Energy methods and global well-posedness for the stochastic Navier-Stokes equation

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The equation

The equation we are going to work with (SNS with additive noise) is:

$$\mathrm{d}u - \mathbf{P}\Delta u + \mathbf{P}(u \cdot \nabla)u = \mathbf{P}\,\mathrm{d}W_t \tag{1}$$

on a smooth bounded domain $\mathcal{O} \subset \mathbb{R}^2$, with Dirichlet boundary conditions $u|_{\partial\mathcal{O}} = 0$.
W_t represents an $L^2(\mathcal{O})$-valued Wiener process and $\mathbf{P}$ denotes the Leray-Helmholtz projection onto divergence-free fields. It is a representation of spacetime white noise (the gaussian process with covariance $\mathbb{E}(\xi(x,t)\xi(y,s)) = \delta_{x,y}\delta_{s,t}$) which is purely derived from the Wiener process on $L^2(\mathcal{O})$.

Solution spaces and regularity

We will denote by $\mathbb{H}^{\alpha,p}$ and $\mathbb{H}^{\alpha,p}_{\sigma}$, respectively, the domains of fractional powers of the Dirichlet laplacian $(-\Delta)^{\alpha/2}$ and the Stokes operator, $(-\mathbf{P}\Delta)^{\alpha/2}$. They are normed spaces with the norms $|v|_{\mathbb{H}^{\alpha,p}} := |(-\Delta)^{\alpha/2} v|_{L^p}$, $|v|_{\mathbb{H}^{\alpha,p}_{\sigma}} := |(-\mathbf{P}\Delta)^{\alpha/2} v|_{L^p}$
They are closely related to Sobolev spaces with or without the divergence-free condition. (For example, $\mathbb{H}^{2,2} = \mathbb{W}^{2,2} \cap \mathbb{W}^{1,2}_0$)

The negative regularity spaces contain distributions (that are not functions) and the Leray projection extends to the $\mathbb{H}^{\alpha}$ spaces for at least all $\alpha \geq -2$.

Our initial condition is taken to be $u_0 \in \mathbb{H}^{-\varepsilon}$ for some fixed, small $\varepsilon > 0$.

Singular equation

Let's now estimate the regularity of the solution to equation (1). Start with the ansatz $u \in L^{\infty}_{[0,T]}\mathbb{H}^{\alpha}$:

$$\underbrace{\mathrm{d}u - \mathbf{P}\Delta u}_{\alpha-2} + \underbrace{\mathbf{P}(u\cdot\nabla)u)}_{2\alpha-1} = \underbrace{\mathbf{P}\,\mathrm{d}W_t}_{-2-\varepsilon} \tag{2}$$

So we expect $\alpha = -\varepsilon$, that is, $u \in L^{\infty}_{[0,T]}\mathbb{H}^{-\varepsilon}$ is not a function. But then the product $(u\cdot\nabla)u$ is not well-defined! So the equation is singular.
(This already suggests an approach using regularity structures, but it turns out they are not strictly needed in this framework, barely, as the same techniques extend to usual paracontrolled calculus with a small modification.)

The work of Da Prato and Debussche

In the now-classical work of Da Prato and Debussche, 2002, they study stochastic Navier-Stokes with periodic boundary conditions, and solve the singular problem by first substituting the nonlinearity (using the divergence-free condition), $(u \cdot \nabla)u = \mathrm{div}(u^{\otimes 2})$, then considering the formal decomposition $u = v + Z$ where Z solves the linear equation $\mathrm{d}Z - \mathbf{P}\Delta Z = \mathrm{d}W_t$. The resulting equation is:

$$dv - \Delta v + \mathbf{P}\mathrm{div}(v + Z)^{\otimes 2}) = 0 \tag{3}$$

where the quadratic term $\mathbf{P}\mathrm{div}Z^{\otimes 2}$ acts as the new noise and the only singular term. Then, they use that $\mathbf{P}\mathrm{div}Z^{\otimes 2} = \mathbf{P}\mathrm{div}(Z^{\otimes 2} - \mathbb{E}Z^{\otimes 2})$ and the latter has a limit under, for example, mollification of the noise.

Some caveats

In our case, for generic smooth bounded domains, we lose the homogeneity that allows us to have $\mathbb{E}Z^{\otimes 2}$ constant in space (and therefore vanish from the equation); we need extra computations and also need to change the equation to solve the singular term.

It is also worth noting that we will need improvements in regularity. The Da Prato-Debussche trick $v = u + Z$ will not work here twice; because the linear term has the same regularity as the additive noise, any cancellings in the latter are nullified by the extra terms from the former.

Local Well-Posedness

Stochastic estimates

Fix $1 \leq p \leq \infty$ and $2 \leq r < \infty$, and $T > 0$, $\beta > 0$. Then:

- The process Z belongs in $L^r(0, T; \mathbb{H}^{-\beta,p})$ almost surely.
- For certain variables $X_{t,\varepsilon}$ that are equal in law to $e^{-\varepsilon A}Z_t$, the limit $\lim_{\varepsilon \to 0}(X_{\varepsilon,t})^{\otimes 2} - \mathbb{E}((X_{\varepsilon,t})^{\otimes 2}) = Y$ exists in law in the same space. This can replace the square in equation (3).

Picard iteration

Assume $2 < q < \infty$, $2 < p < \infty, \varepsilon < \min(\frac{1}{2}, 1 - \frac{2}{q})$. Define the solution space $S_T^{q,\varepsilon} := L^q(0, T; \mathbb{H}_\sigma^{1-\varepsilon,p}) \cap C^0([0, T]; \mathbb{B}_{q,\sigma}^{\alpha,p})$. Then, for any fixed initial condition $v_0 \in \mathbb{H}_\sigma^{1-\varepsilon,p} \cap \mathbb{B}_{q,\sigma}^{\alpha,p}$, equation (3), after replacing Z^2 by Y, has a unique strong solution in $S_T^{q,\varepsilon}$.

The energy method - Smooth case

When the noise in the equation (3) is sufficiently smooth, we can estimate the energy growth by computing:

$$\frac{\partial}{\partial t}|v|^2_{\mathbb{L}^2} - \langle v, \mathbf{P}\Delta v\rangle + \langle v, \mathbf{P}\mathrm{div}(v^2 + 2vZ)\rangle + \langle v, Z^2\rangle = 0 \quad (4)$$

The projection is self-adjoint and so it vanishes in the inner product, the term that is cubic in v is antisymmetric so it vanishes identically, and the linear and bilinear terms can be bounded quadratically in terms of the norm of v - in case of $\langle v, 2v\mathrm{div}(Z)\rangle$, for example, this follows from $2v\mathrm{div}(Z) \leq \varepsilon v^2 + \frac{1}{\varepsilon}\mathrm{div}(Z)^2$ Unfortunately, in our case, Z is too rough for these estimates and also some of these pairings are not well-defined. We need better regularity control on the solution.

Paraproducts

In order to isolate the next worst regularity term, we introduce paraproducts, which are convergent pieces of a product. They follow the rules, for $f \in C^\alpha$, $g \in C^\beta$:

$$|f \circledless g|_{C^\beta} \lesssim |f|_{C^\alpha \cup L^\infty} |g|_{C^\beta}$$

They also satisfy commutator estimates such as

$$\partial_t (f \circledless g) \approx f \circledless \partial_t g, \text{ up to terms with regularity} \geq C^\beta$$

In the case of bounded domains (no Fourier transform) we follow Bailleul and Bernicot, 2016 and add the restriction of working on the regularity interval $(-2, 1)$.

The energy method - Rosati and Hairer

The analysis proceeds with the ansatz that the next worst term of the solution v is given by $\mathbf{P}\mathrm{div}(v \circledless Q)$ where $(\partial_t - \Delta)Q = 2Z$. Removing smooth terms should not decrease the regularity of the remainder, so we remove the low frequencies. This does not change roughness but it changes size, so we can perform such a decomposition to make this term $\mathbf{P}\mathrm{div}(v \circledless Q)$ become small. Due to the paraproduct, we resign ourselves to solving a system of 2 equations rather than try and undo the paraproduct. Our new system looks like:

$$\begin{aligned} v &= \mathbf{P}\mathrm{div}(v \circledless Q) + w \\ \partial_t w &= \Delta w + \mathbf{P}\mathrm{div}(w^2 + (2X \cdot v - 2X \circledgtr v) + R(w) + Y) \end{aligned} \tag{5}$$

where Y represents the replaced square and R represents the more regular leftover terms from the commutator estimates.

Then, the final steps are:

- Prove an L^2 energy blowup criterion (solution explodes if and only if L^2 norm explodes)
- Choose the projection P into high frequencies appropriately (in this case, discretely and with cutoff frequency proportional to the cube of solution norm)
- Bound the energy pairings using both smallness of the norm of the component $v - w$, and analysis of the terms involving only w by bounding the highest eigenvalue of the quadratic form
- Deduce that all of the worst terms can be bounded by $|w|^2 \log(1 + |w|^2)$ and deduce that $\partial_t |w|^2 \leq C|w|^2 \log(1 + |w|^2)$ which implies no blow-up of L^2 norm in finite time

This has been done for the periodic boundary case in Hairer and Rosati, 2024.

Details on Projections

As mentioned before, the framework for proving global well-posedness requires projections $\mathcal{L}_\lambda, \mathcal{H}_\lambda$ into high and low frequencies, respectively, and more importantly λ needs to change throughout the lifetime of the solution.
As we would rather not differentiate it, we instead discretize the projection level $\lambda_t = (1+i)^3$ whenever $i \leq \sup\limits_{[0,t],\omega} \|w\|_{L^2} \leq i+1$.
Then, define the high-frequency component, which will be given by $w^{\mathcal{H}} = \mathbf{P}\mathrm{div}(w \bigcirc\!\!\!\!< (\mathcal{H}_{\lambda_t} Q))$. (Then the low-frequency component is just $w^{\mathcal{L}} = w - w^{\mathcal{H}}$.)

Estimate for the high projected component

$$\|w^{\mathcal{H}}\|_{\mathbb{H}^{1-2\kappa-\delta}} \leq C(\delta, W)(1 + \|w\|_{L^2})^{1-3\delta}$$

where $\kappa > 0$ is such that the initial condition $u_0 \in L^2 \cap C^{-1+\kappa}$, and $\delta > 0$ is arbitrary, and C depends on the stochastic noise up to time t.

Additional property of the low projected component

$$\|w^{\mathcal{L}}\|_{\mathbb{H}^{\beta}} \leq C\lambda^{\beta-\alpha}\|w\|_{\mathbb{H}^{\alpha}}$$

C is uniform in $\lambda = \lambda_t$. This is an important source of decay that can help utilize the differences in norms taken, when the usual estimates are applied.

L^2 energy estimates

$$\partial_t \|w^{\mathcal{L}}\|^2 + \frac{1}{2}\|w^{\mathcal{L}}\|^2_{\mathbb{H}^1} \leq C((1+\log(\lambda_t))\|w^{\mathcal{L}}\|^2 + 1)$$

As above, C depends continuously on the parameter κ and the noise up to time t. This implies:

$$\sup_{T_i \leq t \leq T_{i+1}} \|w^{\mathcal{L}}\|^2 + \int_{T_i}^{T_{i+1}} \frac{1}{2}\|w^{\mathcal{L}}\|^2_{\mathbb{H}^1} dt \leq$$

$$\leq (\|w^{\mathcal{L}}_{T_i}\|^2 + C) \cdot e^{C(T_{i+1}-T_i)(1+\log(\lambda_{T_i}))}$$

By the definition of the stopping times T_i we obtain an inequality up to the finitude time of the solution:

$$T_{i+1} - T_i \leq \frac{1}{C(1+\log(1+i))} \frac{\log(i^2+2i-C)}{i^2+C}$$

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