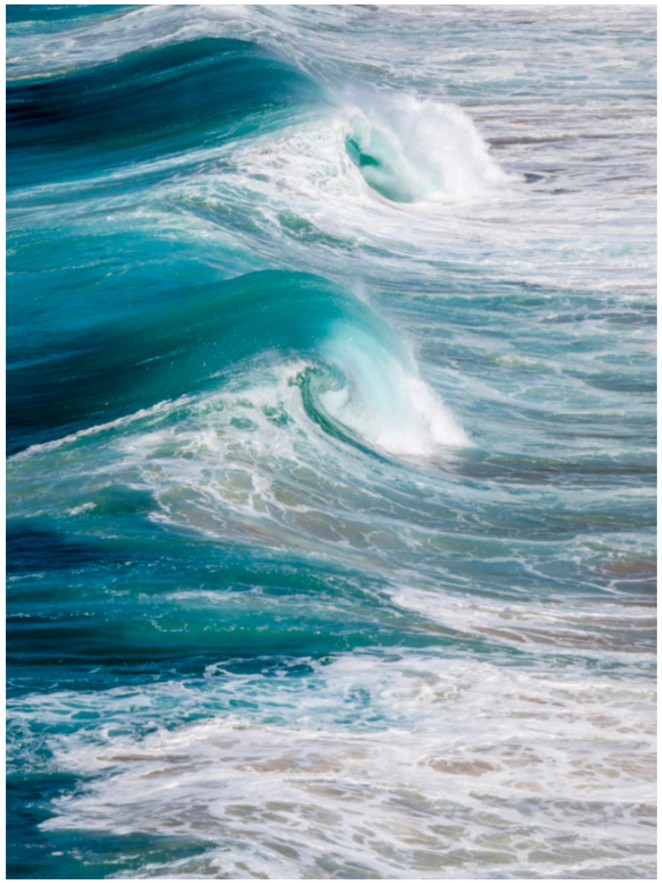
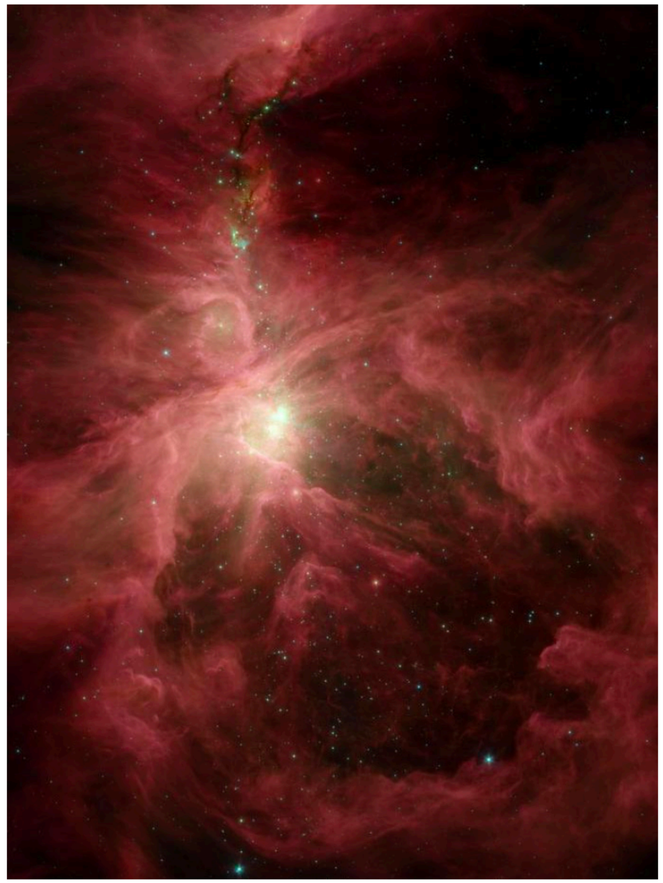
Spontaneous Stochasticity: Randomness Without Noise

Alexei A. Mailybaev
(IMPA, Rio de Janeiro)

Artem Raibekas
(UFF, Rio de Janeiro)





Figures from PhD thesis of Erika Bernal, defense on 6/03/2026 at 10am, IMPA

I. Turbulence: modelling and phenomenology

Decaying Turbulence in a Tea Cup

by Alban Pothérat

Developed turbulence

Initial value problem for the **3D stochastic Navier-Stokes equations**:

Euler equations (ideal fluid) — viscosity — space-time white noise

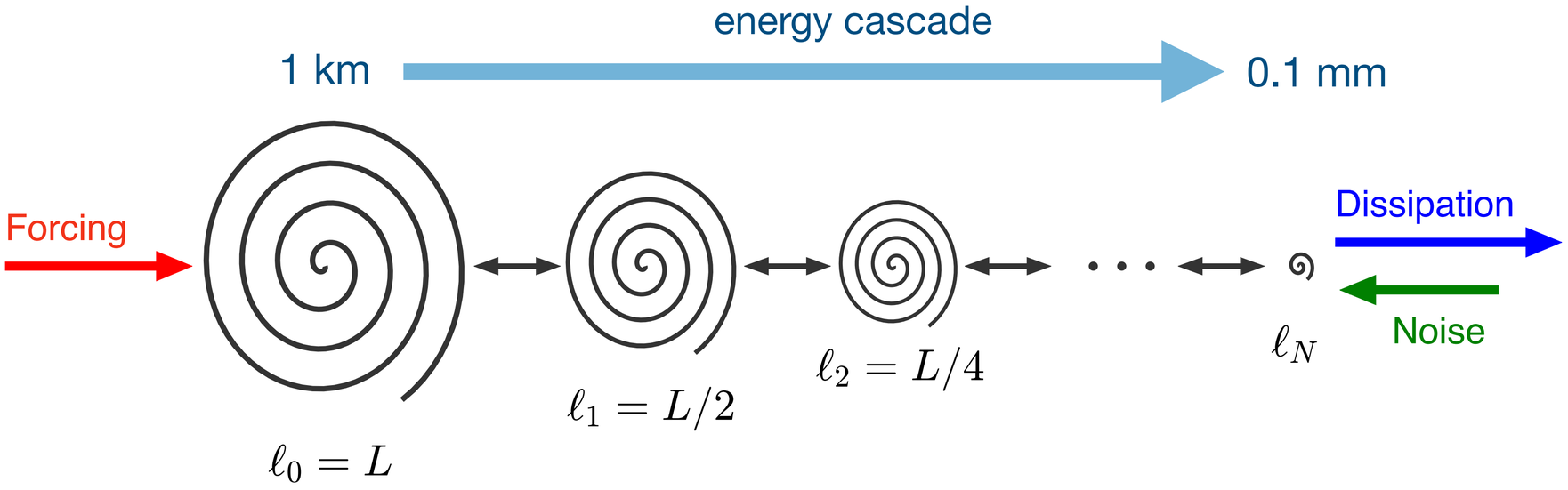
$$\boxed{\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \mathbf{f}} + \boxed{\mathrm{Re}^{-1}\Delta\mathbf{u}} + \boxed{\sqrt{\Theta}\,\nabla\cdot\boldsymbol{\xi}}, \quad \nabla\cdot\mathbf{u} = 0, \quad \mathbf{u}\big|_{t=0} = \mathbf{u}_0(\mathbf{x}).$$

$\mathrm{Re}^{-1} \sim 10^{-8}$ → 0

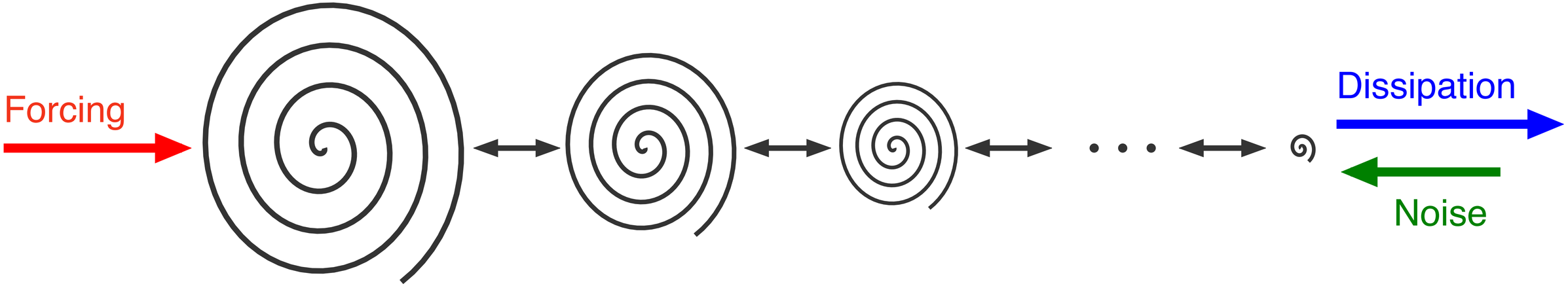
$\sqrt{\Theta} = 10^{-19}$ → 0

+ Galerkin regularization at molecular scale

$\delta = 10^{-12}$



Phenomenological picture



$\ell_0 = 1$ $\quad$ $\ell_1 = 1/2$ $\quad$ $\ell_2 = 1/4$ $\quad$ $\ell_N = 2^{-N}$

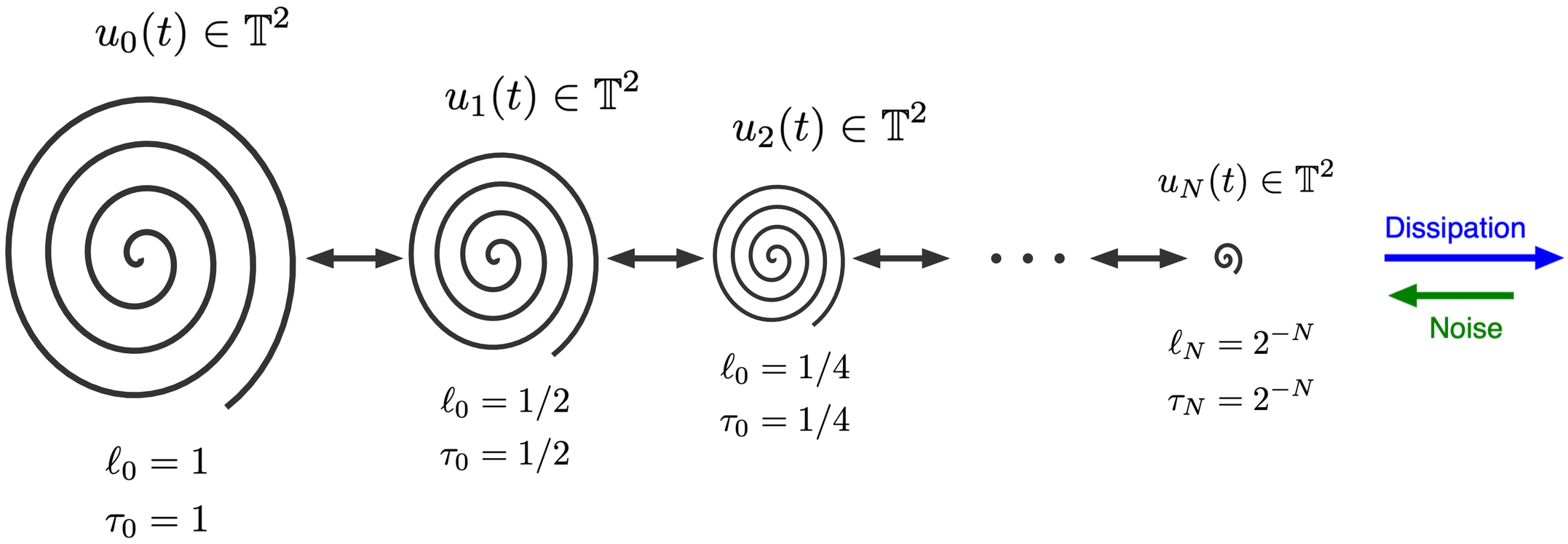
$\tau_0 = 1$ $\quad$ $\tau_1 = 1/2$ $\quad$ $\tau_2 = 1/4$ $\quad$ $\tau_N = 2^{-N}$

Scaling symmetry of ideal fluid dynamics (Euler equations):

$$t,\ \mathbf{x},\ \mathbf{u} \ \mapsto\ \lambda^{1-h}t,\ \lambda\mathbf{x},\ \lambda^{h}\mathbf{u} \qquad (\lambda = 2,\ h = 0)$$

II. Solvable toy model: multiscale Arnold’s cat

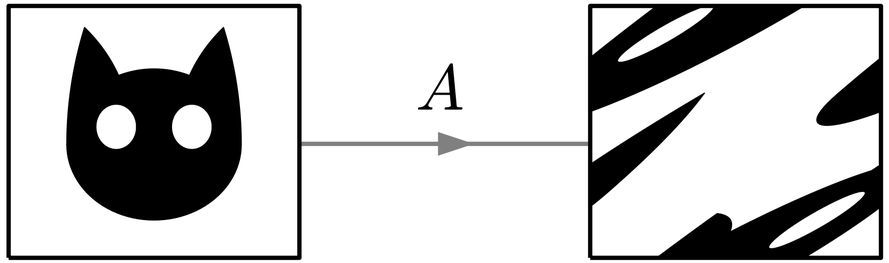
Variables

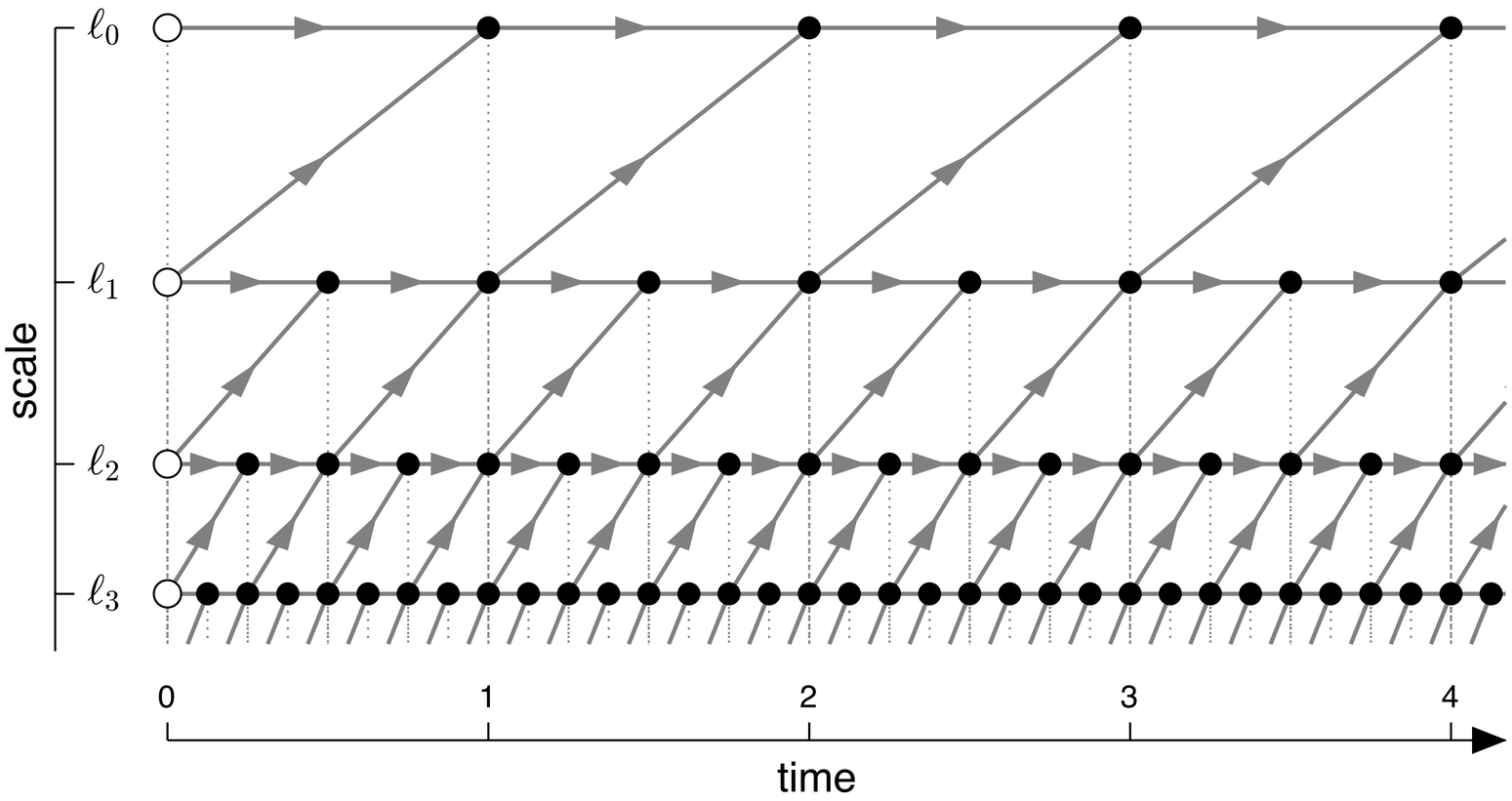


Discrete dynamics: each eddy evolves with its own time step = turnover time $\tau_n = 2^{-n}$

Chaotic dynamics: Arnold's cat map

Multiscale Arnold's cat





Space-time (fractal) lattice:

$$\mathcal{L} = \{(n,t) : n \in \mathbb{Z}_+,\ t \in \tau_n \mathbb{Z}_+\}$$

Variables:

$$u_n(t) = (x,y) \in \mathbb{T}^2$$

Dynamics:

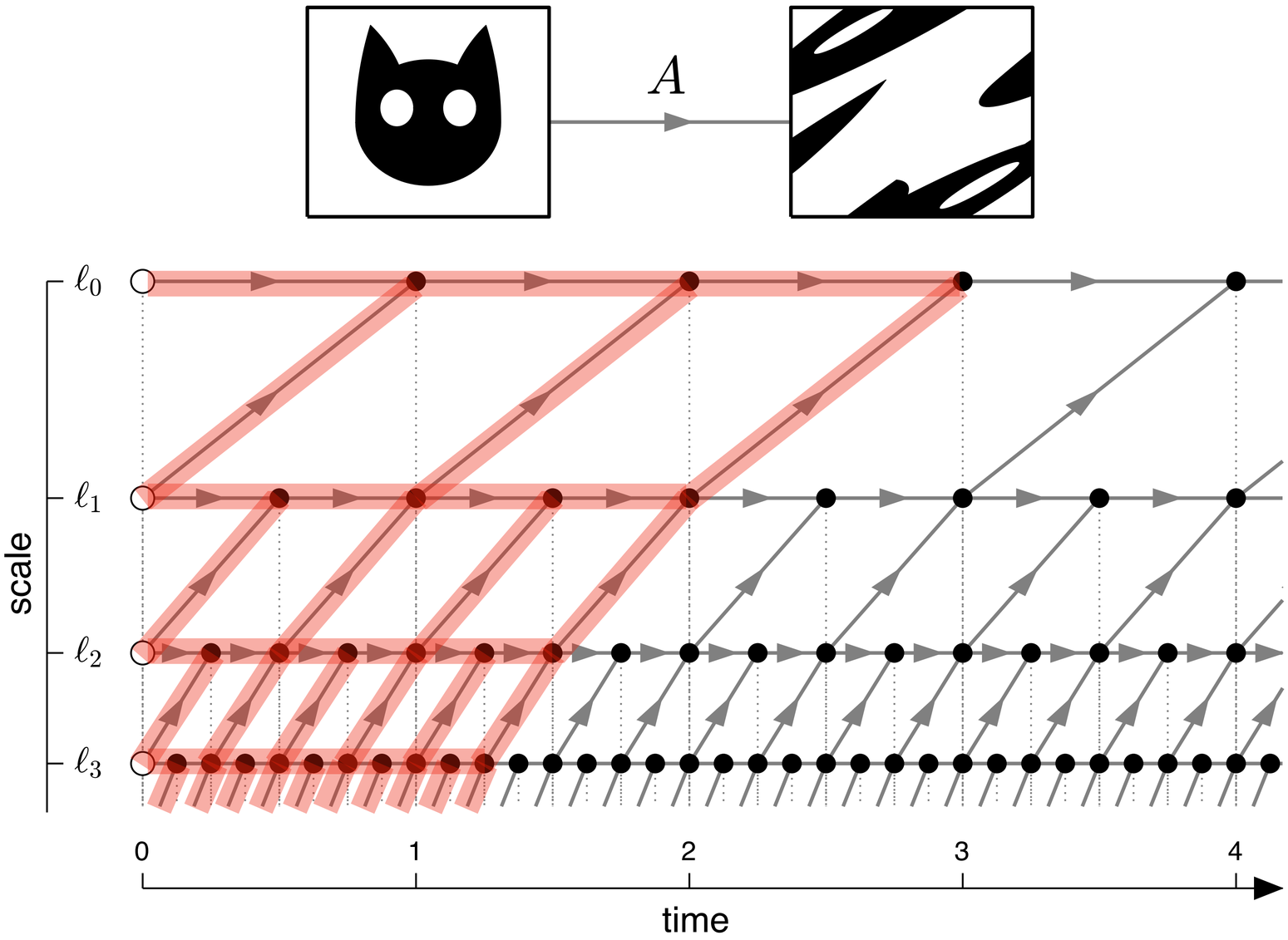
$$u_n(t+\tau_n) = A\big(u_n(t) + u_{n+1}(t)\big)$$

$$A:\ (x,y) \mapsto (2x+y,\ x+y) \bmod 1$$

Initial conditions:

$$u_n(0) = a_n, \quad n \in \mathbb{Z}_+$$

Multiscale Arnold's cat



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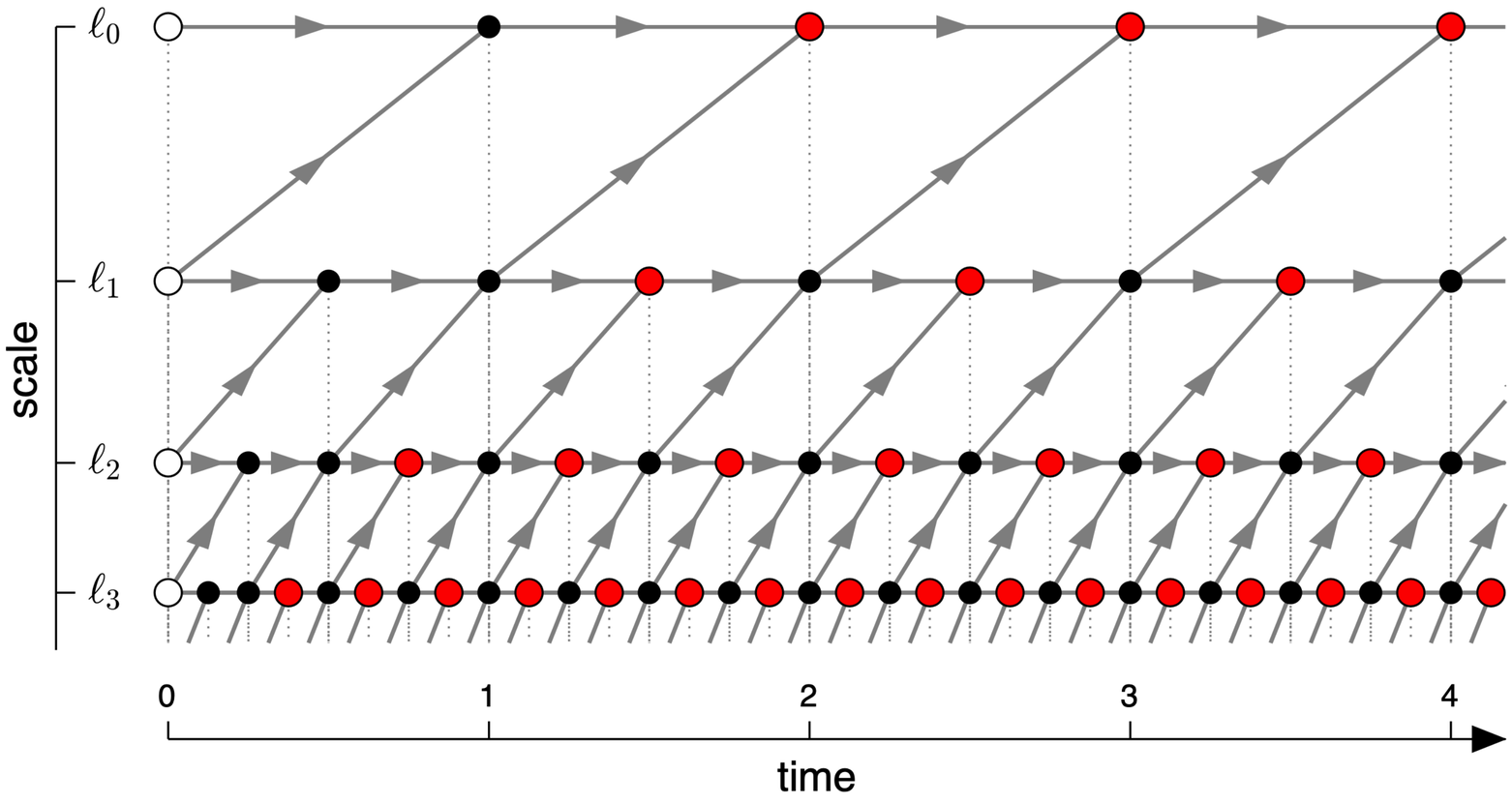
Nonuniqueness of solutions

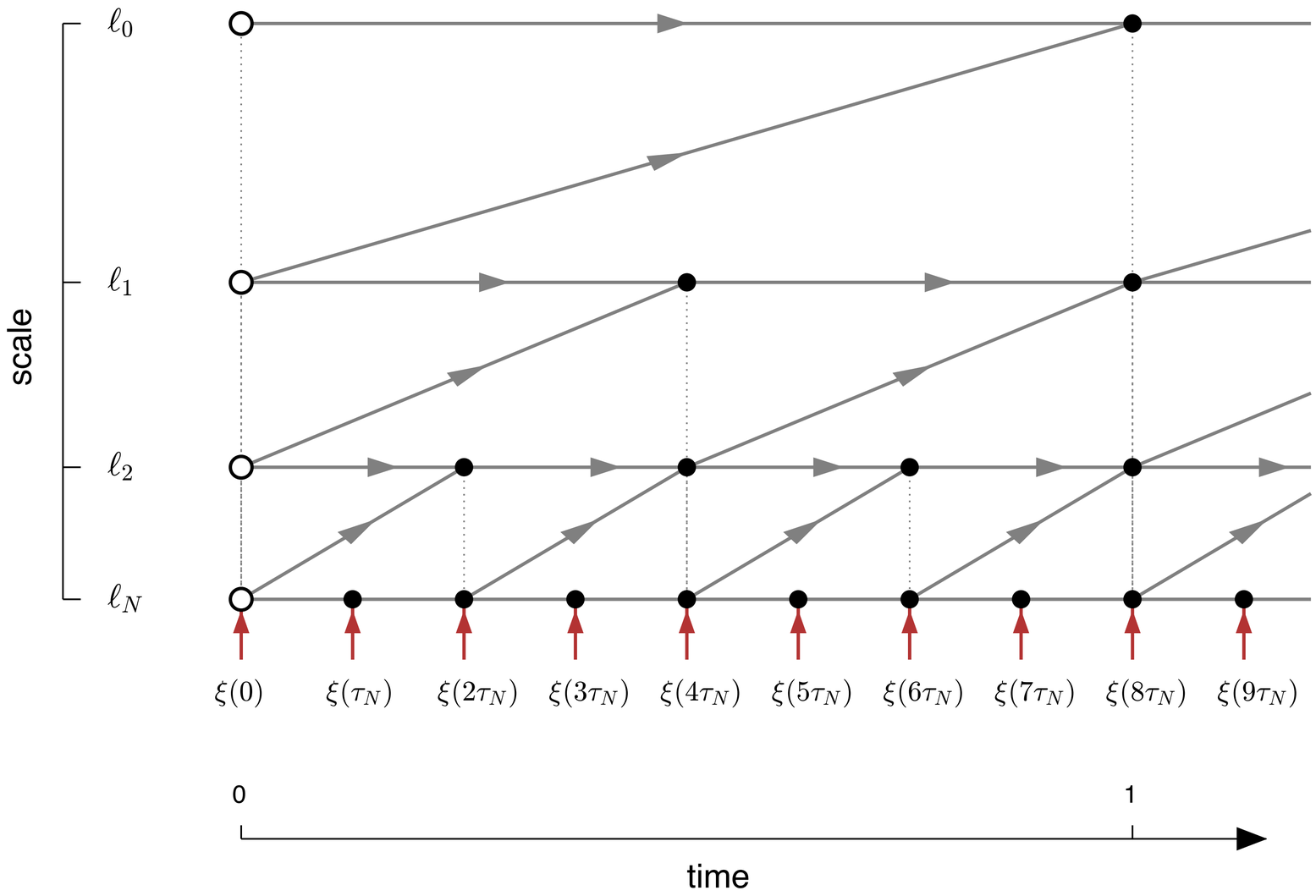


Theorem 1

For any initial condition, the system admits infinitely many solutions.

Namely, there is one solution for any values $u_n(t)$ at the red nodes.

Regularization



Cutoff:

$$u_n(t) = (0,0) \quad \text{for} \quad n > N$$

Noise:

$$u_N(t+\tau_N) = A\big(u_N(t) + \xi(t)\big)$$

i.i.d.
random

Well-posedness:

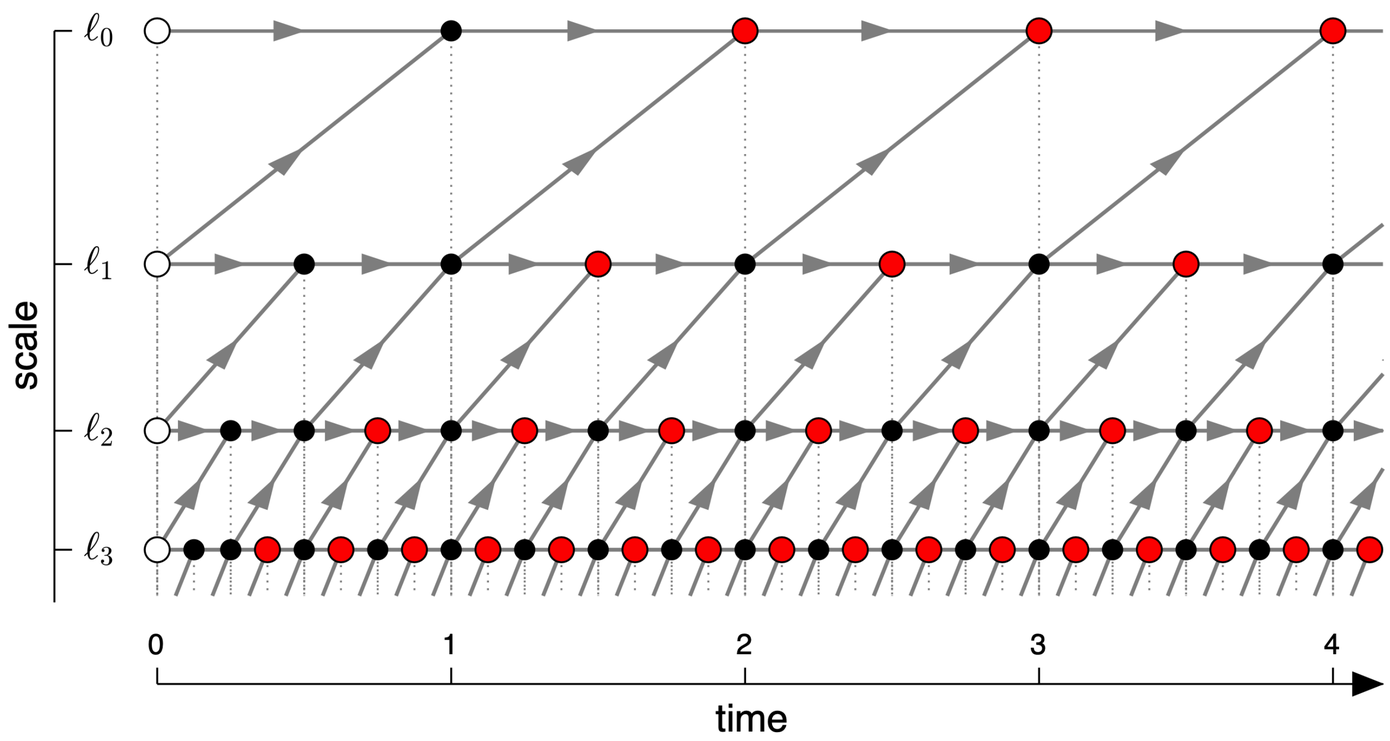
unique stochastic solution

for every N.

Turbulence:

Ideal limit: $N \to \infty$

Spontaneously stochastic ideal limit



Theorem 2

For any initial condition, regularized solutions converge in law as $N \to \infty$.

The limit is a stochastic process solving the ideal initial value problem.

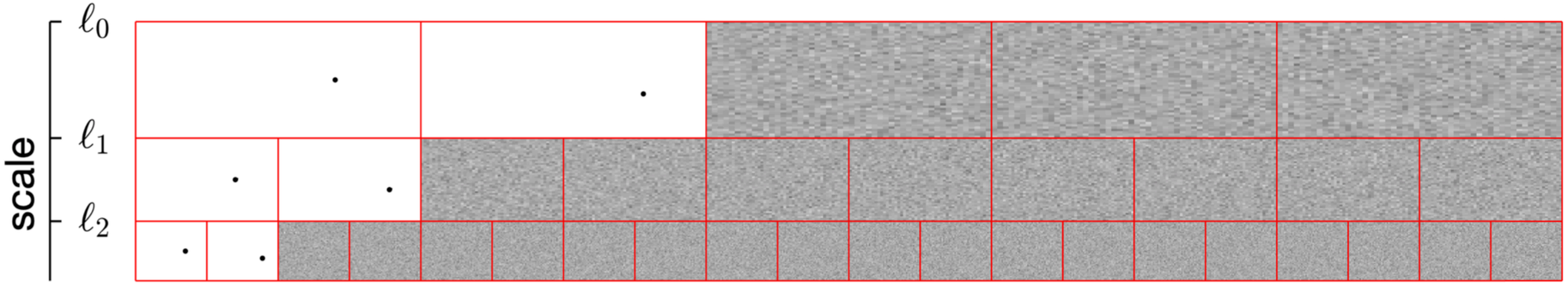
This limiting process is uniformly distributed on nonunique solutions of the ideal problem.

It is independent of the specific choice of the (absolute continuous) noise.

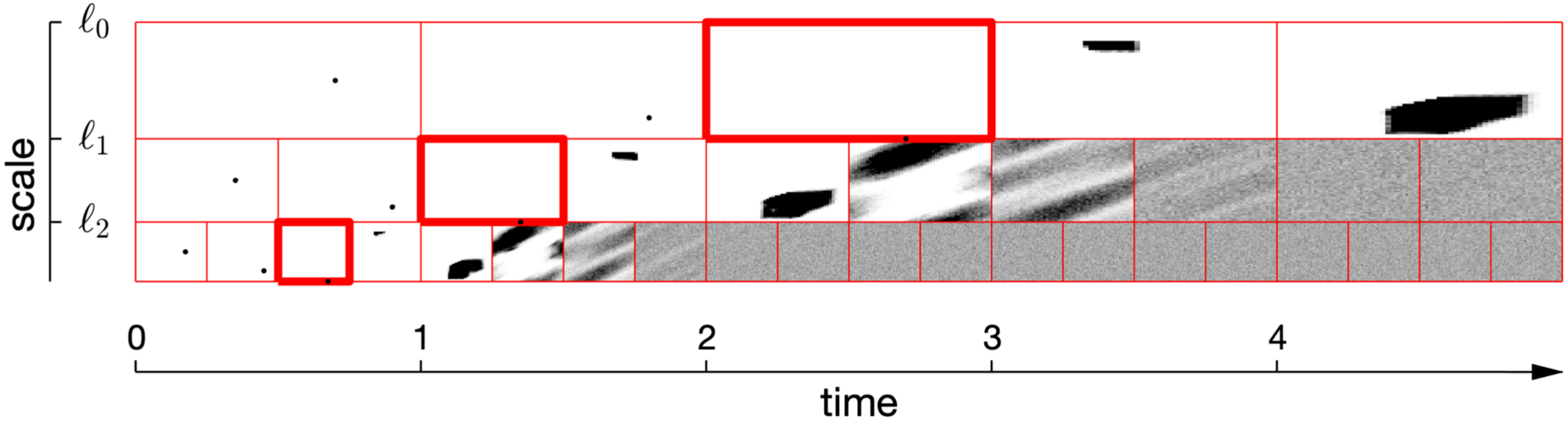
convergence • stochasticity • spontaneity • universality

Gerenalizations of the model

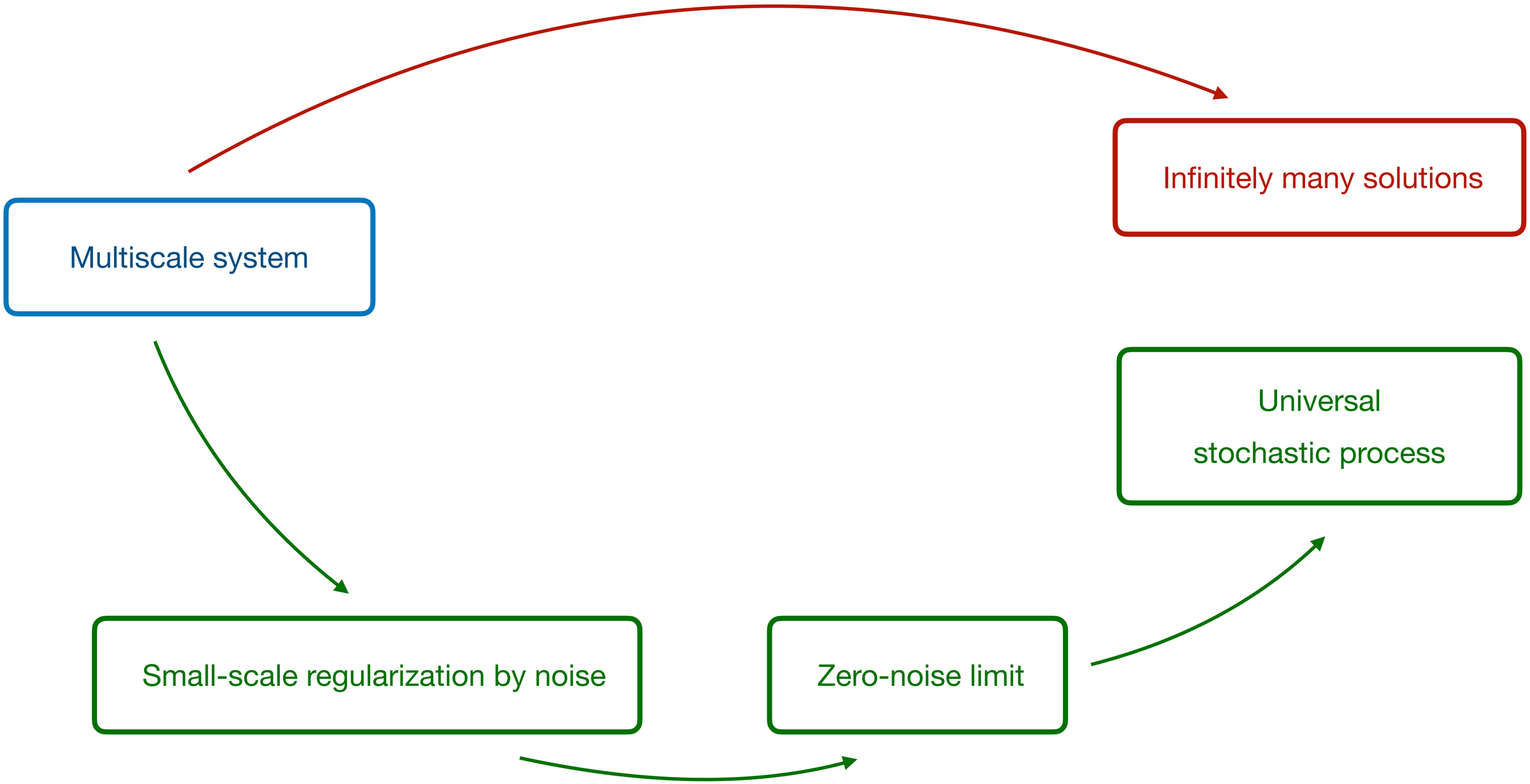
$$u_n(t+\tau_n) = Au_n(t) + Au_{n+1}(t)$$



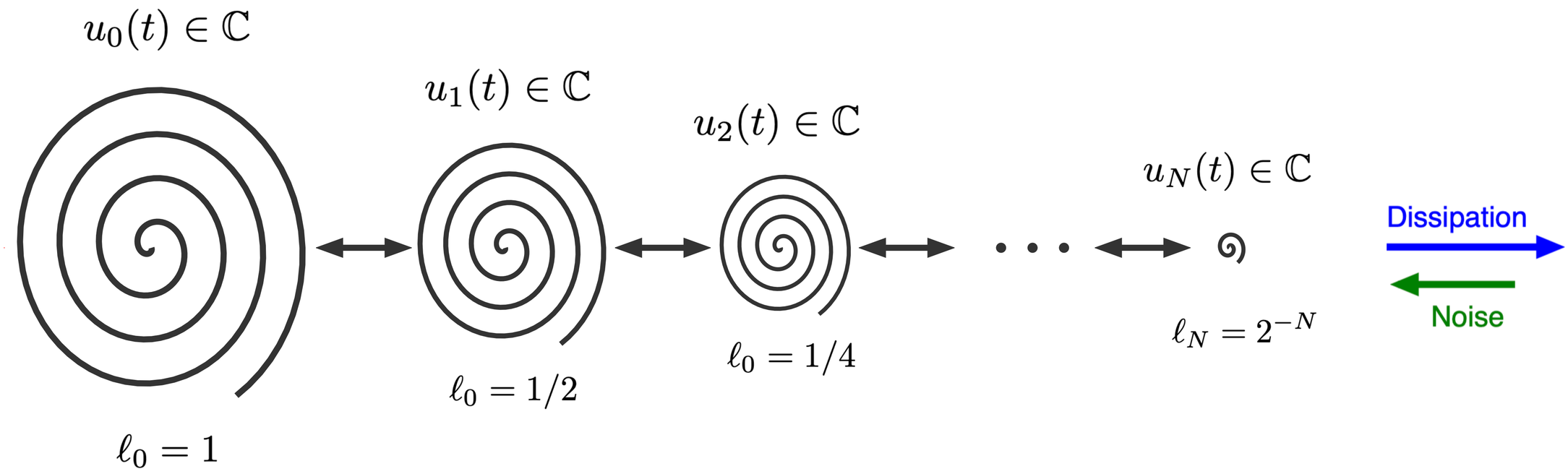
$$u_n(t+\tau_n) = Au_n(t) + 0.1\,(\cos x_{n+1}(t), \cos y_{n+1}(t))$$



stochastic (Markov) process


Multiscale system
Infinitely many solutions
Small-scale regularization by noise
Zero-noise limit
Universal
stochastic process

III. Spontaneous stochasticity in shell models of turbulence



Ideal system:

$$\boxed{\frac{du_n}{dt} = B_n, \quad n = 1, 2, \ldots}$$

Quadratic interaction term:

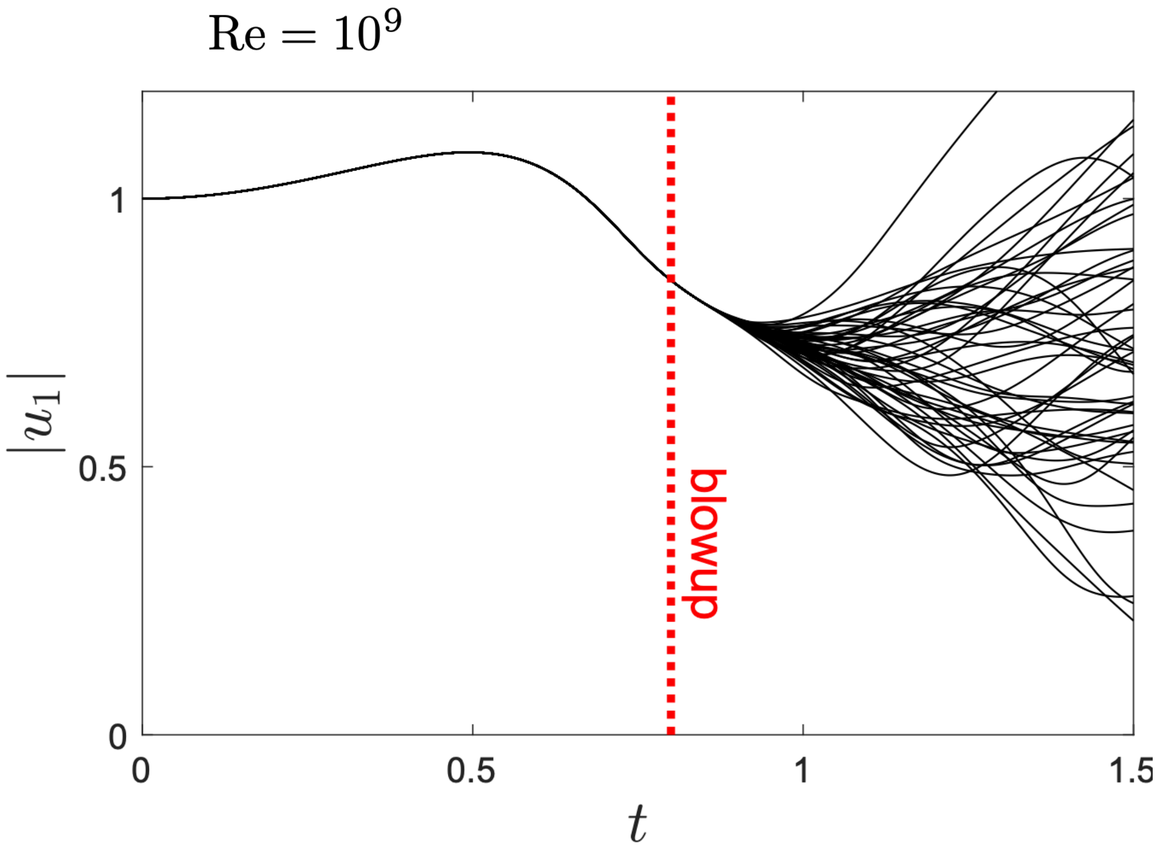
$$B_n = i2^n \left(2u_{n+2}u_{n+1}^* - \frac{1}{2}u_{n+1}u_{n-1}^* + \frac{1}{4}u_{n-1}u_{n-2} \right)$$

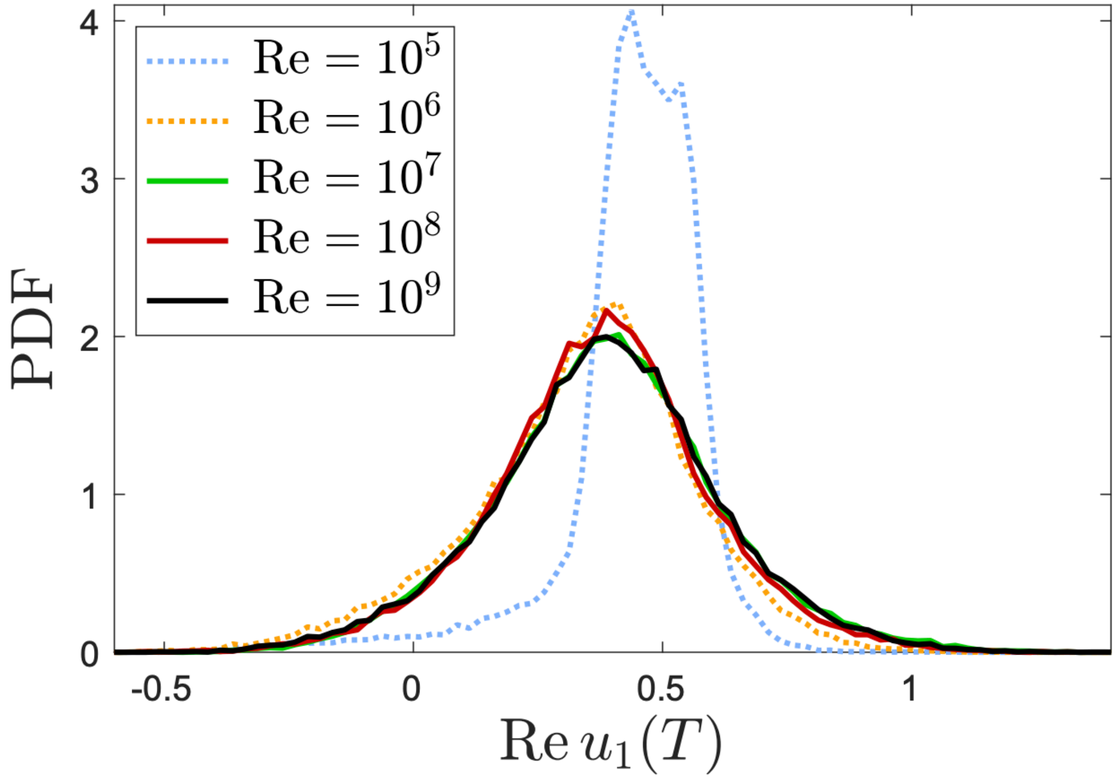
Regularized SDE system with noise:

$$\frac{du_n}{dt} = B_n - \mathrm{Re}^{-1}2^{2n}u_n + \sqrt{\Theta}\, 2^n \eta_n, \quad n = 1, \ldots, N$$

Ideal limit: $\mathrm{Re}^{-1} \to 0, \quad \Theta = \mathrm{Re}^{-\beta} \to 0, \quad \ell_N = 2^{-N} \sim \mathrm{Re}^{-\gamma} \to 0$

Numerical simulations





convergence • stochasticity • spontaneity • universality

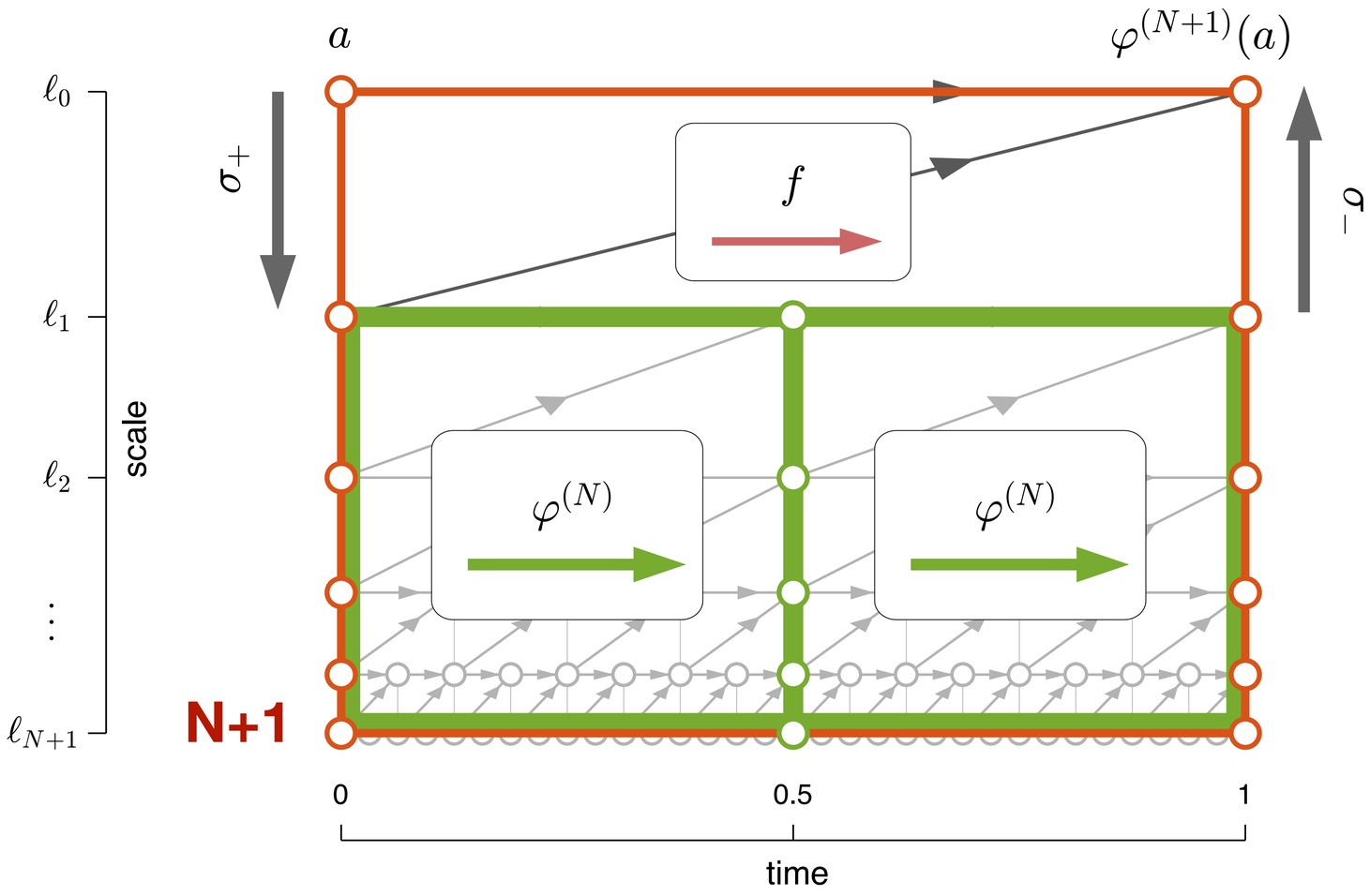
- Why zero-noise limit converges?
- Why it converges to a stochastic process?
- Why this limiting process is universal?

These properties do not follow from the nonuniqueness property alone!

IV. Renormalization-group (RG) approach

Renormalization group (RG) operator

deterministic case (no noise, just cutoff)



Flow map at unit time:

$$\varphi^{(N)}(a) = u(1)$$

Relation between flow maps:

$$\boxed{\varphi^{(N+1)} = \sigma_- \circ \varphi^{(N)} \circ \varphi^{(N)} \circ \sigma_+ + f}$$

Auxiliary maps:

$$f(u) = \big(A(u_0 + u_1), 0, 0, \dots\big),$$

$$\sigma_+(u) = (u_1, u_2, \dots), \quad \sigma_-(u) = (0, u_0, u_1, \dots)$$

Renormalization group (RG) operator

stochastic case (noise and cutoff)

Flow (Markov) kernel at unit time:

$$\Phi^{(N)}(\mathcal{U}\,|\,a) := \mathbb{P}\big(u(1) \in \mathcal{U}\,\big|\,u(0) = a\big)$$

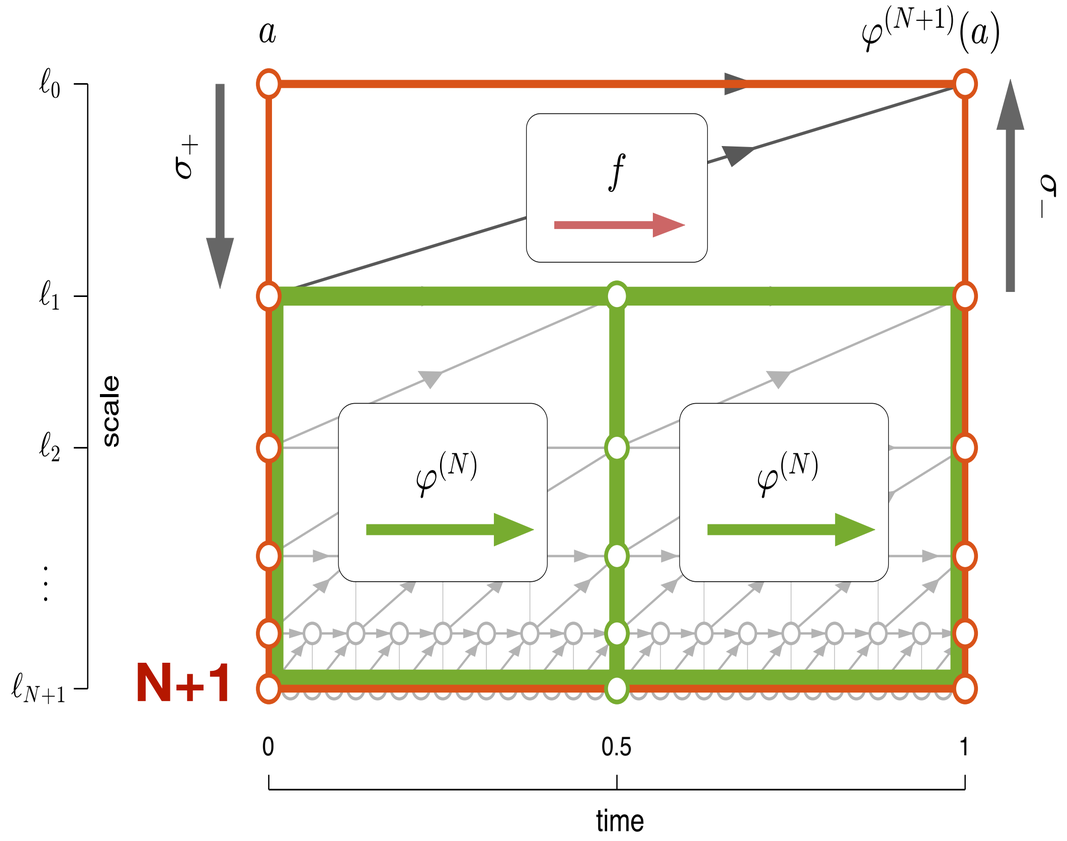
RG operator:

$$\Phi^{(N+1)} = \mathcal{R}[\Phi^{(N)}],$$

$$\mathcal{R}[\Phi] = \big(\Sigma_- \circ \Phi \circ \Phi \circ \Sigma_+\big) * F$$

Ideal limit

$N \to \infty$ → RG dynamics



Initial condition for RG dynamics: $\Phi^{(0)}(\cdot\,|\,a) = \nu_{a_0} \times \delta_0 \times \delta_0 \times \cdots,$

Fixed-point RG attractor

RG dynamics: $\Phi^{(N+1)} = \mathcal{R}[\Phi^{(N)}],$

RG fixed point attractor:

$$\Phi^{(N)} \to \Phi^* \quad \text{as} \quad N \to \infty$$

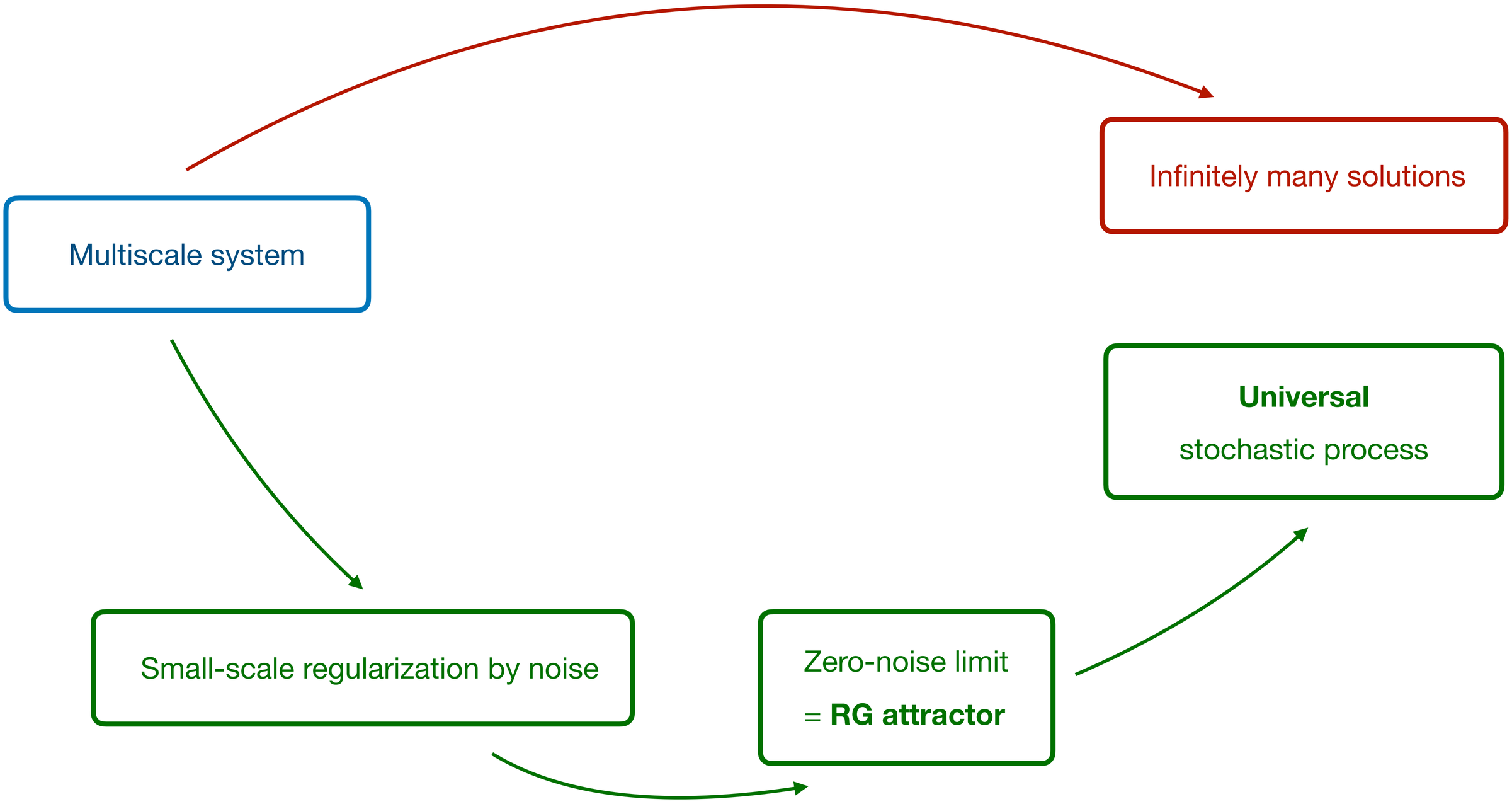
$$\Phi^* = \mathcal{R}[\Phi^*]$$

Ideal System	→	$\mathcal{R}$
Regularization & Noise	→	$\Phi^{(0)}$
Ideal limit	→	Φ^*

Corollary: there is a fixed-point RG attractor

$$\Phi^*(\cdot \,|\, a) = \delta_{A(a_0+a_1)} \times \lambda \times \lambda \times \cdots$$

RG attractor ⇒ Universality of spontaneous stochasticity


Multiscale system
Infinitely many solutions
Small-scale regularization by noise
Zero-noise limit
= **RG attractor**
Universal
stochastic process

Spontaneous stochasticity conjecture for turbulence

Initial value problem for the **3D stochastic Navier-Stokes equations**:

Euler equations (ideal fluid) — viscosity — space-time white noise

$$\boxed{\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \mathbf{f}} + \boxed{\mathrm{Re}^{-1}\Delta\mathbf{u}} + \boxed{\sqrt{\Theta}\,\nabla\cdot\boldsymbol{\xi}}, \quad \nabla\cdot\mathbf{u} = 0, \quad \mathbf{u}\big|_{t=0} = \mathbf{u}_0(\mathbf{x}).$$

$\mathrm{Re}^{-1} \sim 10^{-8} \to 0$

$\sqrt{\Theta} = 10^{-19} \to 0$

+ Galerkin regularization at molecular scale

$\delta = 10^{-12}$

Ideal limit:

$$\mathrm{Re}^{-1} \to 0$$

$$\sqrt{\Theta} = \mathrm{Re}^{-\beta} \to 0$$

$$\delta = \mathrm{Re}^{-\gamma} \to 0$$

Spontaneously stochastic solution in the ideal limit?

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \mathbf{f}$$

$$\nabla\cdot\mathbf{u} = 0, \quad \mathbf{u}\big|_{t=0} = \mathbf{u}_0(\mathbf{x})$$

Mathematics implications

Inviscid Burgers equation

$$u_t + u u_x = 0$$

nonunique weak solutions

add viscous term

$$u_t + u u_x = \varepsilon\, u_{xx}$$

take zero viscosity limit

Uniquely selected weak solution (shock)

ideal system with nonunique solutions

stochastic regularization + zero noise limit

Spontaneously stochastic solution

Classical stochastic analysis

RG + dynamical systems

V. Relation to other RG theories

Multiscale Arnold's cat

$$\varphi \;\rightarrow\; \sigma_- \circ \varphi \circ \varphi \circ \sigma_+ + f$$

scaling

scaling

extra term - interaction

$$\varphi(x) \;\rightarrow\; -\alpha\, \varphi \circ \varphi\, (-x/\alpha)$$

Feigenbaum

Feigenbaum RG equation

Feigenbaum-Cvitanovic-Coullet-Tresser

Dynamics of a (unimodal) map:

$$x_{m+1} = f(x_m), \quad m = 0, 1, 2, \ldots,$$

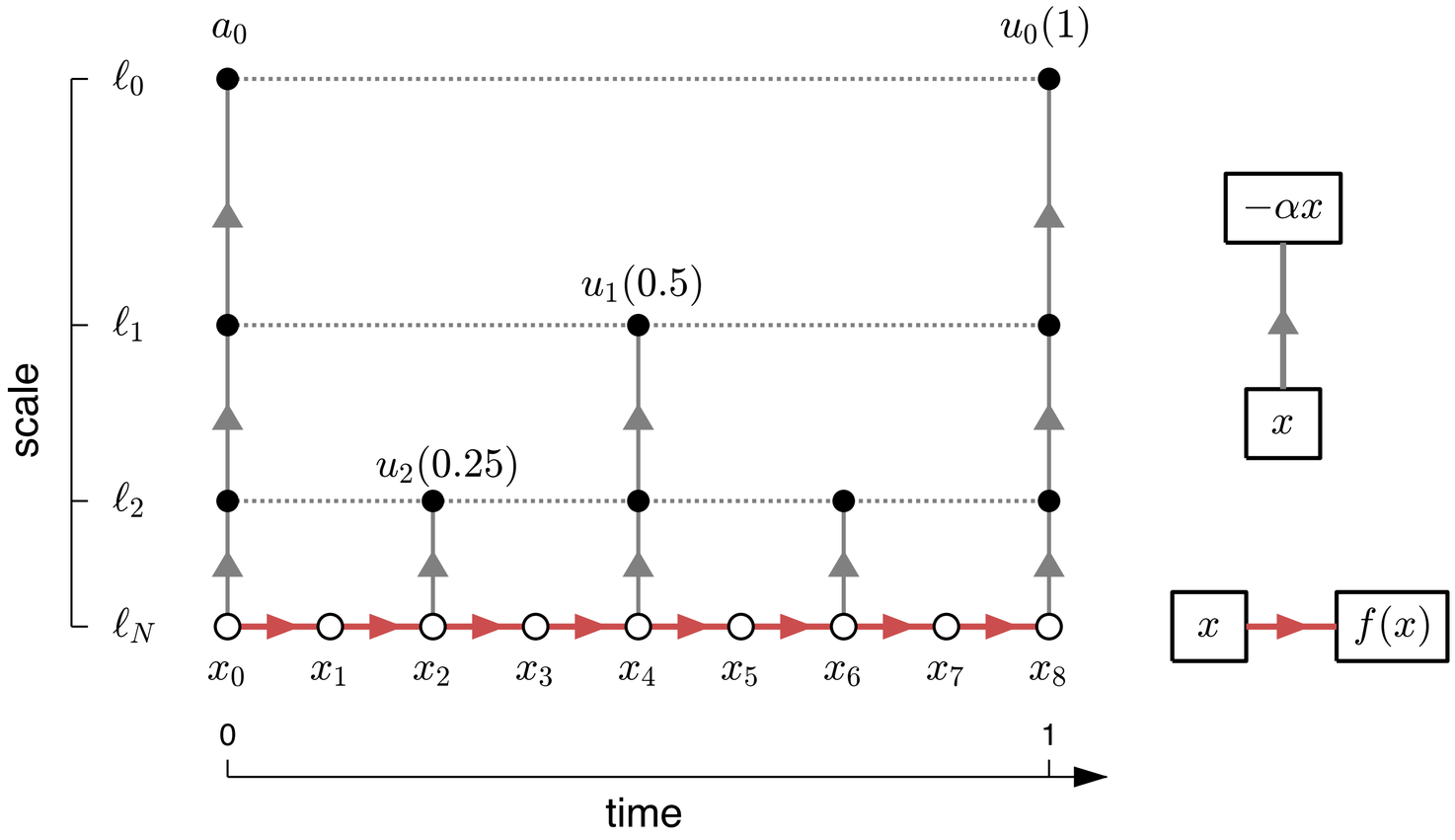
$$f : \mathbb{R} \to \mathbb{R}$$

Cutoff dynamics:

$$u_N(t + \tau_N) = f\big(u_N(t)\big)$$

Ideal (large-scale) dynamics:

$$u_n(t) = -\alpha\, u_{n+1}(t)$$



Flow map of the lattice:

$$\varphi^{(N)}(a_0) = u_0(1)$$

RG dynamics: $\varphi^{(N+1)} = \mathcal{R}[\varphi^{(N)}], \quad \mathcal{R}[\varphi](x) = -\alpha\, \varphi \circ \varphi(-x/\alpha).$

RG fixed point: universal Feigenbaum map

RG of the central limit theorem

I.i.d. random numbers $x_1, x_2, x_3, \ldots$

(zero mean and unit variance)

CLT: $$\frac{1}{\sqrt{m}} \sum_{i=1}^{m} x_m \to \mathcal{N}(0,1)$$

Cutoff noise:

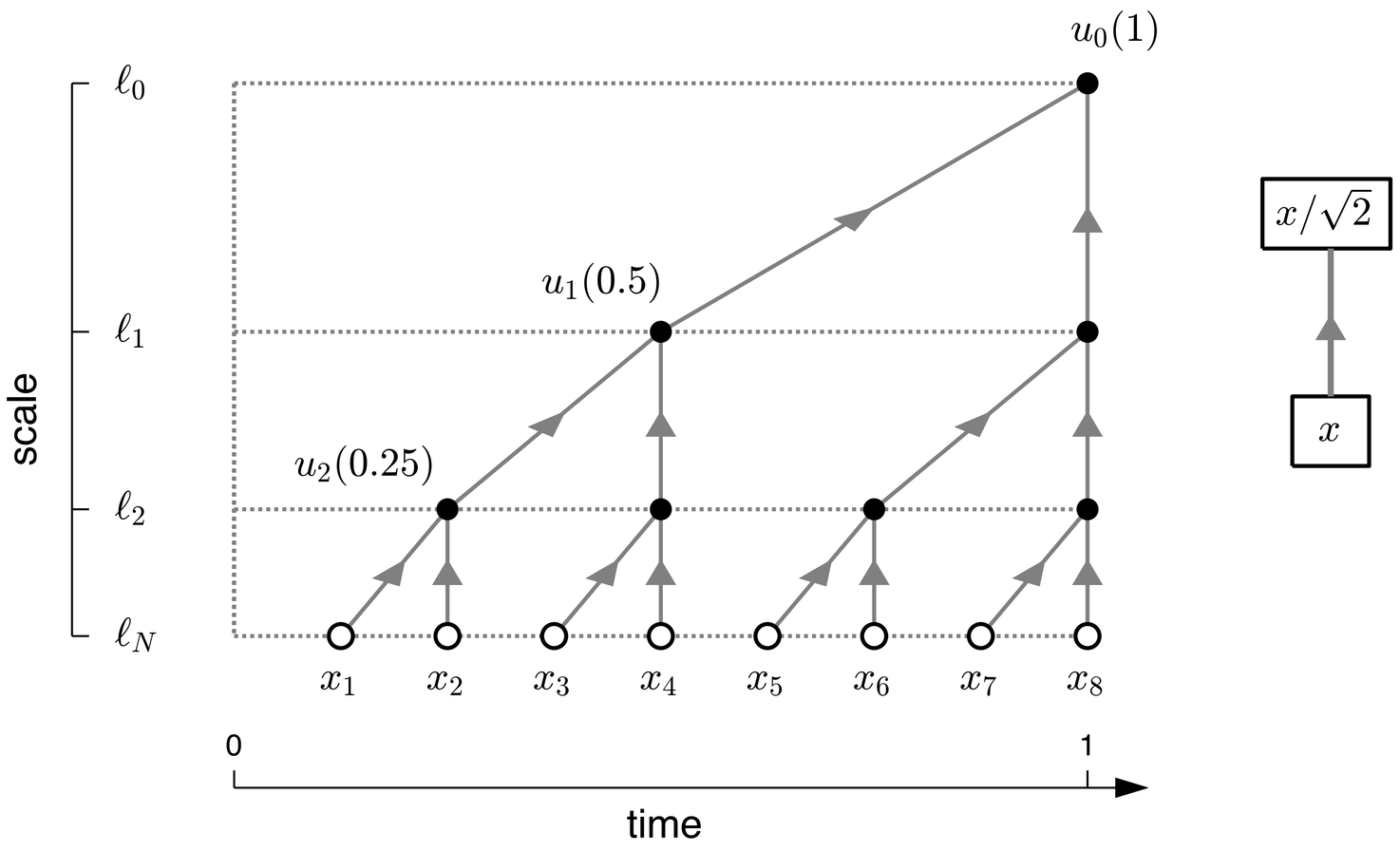
$$u_N(m\tau_N) = x_m, \qquad m = 1, 2, \ldots$$

Ideal (large-scale) dynamics:

$$u_n(t) = \frac{u_{n+1}(t) + u_{n+1}(t - \tau_{n+1})}{\sqrt{2}}$$

Large-scale distribution:

$$\varphi^{(N)} = \mathrm{Dist}\left[u_1(0)\right]$$



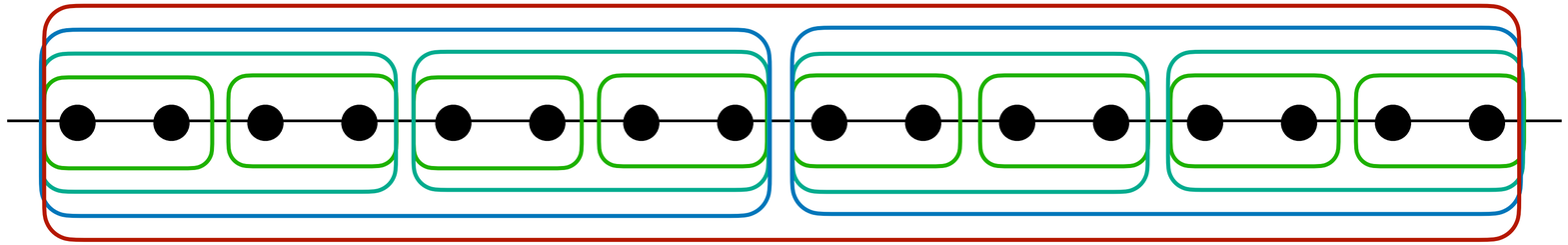
RG dynamics: $\varphi^{(N+1)} = \mathcal{R}[\varphi^{(N)}], \quad \mathcal{R}[\varphi](x) = \sqrt{2} \int_{\mathbb{R}} \varphi(\sqrt{2}\,x - u)\, \varphi(u)\, du.$

Fixed point RG attractor: $\mathcal{N}(0,1)$

Sinai's book, 1991

RG for a hierarchical spin model

Spins (real or discrete): $x_1, x_2, x_3, \ldots$



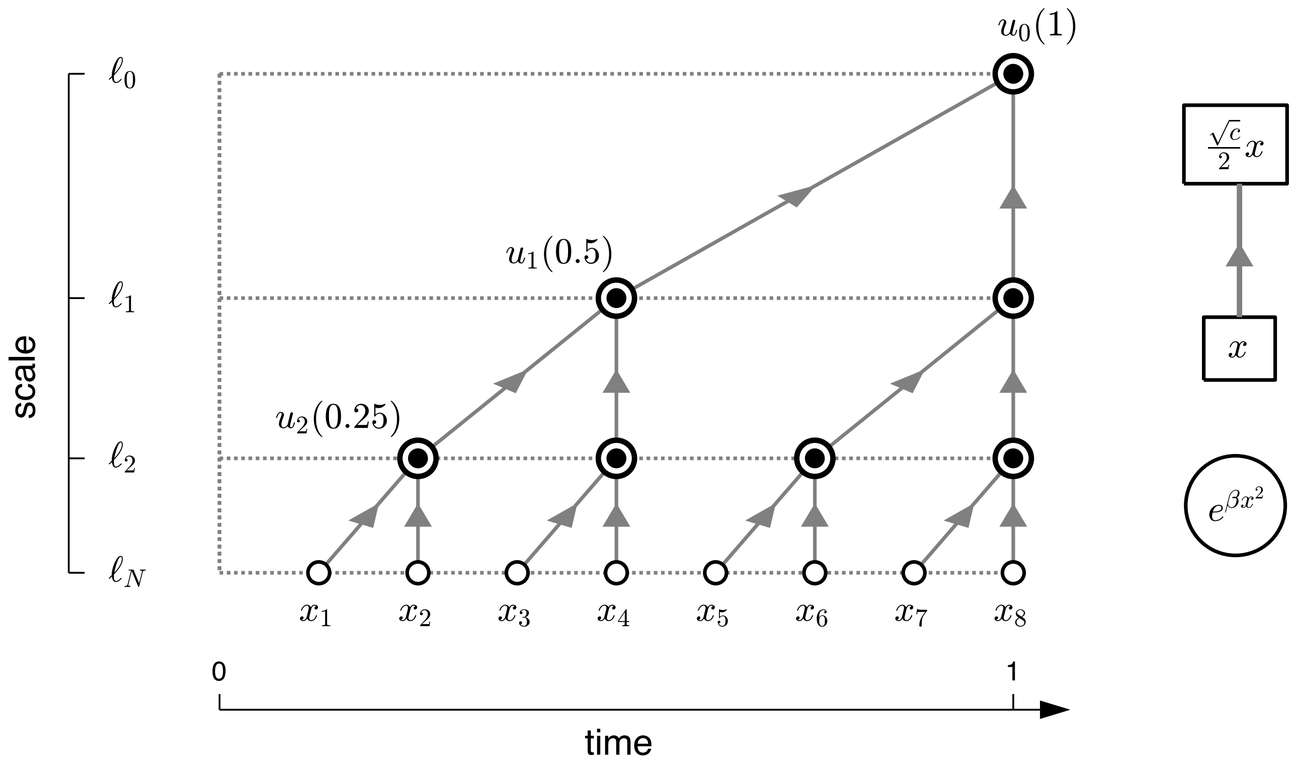
Block spins: $$S_{l,r} = \sum_{m=(r-1)2^l+1}^{r2^l} x_m$$

Hamiltonian:

$$H_N(x_1, \ldots, x_{2^N}) = -\sum_{l=1}^{N} \sum_{r=1}^{2^{N-l}} \left(\frac{\sqrt{c}}{2}\right)^{2l} S_{l,r}^2,$$

Gibbs measure:

$$d\mu_N(x_1, \ldots, x_{2^N}) = Z_N^{-1} e^{-\beta H_N(x_1,\ldots,x_{2^N})} \prod_{m=1}^{2^N} d\mu_0(x_m)$$



Macroscopic distribution:

$$\varphi^{(N)} = \mathrm{Dist}\left[u_1(0)\right]$$

RG dynamics:

$$\varphi^{(N+1)} = \mathcal{R}[\varphi^{(N)}], \quad \mathcal{R}[\varphi](x) = z\, e^{\beta x^2} \int_{\mathbb{R}} \varphi\left(\frac{2x}{\sqrt{c}} - u\right) \varphi(u)\, du$$

alexei.impa.br

Thank you!

- *A. A. Mailybaev and A. Raibekas, Spontaneous stochasticity and renormalization group in discrete multi-scale dynamics, Communications in Mathematical Physics 401, 2643–2671 (2023).*
- *A. A. Mailybaev and A. Raibekas, Spontaneously stochastic Arnold's cat, Arnold Mathematical Journal **9**, 339-357 (2023).*
- *D. Bandak et al., Spontaneous stochasticity amplifies even thermal noise to the largest scales of turbulence in a few eddy turnover times, Physical Review Letters 132, 104002 (2024).*
- *A. A. Mailybaev, RG theory of spontaneous stochasticity for Sabra model of turbulence. Physical Review Fluids **11**, 034605 (2026).*
- *A.A. Mailybaev and L. Moriconi, Renormalization-group perspective on spontaneous stochasticity, Preprint arXiv:2602.24221 (2026).*