Fractional interacting particle system: drift parameter estimation via Malliavin calculus

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Workshop on Stochastic Analysis: irregularity, mean-fields and related topics

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Plan of the Talk

- Introduction to the Model
- Motivation and Previous Literature
- Construction of Estimators

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- Asymptotic Results
- Numerical Illustration
- Conclusion and Comments

Model

Interacting Particle System driven by fractional Brownian motion:

$$dX_t^{i,N} = \theta b(X_t^{i,N}, \mu_t^N)dt + \sigma dB_t^{i,H}, \quad i = 1, \ldots, N, \; t \in [0, T].$$

- **Empirical Measure of the System:**

$$\mu_t^N := \frac{1}{N} \sum_{i=1}^{N} \delta_{X_t^{i,N}}.$$

- **Driving Noise:** Fractional Brownian motion with Hurst parameter $H \in (0, 1)$.
- **Known Parameters:** b, σ, and H are assumed to be known.

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- **Known Parameters:** b, σ, and H are assumed to be known.

Goal: Estimate $\theta \in \Theta \subset \mathbb{R}$ [in the paper, in $\mathbb{R}^p$], from continuous observations over the fixed interval $[0, T]$, as $N \to \infty$.

Motivation

fBm captures *long-range dependence* and *self-similarity*, making it ideal for modeling systems with memory effects. Application in various fields:

- **Economics and Finance:** Forecasting stock prices, exchange rates, and commodity prices. (e.g., *Chronopoulou-Viens, 2012; Comte-Coutin-Renault, 2012; Fleming-Kirby, 2011; Rostek, 2009* just to name a few).
- **Geophysics and Hydrology:** Modeling natural processes like *earthquake occurrences* *Eftaxias, 2008*, *river flows*, and *precipitation patterns* *Hurst, 1951*.
- **Data Networks:** Analyzing traffic patterns in high-speed data networks. (e.g., *Caglar, 2004; Norros, 1995; Willinger-et-al, 1995*).
- **Medical Research:** Studying biological systems and health-related phenomena. (e.g., *Lai-Davis-Hardy,2000*).

Motivation for IPS driven by fBm

IPS are used to model systems of multiple interacting components. Applications in:

- **Mathematical Biology:** Modeling large societies of animals or neural networks.
- **Social Sciences:** Studying opinion dynamics, consensus in multi-agent systems.
- **Finance:** Understanding systemic risks and calibration of volatility smiles.
- **Mean-Field Games:** Modeling systems with many interacting players.
- ...

By incorporating **fBm** into IPS, we can model systems with:

- More or less *memory* (as H varies).
- Dynamic interactions with memory effects controlled by the Hurst parameter $H \in (0, 1)$.

Previous Literature on IPS Driven by Standard BM

Main References:

Kasonga (1990), Giesecke-Schwenkler-Sirignano (2020), Genon-Catalot-Larédo (2021, 2023), Chen (2021), Pavliotis-Zanoni (2022), Sharrock-Kantas-Parpas-Pavliotis (2023), DellaMaestra-Hoffmann (2023), Amorino-Heidari-Pilipauskaitė-Podolskij (2023), ...

Remark:

- **Historical Context:** Except for *Kasonga (1990)*, most contributions in this area are quite recent.
- **Statistical Framework:** Observations of interacting particles or associated McKean-Vlasov equation under various asymptotic regimes, all with $N \to \infty$: T fixed, $T \to \infty$, small noise regimes...
- **Methodological Foundation:** Approaches leverage the Markovianity and semimartingale properties of the standard Brownian framework.

Previous Literature: Classical SDEs Driven by fBm

- *Tudor-Viens (2007)*: Linear model with additive fractional noise. Consistency of the MLE for any $H \in (0,1)$ via Malliavin calculus.
- *Hu-Nualart (2010), Hu-Nualart-Zhou (2019)*: Analysis of LSE. Asymptotic normality established only for fractional OU processes, using a CLT for multiple Wiener integrals.

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- *Neuenkirch-Tindel (2013)*: Strong consistency for least square-type estimator with $H > \frac{1}{2}$ under discrete observations. General model: b not necessarily linear in the parameter.
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- *Comte-Marie (2019)*: N i.i.d. fractional SDEs. Consistency for Nadaraya-Watson drift estimator using Skorohod integrals. Associated computable estimators in *Marie (2023)*.

The Hunt for Estimators

Least-squares-inspired "estimator":

$$\tilde{\theta}_N = \frac{\sum_{i=1}^N \int_0^T b(X_t^{i,N}, \mu_t^N)\, dX_t^{i,N}}{\sum_{i=1}^N \int_0^T b^2(X_t^{i,N}, \mu_t^N)\, dt} = \theta_0 + \sigma \frac{\sum_{i=1}^N \int_0^T b(X_t^{i,N}, \mu_t^N)\, dB_t^{i,H}}{\sum_{i=1}^N \int_0^T b^2(X_t^{i,N}, \mu_t^N)\, dt}.$$

Remarks:

1. For $H = \frac{1}{2}$, the divergence integral simplifies to the standard Itô integral $\Rightarrow$ approximated using forward Riemann sums.
2. For $H > \frac{1}{2}$, the divergence integral is the limit of Riemann sums involving the Wick product.

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3. **Key issue:** Observations depend on the Skorohod integral, which is not directly observable $\Rightarrow$ referred to as the "fake-stimator" or "estim-actor".
4. **Goal:** Analyze the asymptotic behavior of **computable** estimators as $N \to \infty$.
5. **Key Tools:** Malliavin calculus, propagation of chaos, Lions derivative.

The Hunt for Estimators

Relationship Between Skorohod and Stratonovich (Pathwise) Integral for $H > \frac{1}{2}$:

$$\int_0^T b(X_t^{i,N}, \mu_t^N)\, dB_t^{i,H} = \int_0^T b(X_t^{i,N}, \mu_t^N) \circ dB_t^{i,H} - \int_0^T \int_0^T D_s^i b(X_t^{i,N}, \mu_t^N)\, \phi(t,s)\, ds\, dt,$$

where $\phi(t,s) := H(2H-1)|t-s|^{2H-2}$.

Idea: Replace this relationship in the fake-stimator.

Problem: $D_s^i b(X_t^{i,N}, \mu_t^N)$ is not observable either.

How can we approximate it?

Propagation of Chaos

Propagation of Chaos: As $N \to \infty$, the system converges to the McKean-Vlasov SDE (independent particles):

$$d\bar{X}_t = \theta b(\bar{X}_t, \mu_t)dt + \sigma dB_t^H, \quad \mu_t = \mathrm{Law}(\bar{X}_t).$$

- **Well-studied for $H = \frac{1}{2}$:** See, for example, *Chaintron-Diez, 2021*.
- **For $H \in (0, 1)$:** Proven for additive noise and singular drift linear in the measure in *Galeati-Lê-Mayorcas, 2024*.
- **Current Context:** Propagation of chaos remains to be established for our specific setting.

Propagation of Chaos

A1: b is Lipschitz, and $|b(0, \delta_0)| \leq c$, where δ_0 is the Dirac delta.

A2: *Boundedness of moments:* For any $k \geq 1$,

$$\int_{\mathbb{R}} |x|^k \mu_0(dx) \leq C_k.$$

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Theorem [A., Nourdin, Shevchenko]
Assume **A1**-**A2**. Then, for each $p \geq 2$ and $i \in \{1,\ldots,N\}$:

$$\mathbb{E}\left[\sup_{t\in[0,T]} |X_t^{i,N} - \bar{X}_t^i|^p\right] \leq c\left(\frac{1}{\sqrt{N}}\right).$$

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Question: Does this result extend to Malliavin derivatives?

Formal computation of Malliavin Derivatives

For any $i, j \in \{1, \dots, N\}$ and $s > t$ $D^j_s \bar{X}^i_t = 0 = D^j_s X^{i,N}_t$. For $s \leq t$,

$$D^j_s \bar{X}^i_t = 1_{\{i=j\}} \left(\sigma + \int_s^t \theta_0 \partial_x b(\bar{X}^i_r, \bar{\mu}_r) D^i_s \bar{X}^i_r \, dr \right),$$

$$\begin{aligned} D^j_s X^{i,N}_t = 1_{\{i=j\}} \sigma + \int_s^t \theta_0 \Big(& \partial_x b(X^{i,N}_r, \mu^N_r) D^j_s X^{i,N}_r \\ & + \frac{1}{N} \sum_{k=1}^N (\partial_\mu b)(X^{i,N}_r, \mu^N_r)(X^{k,N}_r) D^j_s X^{k,N}_r \Big) \, dr. \end{aligned}$$

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Previous results on POC for Malliavin derivatives?

Only in *(DosReis - Wilde, 2023)*.
For $H = \frac{1}{2}$, $D^j X^{i,N}_t \to D^j \bar{X}^i_t$ in L^2, but no quantitative results.

Propagation of Chaos for Malliavin Derivatives

A3: $\partial_x b$, $\partial_\mu b$ are bounded and Lipschitz continuous.

Theorem [A., Nourdin, Shevchenko]
Let $H \in (0,1)$. Assume A1-A2-A3. Then, for each $p \geq 2$, $i \in \{1, \ldots, N\}$:

1. For each $p \geq 1$,
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Remarks:

- **Off-diagonal elements:** Almost sure bound thanks to the boundedness of b's derivatives. The convergence rate is faster than the classical poc.
- **Diagonal elements:** Classical poc for particles is required in the proof, explaining the L^p result and the classical rate.

Asymptotic Behavior of the Fake-stimator

We want to use the results above to study:

$$\tilde{\theta}_N - \theta_0 = \frac{\frac{\sigma}{N}\sum_{i=1}^{N}\int_0^T b(X_t^{i,N}, \mu_t^N)dB_t^{i,H}}{\frac{1}{N}\sum_{i=1}^{N}\int_0^T b^2(X_t^{i,N}, \mu_t^N)dt}.$$

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- Proof based on propagation of chaos and the law of large numbers.

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- Proof based on propagation of chaos and the law of large numbers.
- What about the numerator?

Consistency of the Fake-stimator

Theorem [A., Nourdin, Shevchenko]
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$$\frac{\sigma}{N}\sum_{i=1}^{N}\int_0^T b(X_t^{i,N},\mu_t^N)dB_t^{i,H} \xrightarrow{\mathbb{L}^2} 0.$$

- Proof based on a consequence of Meyer's inequality and poc for Malliavin derivatives.
- As a consequence of the last two theorems, it follows that $\tilde{\theta}_N \xrightarrow{\mathbb{P}} \theta_0$.

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- Question: Can we obtain asymptotic normality?

Hope and Despair

Let us introduce

$$\bar{Z}_N := \sum_{i=1}^{N} \int_0^T b(\bar{X}_t^i, \bar{\mu}_t)\sigma dB_t^{i,H}, \qquad Z_N := \sum_{i=1}^{N} \int_0^T b(X_t^{i,N}, \mu_t^N)\sigma dB_t^{i,H}.$$

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- **Ok for independent particles:** $\frac{1}{\sqrt{N}} \bar{Z}_N \xrightarrow{\mathcal{L}} N(0, v^2)$ for some v :)
- **This does not imply:** $\frac{1}{\sqrt{N}} Z_N \xrightarrow{\mathcal{L}} N(0, v^2)$, as the propagation of chaos guarantees that the error in transitioning from $\bar{Z}_N$ to Z_N is of order $\frac{1}{\sqrt{N}}$: (

Asymptotic Normality of Z_N

Quite surprising: Let $H > \frac{1}{2}$. Faster propagation of chaos of Malliavin derivatives off-diagonal is sufficient for obtaining asymptotic normality of $\frac{1}{\sqrt{N}} Z_N$:)

Key tool: Multidimensional isometry for Skorohod integral (see Theorem 3.11.1 of *(Biagini et al - 2008)*).

Indeed, using empirical projection, we can write Z_N as:

$$Z_N = \int_0^T B^N(\boldsymbol{X}_s)\, d\boldsymbol{B}_s^H,$$

$$B^N(\boldsymbol{x}) = \left(b_1^N(\boldsymbol{x}), \ldots, b_N^N(\boldsymbol{x})\right)^T = \left(b(x^1, \mu^N), \ldots, b(x^N, \mu^N)\right)^T,$$

and $\boldsymbol{B}^H$ is an N-dimensional fractional Brownian motion.

Asymptotic Normality of Z_N, Key Ideas

Theorem 3.11.1 of *(Biagini et al - 2008)* ensures:

$$\mathbb{E}\left[\left(\frac{1}{\sqrt{N}}Z_N\right)^2\right] = \frac{1}{N}\sigma^2\sum_{i=1}^{N}\int_0^T\int_0^T \mathbb{E}[b(X_s^i,\mu_s^N)b(X_t^i,\mu_t^N)]\phi(s,t)\,ds\,dt$$

$$+\frac{1}{N}\sigma^2\sum_{i,j=1}^{N}\int_0^T\int_0^T\int_0^T\int_0^T \mathbb{E}[D_v^i b(X_s^j,\mu_s^N)D_u^j b(X_t^i,\mu_t^N)]\phi(v,s)\phi(u,t)\,dv\,du\,ds\,dt$$

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Remarks:

- In the last sum, the terms where $j \neq i$ are negligible thanks to the propagation of chaos for Malliavin derivatives off-diagonal, of order $\frac{1}{N}$.
- The poc implies:

$$\mathbb{E}\left[\left(\frac{1}{\sqrt{N}}Z_N\right)^2\right] = \mathbb{E}\left[\left(\frac{1}{\sqrt{N}}\bar{Z}_N\right)^2\right] =: v^2,$$

i.e., the same limit as for independent particles.

Asymptotic Normality of Z_N, Key Ideas

Let $Z := \mathcal{N}(0,1)$. The Wasserstein distance satisfies:

$$W_1\left(\frac{1}{\sqrt{Nv^2}} Z_N, Z\right) \leq W_1\left(\frac{1}{\sqrt{Nv^2}} Z_N, \frac{1}{\sqrt{Nv^2}} \bar{Z}_N\right) + W_1\left(\frac{1}{\sqrt{Nv^2}} \bar{Z}_N, Z\right).$$

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From the Berry-Esseen theorem for i.i.d. sums of random variables, we have:

$$W_1\left(\frac{1}{\sqrt{Nv^2}} \bar{Z}_N, Z\right) \leq \frac{c}{\sqrt{N}}.$$

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By definition of the Wasserstein distance:

$$W_1\left(\frac{1}{\sqrt{Nv^2}} Z_N, \frac{1}{\sqrt{Nv^2}} \bar{Z}_N\right) \leq \frac{1}{\sqrt{Nv^2}} \mathbb{E}\left[(Z_N - \bar{Z}_N)^2\right]^{\frac{1}{2}}.$$

Using multidimensional isometry for the Skorohod integral, we can prove:

$$\frac{1}{\sqrt{Nv^2}} \mathbb{E}\left[(Z_N - \bar{Z}_N)^2\right] \leq \frac{c}{\sqrt{N}}.$$

Asymptotic Normality for Z_N, The End

Putting the pieces together:

Theorem [A., Nourdin, Shevchenko]
Let $H \geq \frac{1}{2}$. Assume A1-A2-A3. Then

$$W_1\left(\frac{1}{\sqrt{Nv^2}} Z_N, Z\right) \leq cN^{-\frac{1}{4}}.$$

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Putting the pieces together:

Theorem [A., Nourdin, Shevchenko]
Let $H \geq \frac{1}{2}$. Assume A1-A2-A3. Then

$$W_1\left(\frac{1}{\sqrt{Nv^2}}Z_N, Z\right) \leq cN^{-\frac{1}{4}}.$$

Remarks

1. Propagation of chaos for Malliavin derivatives provides a CLT at the usual rate of $\frac{1}{\sqrt{N}}$.
2. The theorem above goes a step further by providing the convergence rate of the fluctuations.

Asymptotic Behavior of the Estim-actor

A4: $\mathbb{P}\left(\int_0^T b^2(\bar{X}_s, \bar{\mu}_s)ds > 0\right) > 0.$

Recall that

$$\tilde{\theta}_N - \theta_0 = \frac{\sum_{i=1}^N \int_0^T b(X_t^{i,N}, \mu_t^N)\sigma dB_t^{i,H}}{\sum_{i=1}^N \int_0^T b^2(X_t^{i,N}, \mu_t^N)dt}.$$

Theorem [A., Nourdin, Shevchenko]

Let $H \geq \frac{1}{2}$. Assume A1-A2-A3-A4. Then:

1. **Consistency in Probability:** $\tilde{\theta}_N \xrightarrow{\mathbb{P}} \theta_0$.
2. **Asymptotic Normality:** $\sqrt{N}(\tilde{\theta}_N - \theta_0) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \tilde{v}^2)$.

- $\tilde{v}^2 := \left(\frac{v}{\int_0^T \mathbb{E}[b^2(\bar{X}_s, \bar{\mu}_s)]\, ds}\right)^2$, where v^2 is the variance of particles as defined earlier.

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- Beautiful... but useless ! (At least so far).

What About Computable Estimators?

Relationship between the Stratonovich and Skorohod integrals:

$$\int_0^T b(X_t^{i,N}, \mu_t^N) \circ dB_t^{i,H} = \int_0^T b(X_t^{i,N}, \mu_t^N) dB_t^{i,H} + \int_0^T \int_0^T D_s^i b(X_t^{i,N}, \mu_t^N) \phi(t,s)\, ds\, dt.$$

We need to approximate:

$$D_s^i b(X_t^{i,N}, \mu_t^N) = \partial_x b(X_t^{i,N}, \mu_t^N) D_s^i X_t^{i,N} + \frac{1}{N} \sum_{k=1}^{N} (\partial_\mu b)(X_t^{i,N}, \mu_t^N)(X_t^{k,N}) D_s^i X_t^{k,N}.$$

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Remarks:

1. Straightforward without interaction; with interaction, also $D_s^i X_t^{k,N}$ for $k \neq i$ contributes.
2. However, thanks to the propagation of chaos off-diagonal, the last term is of order $\frac{1}{N}$.

Approximation $D_s^i X_t^{i,N}$

OBS: Without interaction, for $s \leq t$,

$$D_s X_t = \sigma + \int_s^t \theta_0 \partial_x b(X_r) D_s X_r \, dr = \sigma \exp(\theta_0 \int_s^t \partial_x b(X_r) dr).$$

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Proposition [A., Nourdin, Shevchenko]
Let $H \geq \frac{1}{2}$. Assume A1-A2-A3. Then

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It justifies we consider the fixed-point estimator

$$\hat{\theta}^{(fp)}_N = -\frac{\sum_{i=1}^N \int_0^T \int_0^t \partial_x b(X^{i,N}_t, \mu^N_t) \exp\left(\hat{\theta}^{(fp)}_N \int_s^t \partial_x b(X^{i,N}_r, \mu^N_r) dr\right) \sigma \phi(t,s) ds dt}{\sum_{i=1}^N \int_0^T b^2(X^{i,N}_t, \mu^N_t) dt}$$

$$+ \frac{\sum_{i=1}^N \int_0^T b(X^{i,N}_t, \mu^N_t) \circ dX^{i,N}_t}{\sum_{i=1}^N \int_0^T b^2(X^{i,N}_t, \mu^N_t) dt} =: F_N(\hat{\theta}^{(fp)}_N)$$

Asymptotic behavior of $\hat{\theta}_N^{(fp)}$

We need F_N to be a contraction.

A6: $|b(x,\mu)| \geq \ell > 0$, $\Theta \subset \mathbb{R}^+$, $\partial_x b(x,\mu) \leq 0$, $C_T := \frac{\|\partial_x b\|_\infty^2}{l^2} \frac{(2H-1)}{(2H+1)} T^{2H} \sigma < 1$.

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Theorem [A., Nourdin, Shevchenko]

Let $H \geq \frac{1}{2}$. Assume A1-A2-A3-A4-A6. Then,

$$\hat{\theta}_N^{(fp)} \xrightarrow{L^1} \theta_0, \qquad \sqrt{N}(\hat{\theta}_N^{(fp)} - \theta_0) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \frac{\tilde{v}^2}{(1-V)^2}\right),$$

$$V := \int_0^T \int_0^t \int_s^t \mathbb{E}[\partial_x b(\bar{X}_t, \bar{\mu}_t) \partial_x b(\bar{X}_r, \bar{\mu}_r) \exp(\int_s^t \theta^0 \partial_x b(\bar{X}_r, \bar{\mu}_r) dr)] \phi(t,s) drdsdt.$$

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1. Time horizon T is fixed $\Rightarrow$ we can choose $T < T_{\max} := \left(\ell \sqrt{\frac{2H+1}{\|\partial_x b\|_\infty (2H-1)}}\right)^{\frac{1}{H}}$, so that $C_T < 1$.
2. Since $V \geq 0$, variance larger than that of the fake-stimator.

Numerical results: drift example ($H = 0.6$)

Model:

$$\int_0^t \theta^0 \Big(2 - \arctan\big(X_s^{i,N} - \tfrac{1}{N}\sum_j X_s^{j,N}\big)\Big)\, ds$$

Simulation parameters:

100 runs, $T = 1$, $\Delta_t = 0.001$, $X_0 = 0$, RMSE (Bias) for $\hat{\theta}^0 \in \{5, 12\}$.

Method / N	30	60
$\hat{\theta}^0 = 5$		
Ratio	1.910 (−0.017)	0.919 (−0.083)
Fixed-point	0.091 (−0.002)	0.061 (−0.007)
Discrete	0.339 (−0.067)	0.247 (0.004)
$\hat{\theta}^0 = 12$		
Ratio	0.623 (−0.044)	0.230 (0.005)
Fixed-point	0.093 (−0.001)	0.066 (−0.001)
Discrete	0.284 (−0.036)	0.208 (−0.059)

Numerical results: drift example ($H = 0.8$)

RMSE (Bias), $H = 0.8$, T=0.79.

Method / N	30	60
	$\hat{\theta}^0 = 5$	
Ratio	2.607 (0.220)	0.646 (0.024)
Fixed-point	0.090 (0.002)	0.063 (0.006)
Discrete	0.106 (−0.045)	0.078 (−0.050)
	$\hat{\theta}^0 = 12$	
Ratio	0.302 (0.082)	0.262 (0.070)
Fixed-point	0.085 (−0.001)	0.065 (−0.003)
Discrete	0.110 (−0.034)	0.073 (−0.024)

1. Ratio faster, fixed point more precise.
2. All computable estimators perform well.

Conclusion

- Probabilistic tools of interest: quantitative propagation of chaos for Malliavin derivatives.
- Consistency and asymptotic normality of fake-stimator $\tilde{\theta}_N$.
- Proposed computable estimators and demonstrated their good asymptotic behavior.
- Highlighted numerical viability of proposed methods.

Future work: statistical inference for...

- Fractional interacting particle system (still)
  - Discrete observations.
  - b not linear in the parameter.
  - Study multiplicative case.
  - $H < \frac{1}{2}$
  - Non parametric estimation

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- Interacting particle system with jumps
- Graphon mean-field particle system

C. Amorino, I. Nourdin, R. Shevchenko.
Fractional interacting particle system: drift parameter estimation via Malliavin calculus,
Stochastic Processes and Applications

Thank you for your attention!