which in that case coincides with the Malliavin matrix.

$\Rightarrow$ when the components of Z lie in a fixed Wiener chaos, the above result can be strengthened.

$\Rightarrow$ in this setting, the almost sure invertibility of Γ_Z is actually *equivalent* to the absolute continuity of the law of Z.

Strategy

$\Rightarrow$ to prove that the finite dimensional vector $(Z_{t_1}, \dots, Z_{t_n})$ has a density, it is enough to show that its Malliavin matrix is almost surely non-singular.

$\Rightarrow$ in the case of Hermite processes, our strategy is to control the determinant of this matrix.

$\Rightarrow$ we combine two ingredients: a *factorization identity* for the Malliavin matrix, and a *strong local nondeterminism* estimate formulated at the level of Malliavin derivatives.

Sketch of the proof when $q = 1$

Let $B^H = (B^H_t)_{t\geq 0}$ denote a fractional Brownian motion with Hurst parameter $H \in (0,1)$. Fix $0 = t_0 < t_1 < \cdots < t_n$ and set $Z_k := B^H_{t_k}$.

Step 1 (determinant and conditional variances). For any Gaussian vector, the determinant of the covariance matrix equals the product of the conditional variances (see Berman):

$$\det \mathrm{Cov}(Z_1, \ldots, Z_n) \; = \; \mathrm{Var}(Z_1) \prod_{k=2}^{n} \mathrm{Var}\left(Z_k \mid Z_1, \ldots, Z_{k-1}\right).$$

Sketch of the proof when $q = 1$: SLND

Step 2 (SLND). Using the self-similarity and the stationarity of the increments, Pitt showed that the fractional Brownian motion enjoys the *strong local nondeterminism* property: there exists $c_H \in (0, 1]$ such that

$$\mathrm{Var}\Big(Z_k \,\Big|\, Z_1, \dots, Z_{k-1}\Big) \;\geq\; c_H\,(t_k - t_{k-1})^{2H}, \qquad k = 2, \dots, n.$$

Sketch of the proof when $q = 1$: conclusion

Step 3 (conclusion). Since $(Z_1, \ldots, Z_n)$ is Gaussian, the non-degeneracy of its covariance matrix implies absolute continuity. Combining Steps 1 and 2, and since $\mathrm{Var}(Z_1) = t_1^{2H}$, we obtain

$$\det \mathrm{Cov}(Z_1, \ldots, Z_n) \;\geq\; c_H^n \prod_{k=1}^{n} (t_k - t_{k-1})^{2H} \;>\; 0,$$

so the law of $(Z_1, \ldots, Z_n)$ has a density.

From $q = 1$ to $q \geq 2$

$\Rightarrow$ in the non-Gaussian case $q \geq 2$ we adopt the same guiding philosophy, but the passage from the Gaussian to the non-Gaussian setting requires substantial innovations: the covariance matrix is replaced by the Malliavin matrix, and the determinant identity for the covariance matrix by the corresponding identity for the Malliavin matrix;

$\Rightarrow$ SLND is replaced by a new analogue at the level of Malliavin derivatives. To establish this extension of SLND, we prove self-similarity and stationarity of increments *for the Malliavin derivative*.

Determinant identity for the Malliavin matrix

Let $Z = (Z_1, \dots, Z_n) \in \left(\mathbb{D}^{1,2}\right)^n$ and let Γ_Z denote its Malliavin matrix. Then, almost surely,

$$\det(\Gamma_Z) = \|DZ_1\|^2_{\mathfrak{H}} \prod_{j=2}^{n} \left\|DZ_j - \mathrm{proj}_{E_{j-1}}(DZ_j)\right\|^2_{\mathfrak{H}},$$

where E_{j-1} is the linear span in $\mathfrak{H}$ of $\{DZ_1, \dots, DZ_{j-1}\}$ and $\mathrm{proj}_{E_{j-1}}$ denotes the orthogonal projection onto E_{j-1}.

$\Rightarrow$ the proof is based on the classical argument used to compute the determinant of a Gram matrix.

Next step

Next step: establish the analogue of the strong local nondeterminism estimate for Hermite processes.

$\Rightarrow$ in the Gaussian case (fBm), the proof is based on the self-similarity and on the stationarity of the increments of the fBm;

$\Rightarrow$ we therefore first need to strengthen the classical properties of self-similarity and stationarity of increments, now at the level of the Malliavin derivative.

Self-similarity of the Malliavin derivative

Let $Z = \{Z_t^{(q,H)},\ t \in \mathbb{R}\}$ be the Hermite process of order $q \geq 1$ and self-similarity parameter $H \in \left(\frac{1}{2}, 1\right)$.
Then the Malliavin derivative of $Z^{(q,H)}$ inherits both stationarity of increments and self-similarity: for all $a \in \mathbb{R}$ and $c > 0$,

$$\left(\langle DZ_{s+a}^{(q,H)} - DZ_a^{(q,H)},\ DZ_{t+a}^{(q,H)} - DZ_a^{(q,H)} \rangle_{\mathfrak{H}} \right)_{s,t\in\mathbb{R}} \stackrel{\text{law}}{=} \left(\langle DZ_s^{(q,H)},\ DZ_t^{(q,H)} \rangle_{\mathfrak{H}} \right)_{s,t\in\mathbb{R}},$$

and

$$\left(\langle DZ_{cs}^{(q,H)},\ DZ_{ct}^{(q,H)} \rangle_{\mathfrak{H}} \right)_{s,t\in\mathbb{R}} \stackrel{\text{law}}{=} c^{2H} \left(\langle DZ_s^{(q,H)},\ DZ_t^{(q,H)} \rangle_{\mathfrak{H}} \right)_{s,t\in\mathbb{R}}.$$

SLND for Hermite processes

Let $Z = \{Z_t^{(q,H)},\ t \in \mathbb{R}\}$ be the Hermite process of order $q \geq 1$ with self-similarity parameter $H \in \left(\frac{1}{2}, 1\right)$. For any $j \in \mathbb{N}^*$ and any time grid

$$0 = t_0 < t_1 < \cdots < t_{j-1} < t_j,$$

one has

$$\left\| DZ_{t_j}^{(q,H)} - \mathrm{proj}_{E_{j-1}} \left(DZ_{t_j}^{(q,H)} \right) \right\|_{\mathfrak{H}}^2 \overset{\text{law}}{\geq} (t_j - t_{j-1})^{2H} \left\| DZ_1^{(q,H)} \right\|_{\mathfrak{H}_1}^2 ,$$

where $\mathfrak{H}_1 = L^2([0,1])$ and E_{j-1} is the span of $\{DZ_{t_1}^{(q,H)}, \ldots, DZ_{t_{j-1}}^{(q,H)}\}$.

Stochastic domination

The notation

$$X \overset{\text{law}}{\geq} Y$$

means that

$$\mathbb{E}[f(X)] \geq \mathbb{E}[f(Y)]$$

for all non-decreasing bounded functions $f : \mathbb{R}_+ \to \mathbb{R}$.

Proof of the absolute continuity

Let $Z = (Z^{(q,H)}_{t_1}, \dots, Z^{(q,H)}_{t_n})$ and denote by Γ_Z its Malliavin matrix. By the Bouleau–Hirsch criterion, it is enough to prove that

$$\mathbb{P}(\det(\Gamma_Z) > 0) = 1.$$

We factorize the determinant as

$$\det(\Gamma_Z) = \left\| DZ^{(q,H)}_{t_1} \right\|^2_{\mathfrak{H}} \prod_{j=2}^{n} \left\| DZ^{(q,H)}_{t_j} - \mathrm{proj}_{E_{j-1}}(DZ^{(q,H)}_{t_j}) \right\|^2_{\mathfrak{H}},$$

where $E_{j-1} = \mathrm{span}\{DZ^{(q,H)}_{t_1}, \dots, DZ^{(q,H)}_{t_{j-1}}\}$.

Proof of the absolute continuity (continued)

By the self-similarity of the Malliavin derivative,
$\|DZ_{t_1}\|^2_{\mathfrak{H}} \overset{\text{law}}{=} t_1^{2H}\,\|DZ_1\|^2_{\mathfrak{H}}$, so the first factor is almost surely positive.

For each $j = 2, \dots, n$,

$$\left\|DZ_{t_j} - \mathrm{proj}_{E_{j-1}}(DZ_{t_j})\right\|^2_{\mathfrak{H}} \overset{\text{law}}{\geq} (t_j - t_{j-1})^{2H}\,\|DZ_1\|^2_{\mathfrak{H}_1},$$

with $\mathfrak{H}_1 = L^2([0,1])$.

$\Rightarrow$ since $\|DZ_1\|^2_{\mathfrak{H}_1} > 0$ almost surely, applying the definition of $\overset{\text{law}}{\geq}$ with $f = 1_{(0,\infty)}$ shows that each factor is almost surely positive.

Conclusion: existence of the density

Therefore $\det(\Gamma_Z) > 0$ almost surely,

so

the law of

$$(Z_{t_1}, \dots, Z_{t_n})$$

admits a density with respect to the Lebesgue measure on $\mathbb{R}^n$, for every $q \geq 1$, every $H \in \left(\frac{1}{2}, 1\right)$ and every choice of the times.

Why the joint law matters

Absolute continuity of the finite dimensional distributions is the entry point for the fine analysis of the sample paths.

$\Rightarrow$ with (bounds on) the joint densities one can address:

- existence and joint continuity of the *local time* (Berman's criterion);
- Hausdorff dimension of the level sets, of the image and of the graph;
- hitting probabilities, polarity of points;
- Chung-type laws of the iterated logarithm and small-ball estimates;
- exact moduli of continuity.

$\Rightarrow$ and, on the statistical side, likelihood-based inference for long-memory data requires the density of the observed vector $(Z_{t_1}, \ldots, Z_{t_n})$.

Smoothness: the criterion

Recall the *Bouleau–Hirsch criterion*: if Γ_F is almost surely invertible, then F admits a density with respect to the Lebesgue measure on $\mathbb{R}^m$.

$\Rightarrow$ there are also criteria to check whether this density is "smooth".

$\Rightarrow$ the random vector F is said to be *nondegenerate* if its components are infinitely Malliavin differentiable and if

$$(\det \Gamma_F)^{-1} \in L^p(\Omega) \quad \text{for every } p \geq 1.$$

Smoothness: consequence

In this case its density is smooth and belongs to the Schwartz space of rapidly decreasing C^∞ functions on $\mathbb{R}^m$.

$\Rightarrow$ hence, obtaining a smooth density is tightly linked to the existence of negative moments of the Malliavin derivatives of the components.

The case $q = 2$

$\Rightarrow$ establishing these negative moments appears to remain a challenging problem for general q;

$\Rightarrow$ however, we succeed in proving nondegeneracy for $q = 2$, that is for *Rosenblatt vectors*. This relies crucially on the specific structure of the random variables in the second Wiener chaos;

$\Rightarrow$ in this case $\|DZ_1\|^2_{\mathfrak{H}_1}$ can be written as an infinite sum of weighted independent centered chi-square random variables.

Second chaos expansion

Since Z_t belongs to the second Wiener chaos for every $t \neq 0$, it admits a series expansion as a weighted sum of independent centered chi-square random variables:

$$Z_t = \sum_{j \geq 1} \lambda_{j,t} \, (N_j^2 - 1),$$

where $(N_j)_{j \geq 1}$ is a sequence of i.i.d. $N(0,1)$ random variables.

$\Rightarrow$ the above series converges in $L^2(\Omega)$ and almost surely;

$\Rightarrow$ the coefficients $\lambda_{j,t}$ can be expressed in terms of contractions of the kernel L_t.

Negative moments of the Malliavin derivative

As a first step, we show that the norm of the Malliavin derivative of a Rosenblatt random variable admits negative moments of all orders.

$\Rightarrow$ Let $(Z_t, t \in \mathbb{R})$ be a Rosenblatt process. Then, for every $p \geq 1$,

$$\mathbb{E}\left[\|DZ_1\|_{\mathfrak{H}_1}^{-2p}\right] < \infty.$$

Negative moments of the determinant

Next: deduce the existence of negative moments for the determinant of the Malliavin matrix of Rosenblatt increment vectors.

$\Rightarrow$ **Result.** Let $(Z_t, t \in \mathbb{R})$ be a Rosenblatt process with self-similarity index $H \in \left(\frac{1}{2}, 1\right)$. Let $m \geq 1$ and $0 = t_0 < t_1 < \cdots < t_m$. Consider the random vector

$$Z_\pi = \left(Z_{t_1} - Z_{t_0}, \ldots, Z_{t_m} - Z_{t_{m-1}}\right).$$

Then, for any integer $k \geq 0$ and any $p \geq 1$,

$$\left\| \left(\det \Gamma_{Z_\pi}\right)^{-1} \right\|_{k,p} \leq C \prod_{j=1}^{m} (t_j - t_{j-1})^{-2H}.$$

Smoothness of the density of the increments

Theorem

Let $(Z_t, t \in \mathbb{R})$ be a Rosenblatt process with self-similarity index $H \in \left(\frac{1}{2}, 1\right)$. Let $m \geq 1$ and $0 = t_0 < t_1 < \cdots < t_m$. Consider the random vector

$$Z_\pi = \left(Z_{t_1} - Z_{t_0}, \ldots, Z_{t_m} - Z_{t_{m-1}}\right).$$

Then Z_π admits a density belonging to the Schwartz space $\mathcal{S}(\mathbb{R}^m)$ of rapidly decreasing smooth functions.

$\Rightarrow$ as a consequence, for every $m \geq 1$ and every $0 \leq s_1 < \cdots < s_m$, the vector $(Z_{s_1}, \ldots, Z_{s_m})$ also has a density in $\mathcal{S}(\mathbb{R}^m)$.

Estimates for the density and its derivatives

$\Rightarrow$ we also obtain an estimate for the density and its derivatives.

$\Rightarrow$ for every $m \geq 1$ and every $n = (n_1, \ldots, n_m) \in \mathbb{N}^m$, there exists $C > 0$ (depending possibly on m and n) such that, for any grid $0 = t_0 < t_1 < \cdots < t_m$, the density p_π of the vector Z_π above satisfies

$$|\partial_x^n p_\pi(x)| \leq C \prod_{j=1}^{m} (t_j - t_{j-1})^{-H(1+n_j)} \, e^{-c \frac{x_j}{(t_j - t_{j-1})^H}},$$

with $c = c(m) > 0$, for every $x = (x_1, \ldots, x_m)$ such that $x_j \geq 2(t_j - t_{j-1})^H$, $j = 1, \ldots, m$.

Open problems and perspectives

- **Negative moments for $q \geq 3$:** does $\mathbb{E}\left[\left\|DZ_1^{(q,H)}\right\|_{\mathfrak{H}_1}^{-2p}\right] < \infty$ hold?
  The diagonalization into weighted chi-squares is available only in the second chaos. A positive answer would give smooth densities for *all* Hermite processes.
- **Local times:** joint continuity and Hölder regularity of the local time of $Z^{(q,H)}$, now accessible through the above density estimates.
- **SPDEs:** regularity of the law of the solution to stochastic (partial) differential equations driven by a Hermite noise.

Thank you!

Obrigado!