Pathwise quantitative particle approximation of nonlinear stochastic Fokker-Planck equations via relative entropy

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Interacting particle system

For $N \in \mathbb{N}$ and $i \in \{1, ..., N\}$ we have for $\omega \in \Omega$

$$X_t^{i,N}(\omega) = \int_0^t \frac{1}{N} \sum_{k=1}^{N} (K * V^N)(X_s^{i,N}(\omega) - X_s^{k,N}(\omega))\, ds$$
$$+ \int_0^t dW_s^{i,N}(\omega)$$
$$+ \int_0^t \sigma_s \, dB_s(\omega).$$

where $W^{i,N}$ and B are independent standard $\mathbb{R}^d$-valued Brownian motions, defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.

Interacting particle system

By Itô's formula to $V^N(x-\cdot)$ for each $i \in \{1,\ldots,N\}$ and $x \in \mathbb{R}^d$

$$
\begin{aligned}
V^N(x - X_t^{i,N}) = \;& V^N(x - X_0^{i,N}) \\
& - \sum_{j=1}^{d} \int_0^t \partial_j V^N(x - X_s^{i,N}) K_j * \rho_s^N(X_s^{i,N})\, ds \\
& + \frac{1}{2} \sum_{j,k=1}^{d} \int_0^t \partial_{jk} V^N(x - X_s^{i,N}) (\sigma\sigma^T)_s^{jk}\, ds \\
& - \sum_{j,k=1}^{d} \int_0^t \partial_j V^N(x - X_s^{i,N}) \sigma_s^{jk}\, dB_s^k \\
& - \sum_{j,k=1}^{d} \int_0^t \partial_j V^N(x - X_s^{i,N})\, dW_s^{i,N,k} \\
& + \frac{1}{2} \sum_{j,k=1}^{d} \int_0^t \partial_{jk} V^N(x - X_s^{i,N})\, ds.
\end{aligned}
$$

Interacting particle system

Then since $\rho^N = V^N * S^N$, we have

$$\begin{aligned}
\rho_t^N(x) = \; & \rho_0^N(x) \\
& - \sum_{j=1}^{d} \int_0^t \left\langle S_s^N, \partial_j V^N(x-\cdot) K_j * \rho_s^N(\cdot) \right\rangle ds \\
& + \frac{1}{2} \sum_{j,k=1}^{d} \int_0^t \left\langle S_s^N, \partial_{jk} V^N(x-\cdot)(\sigma\sigma^T)_s^{jk} \right\rangle ds \\
& - \sum_{j,k=1}^{d} \int_0^t \left\langle S_s^N, \partial_j V^N(x-\cdot)\sigma_s^{jk} \right\rangle dB_s^k \\
& - \frac{1}{N} \sum_{i=1}^{N} \sum_{j,k=1}^{d} \int_0^t \partial_j V^N(x - X_s^{i,N})\, dW_s^{i,N,k} \\
& + \frac{1}{2} \sum_{j,k=1}^{d} \int_0^t \left\langle S_s^N, \partial_{jk} V^N(x-\cdot) \right\rangle ds.
\end{aligned}$$

Interacting particle system

Or more concisely,

$$
\begin{aligned}
d\rho_t^N(x) = & -\nabla \cdot \left\langle S_t^N, V^N(x-\cdot) K * \rho_t^N(\cdot) \right\rangle dt \\
& + \frac{1}{2} \sum_{j,k=1}^{d} \left\langle S_t^N, \partial_{jk} V^N(x-\cdot)(\sigma\sigma^T)_t^{jk} \right\rangle dt \\
& - \nabla \cdot \left\langle S_t^N, V^N(x-\cdot)\sigma_t^{jk} \right\rangle dB_t \\
& - \frac{1}{N} \sum_{i=1}^{N} \nabla V^N(x - X_s^{i,N}) \nu_t(X_t^{i,N})\, dW_t^{i,N} \\
& + \frac{1}{2} \sum_{j,k=1}^{d} \left\langle S_t^N, \partial_{jk} V^N(x-\cdot) \right\rangle dt.
\end{aligned}
$$

Limiting process

$$d\rho_t = \frac{1}{2}\Delta\rho_t + \frac{1}{2}D^2\rho_t(\sigma\sigma^\top)_t\,dt - \nabla\cdot\left(\rho_t K * \rho_t\right)dt$$
$$-\nabla\rho_t\cdot\sigma_t\,dB_t.$$

$$\begin{aligned}
d\rho_t^N &= -\nabla\cdot\left\langle S_t^N, V^N(x-\cdot)K*\rho_t^N(\cdot)\right\rangle dt\\
&\quad+\frac{1}{2}\sum_{j,k=1}^{d}\left\langle S_t^N, \partial_{jk}V^N(x-\cdot)(\sigma\sigma^T)_t^{jk}\right\rangle dt\\
&\quad-\sum_{j,k=1}^{d}\left\langle S_t^N, \partial_j V^N(x-\cdot)\sigma_t^{jk}\right\rangle dB_t^k\\
&\quad-\frac{1}{N}\sum_{i=1}^{N}\nabla V^N(x-X_s^{i,N})\,dW_t^{i,N}\\
&\quad+\frac{1}{2}\sum_{j,k=1}^{d}\left\langle S_t^N, \partial_{jk}V^N(x-\cdot)\right\rangle dt.
\end{aligned}$$

First main result

Take $\delta > 0$ such that $\theta > 0$, where

$$\theta = \min\left(\beta(1-2\gamma); \frac{\beta}{d}\gamma^2; \frac{1}{2} - \beta\left(1+\frac{1}{d}\right)\right) - \delta,$$

with $d \geq 1$ and $\beta \in \left(0, \frac{1}{2\left[1+\frac{1}{d}\right]}\right)$. In addition, assume

$$\lim_{N\to\infty} N^{\theta}\mathcal{H}(\rho_0^N|\rho_0) = 0, \mathbb{P} - a.s.$$

Then, we have

$$\lim_{N\to\infty} N^{\theta} \sup_{t\in[0,T]} \mathcal{H}(\rho_t^N|\rho_t) = 0, \mathbb{P} - a.s.$$

Corollary: Propagation of chaos

Let the same assumptions as in previous Theorem. Then, we have

$$\lim_{N\to\infty} N^{\theta-\delta} \sup_{t\in[0,T]} \left\| S_t^N - \rho_t \right\|_0^2 = 0, \ \ \mathbb{P} - a.s.$$

Second main result

Take $\sigma = 0$ and

$$\theta = \min\left(\beta(1-2\gamma); \frac{\beta}{d}\gamma^2; \frac{1}{2} - \beta\Big(1+\frac{1}{d}\Big)\right),$$

with $d \geq 1$ and $\beta \in \left(0, \frac{1}{2\left[1+\frac{1}{d}\right]}\right)$. In addition, assume for any $m \geq 1$,

$$\sup_{N\in\mathbb{N}} \left\|\mathcal{H}(\rho_0^N|\rho_0)\right\|_{L^m(\Omega)} < \infty.$$

Then, we have

$$\left\|\sup_{t\in[0,T]} \mathcal{H}(\rho_t^N|\rho_t)\right\|_{L^m(\Omega)} \lesssim \left\|\mathcal{H}(\rho_0^N|\rho_0)\right\|_{L^m(\Omega)} + N^{-\theta}.$$

Main novelties

Apply the relative entropy functional in the context of moderate interacting particles;

Introduction of a new class of scaling $V^N \doteq N^\beta V(N^{\frac{\beta}{d}} \cdot)$, where the mollifier V is not compactly supported;

Introduction of a class of singular kernels satisfying $K = \nabla \cdot K_0$, for $K_0 \in C^\gamma$;

Proof: A priori estimates via Ito's formula

$$
\begin{aligned}
\mathcal{H}(\rho_t^N|\rho_t) - \mathcal{H}(\rho_0^N|\rho_0) &= \int_{\mathbb{T}^d}\int_0^t \Big[\ln(\rho_s) - \ln(\rho_s^N)\Big] \Big\langle S_s^N, \nabla V^N(x-\cdot)K * \rho_s^N(\cdot)\Big\rangle \, ds\, dx \\
&+ \int_{\mathbb{T}^d}\int_0^t \Big[-\Big\langle S_s^N, \nabla V^N(x-\cdot)K * \rho_s^N(\cdot)\Big\rangle + \frac{\rho_s^N}{\rho_s}\nabla\cdot(\rho_s K * \rho_s)\Big] \, ds\, dx \\
&+ \frac{1}{2}\int_{\mathbb{T}^d}\int_0^t \Big[\ln(\rho_s^N) - \ln(\rho_s)\Big] (\sigma\sigma^\top)_s D^2 \rho_s^N \, ds\, dx \\
&+ \frac{1}{2}\int_{\mathbb{T}^d}\int_0^t (\sigma\sigma^\top)_s D^2\rho_s^N \, ds\, dx - \frac{1}{2}\int_{\mathbb{T}^d}\int_0^t \frac{\rho_s^N}{\rho_s} D^2\rho_s(\sigma\sigma^\top)_s \, ds\, dx \\
&+ \frac{1}{2}\int_{\mathbb{T}^d}\int_0^t \Big[\ln(\rho_s^N) - \ln(\rho_s)\Big] \Delta\rho_s^N \, ds\, dx \\
&+ \frac{1}{2}\int_{\mathbb{T}^d}\int_0^t \Big[-\frac{\rho_s^N}{\rho_s}\Delta\rho_s + \Delta\rho_s^N\Big] \, ds\, dx + \int_{\mathbb{T}^d}\int_0^t \Big[\ln(\rho_s) - \ln(\rho_s^N)\Big] \sigma_s^\top \nabla\rho_s^N \, dB_s\, dx \\
&+ \int_{\mathbb{T}^d}\int_0^t \sigma_s^\top \nabla\rho_s^N \, dB_s\, dx + \frac{1}{N}\sum_{i=1}^N \int_{\mathbb{T}^d}\int_0^t \Big[\ln(\rho_s) - \ln(\rho_s^N)\Big] \nabla V^N(x - X_s^i)\, dW_s^i \, dx \\
&- \frac{1}{N}\sum_{i=1}^N \int_{\mathbb{T}^d}\int_0^t \nabla V^N(x - X_s^i)\, dW_s^i\, dx + \int_{\mathbb{T}^d}\int_0^t \frac{\rho_s^N}{\rho_s}\sigma_s^\top \nabla\rho_s \, dB_s\, dx \\
&+ \frac{1}{2}\int_{\mathbb{T}^d}\int_0^t \frac{1}{\rho_s^N}\Big|\sigma_s^\top\nabla\rho_s\Big|^2 \, ds\, dx + \frac{1}{2N^2}\sum_{i=1}^N \int_{\mathbb{T}^d}\int_0^t \frac{1}{\rho_s^N}|\nabla V^N(x - X_s^i)|^2 \, ds\, dx \\
&- \int_{\mathbb{T}^d}\int_0^t \frac{1}{\rho_s}\sigma_s^\top \nabla\rho_s^N \sigma_s^\top \nabla\rho_s \, ds\, dx + \frac{1}{2}\int_{\mathbb{T}^d}\int_0^t \frac{\rho_s^N}{(\rho_s)^2}\Big|\sigma_s^\top\nabla\rho_s\Big|^2 \, ds\, dx,
\end{aligned}
$$

Proof: A priori estimates via Ito's formula

$$\mathcal{H}(\rho_t^N|\rho_t) - \mathcal{H}(\rho_0^N|\rho_0) = I_t + II_t + III_t + IV_t + MARTINGALE$$

$$I_t = \int_0^t \int_{\mathbb{T}^d} \rho_s^N \nabla \ln\left(\frac{\rho_s}{\rho_s^N}\right) [K * \rho_s - K * \rho_s^N] \, dx \, ds$$
$$- \int_0^t \int_{\mathbb{T}^d} \nabla \ln\left(\frac{\rho_s}{\rho_s^N}\right) \left\langle S_s^N, V^N(x - \cdot)[K * \rho_s^N(\cdot) - K * \rho_s^N(x)] \right\rangle \, dx \, ds,$$

$$II_t + IV_t = \frac{1}{N^2} \sum_{i=1}^N \int_0^t \int_{\mathbb{T}^d} \frac{1}{\rho_s^N} |\nabla V^N(x - X_s^{i,N})|^2 \, dx \, ds,$$

$$III_t = -\int_0^t \mathcal{I}\left(\rho_s^N|\rho_s\right) \, ds,$$

Proof: $|\nabla V_0| \lesssim V_0$

$$\frac{1}{2N^2}\sum_{i=1}^{N}\int_{\mathbb{T}^d}\int_0^t \frac{1}{\rho_s^N}|\nabla V^N(x-X_s^i)|^2\,ds\,dx.$$

We note

$$\begin{aligned}
\left|\nabla V^N(x-X_s^i)\right| &\le \sum_{k\in\mathbb{Z}^d}\left|\nabla V_0^N\left((x-X_s^i)-k\right)\right| \\
&= \sum_{k\in\mathbb{Z}^d}\left|N^{\frac{\beta}{d}}N^{\beta}(\nabla V_0)\left(N^{\frac{\beta}{d}}((x-X_s^i)-k)\right)\right| \\
&= N^{\frac{\beta}{d}}\sum_{k\in\mathbb{Z}^d}N^{\beta}\left|(\nabla V_0)\left(N^{\frac{\beta}{d}}((x-X_s^i)-k)\right)\right|.
\end{aligned}$$

Proof: $|\nabla V_0| \lesssim V_0$

Now since $|\nabla V_0| \lesssim V_0$, we have

$$
\begin{aligned}
\left|\nabla V^N(x - X_s^i)\right| &\leq N^{\frac{\beta}{d}} \sum_{k \in \mathbb{Z}^d} N^{\beta} \left| V_0 \left(N^{\frac{\beta}{d}} ((x - X_s^i) - k) \right) \right| \\
&= N^{\frac{\beta}{d}} V^N(x - X_s^i) \\
&\leq 2N^{\beta} C_d N^{\frac{\beta}{d}}.
\end{aligned}
$$

Proof: $|\nabla V_0| \lesssim V_0$

It follows that,

$$
\begin{aligned}
&\frac{1}{2N^2}\sum_{i=1}^{N}\int_{\mathbb{T}^d}\int_0^t \frac{1}{\rho_s^N(x)}|\nabla V^N(x-X_s^i)|^2\,ds\,dx \\
&\le C_d\frac{2N^{\beta}N^{\frac{\beta}{d}}}{2N^2}\sum_{i=1}^{N}\int_{\mathbb{T}^d}\int_0^t \frac{1}{\rho_s^N(x)}|\nabla V^N(x-X_s^i)|\,ds\,dx \\
&\le C_d\frac{N^{\beta+2\frac{\beta}{d}}}{N}\int_0^t\int_{\mathbb{T}^d}\frac{1}{\rho_s^N(x)}\frac{1}{N}\sum_{i=1}^{N}V^N(x-X_s^i)\,dx\,ds \\
&= C_d\frac{N^{\beta+2\frac{\beta}{d}}}{N}\int_0^t\int_{\mathbb{T}^d}\frac{1}{\rho_s^N(x)}\rho_s^N(x)\,dx\,ds \\
&\le C_d\,tN^{-1+\beta+2\frac{\beta}{d}}.
\end{aligned}
$$

Proof: $K = \nabla \cdot K_0$, $K \in C^\gamma$

$$\int_0^t \int_{\mathbb{T}^d} \nabla \ln\left(\frac{\rho_s}{\rho_s^N}\right) \left\langle S_s^N, V^N(x-\cdot)[K*\rho_s^N(\cdot) - K*\rho_s^N(x)]\right\rangle dx\, ds$$

First since, $K = \nabla \cdot K_0$ with $K_0 \in C^\gamma$,

$$\begin{aligned}
\left|K*\rho_s^N(\cdot) - K*\rho_s^N(x)\right| &= \left|K_0 * \nabla\rho_s^N(\cdot) - K_0 * \nabla\rho_s^N(x)\right| \\
&\le \int_{\mathbb{T}^d} |K_0(\cdot - y) - K_0(x-y)|\,|\nabla\rho_s^N(y)|\,dy \\
&\le \int_{\mathbb{T}^d} \|K_0\|_\gamma\, |\cdot - x|^\gamma\, |\nabla\rho_s^N(y)|\,dy \\
&= \|K_0\|_\gamma |\cdot - x|^\gamma \int_{\mathbb{T}^d} |\nabla\rho_s^N(y)|\,dy.
\end{aligned}$$

Proof: $|\nabla V_0| \lesssim V_0$

In addition, since $\rho_s^N \in \mathcal{P}\left(\mathbb{T}^d\right)$, $\mathbb{P}$-a.s. and $|\nabla V_0| \lesssim V_0$, by Holder's inequality, we derive

$$\begin{aligned}
\int_{\mathbb{T}^d} |\nabla \rho_s^N|\, dy &= \int_{\mathbb{T}^d} |\nabla \rho_s^N| \frac{\sqrt{\rho_s^N}}{\sqrt{\rho_s^N}}\, dy \\
&\overset{\text{Holder}}{\leq} \left(\int_{\mathbb{T}^d} \frac{|\nabla \rho_s^N|^2}{\rho_s^N}\, dy\right)^{\frac{1}{2}} \left(\int_{\mathbb{T}^d} \rho_s^N\, dy\right)^{\frac{1}{2}} \\
&= \left(\int_{\mathbb{T}^d} \frac{|\nabla \rho_s^N|^2}{\rho_s^N}\, dy\right)^{\frac{1}{2}}.
\end{aligned}$$

Proof: $|\nabla V_0| \lesssim V_0$

Recalling $\Lambda^{-1} \leq \rho \leq \Lambda$, for some $\Lambda > 1$, $\mathbb{P}$-a.s., from Leibniz's rule, we deduce

$$
\begin{aligned}
\int_{\mathbb{T}^d} \frac{|\nabla \rho_s^N|^2}{\rho_s^N}\, dy &= \int_{\mathbb{T}^d} \frac{\left|\nabla\left(\rho_s \frac{\rho_s^N}{\rho_s}\right)\right|^2}{\rho_s^N}\, dy \\
&\leq 2\int_{\mathbb{T}^d} \frac{\left|\nabla \rho_s \left(\frac{\rho_s^N}{\rho_s}\right)\right|^2}{\rho_s^N}\, dy + 2\int_{\mathbb{T}^d} \frac{\left|\rho_s \nabla\left(\frac{\rho_s^N}{\rho_s}\right)\right|^2}{\rho_s^N}\, dy \\
&\leq 2\Lambda^2 \|\nabla \rho_s\|_\infty^2 + 2\int_{\mathbb{T}^d} \rho_s^N \left|\nabla \ln\left(\frac{\rho_s^N}{\rho_s}\right)\right|^2\, dy \\
&= 2\Lambda^2 \|\nabla \rho_s\|_\infty^2 + 2\mathcal{I}(\rho_s^N|\rho_s).
\end{aligned}
$$

Proof: $|\nabla V_0| \lesssim V_0$

So we get

$$\left|\left\langle S_s^N, V^N(x-\cdot)[K*\rho_s^N(\cdot) - K*\rho_s^N(x)]\right\rangle\right|$$
$$\leq \left\langle S_s^N, V^N(x-\cdot)|x-\cdot|^\gamma\right\rangle \|K_0\|_\gamma \left(2\Lambda^2\|\nabla\rho_s\|_\infty^2 + 2\mathcal{I}\left(\rho_s^N|\rho_s\right)\right)^{\frac{1}{2}}.$$

Proof: V_0 is not compactly supported

Now if $|x - \cdot| < N^{-\frac{\beta\gamma}{d}}$ we have

$$\left| \left\langle S_s^N, V^N(x - \cdot)|x - \cdot|^\gamma \right\rangle \right| \leq N^{-\theta_2} \rho_s^N(x)$$

with $\theta_2 \doteq \frac{\beta\gamma^2}{d}$.

Proof: V_0 is not compactly supported

Now, we consider the case that $|x - \cdot| \geq N^{-\frac{\beta\gamma}{d}}$. We note that,

$$
\begin{aligned}
V_0\left(N^{\frac{\beta}{d}}\left(\left(x-\cdot\right)-k\right)\right) &\leq \frac{C_d}{\left|N^{\frac{\beta}{d}}\left(\left(x-\cdot\right)-k\right)\right|^{2d}} \\
&= \frac{C_d}{\left|N^{\frac{\beta}{d}}\right|^{2d}\left|\left(\left(x-\cdot\right)-k\right)\right|^{2d}}, \;\; k \in \mathbb{Z}^d.
\end{aligned}
$$

Proof: V_0 is not compactly supported

For $k = 0$

$$
\begin{aligned}
V_0\left(N^{\frac{\beta}{d}}\left((x-\cdot)\right)\right) &\leq \frac{C_d}{\left|N^{\frac{\beta}{d}}\left((x-\cdot)\right)\right|^{2d}} \\
&\leq \frac{C_d}{\left|N^{\frac{\beta}{d}}\right|^{2d}\left|(x-\cdot)\right|^{2d}} \\
&\overset{|x-\cdot|\geq N^{-\frac{\beta\gamma}{d}}}{\leq} \frac{C_d\left|N^{\frac{\beta\gamma}{d}}\right|^{2d}}{\left|N^{\frac{\beta}{d}}\right|^{2d}} = \frac{N^{2\beta\gamma}}{N^{\beta}}\frac{C_d}{N^{\beta}}.
\end{aligned}
$$

Proof: V_0 is not compactly supported

For $k \neq 0$,

$$\sum_{k\neq 0} V_0\left(N^{\frac{\beta}{d}}\left((x-\cdot)-k\right)\right) \leq \sum_{k\neq 0} \frac{C_d}{\left|N^{\frac{\beta}{d}}\right|^{2d} |((x-\cdot)-k)|^{2d}}$$

$$= N^{-2\beta} \sum_{k\neq 0} \frac{C_d}{|((x-\cdot)-k)|^{2d}} \leq C_d N^{-2\beta}.$$

So, if $|x-\cdot| \geq N^{-\frac{\beta\gamma}{d}}$ and $\gamma \in (0, 1/2)$, we have

$$V^N(x-\cdot) = \sum_{k\in\mathbb{Z}^d} V_0^N((x-\cdot)-k)$$

$$= \sum_{k\in\mathbb{Z}^d} N^{\beta} V_0\left(N^{\frac{\beta}{d}}\left((x-\cdot)-k\right)\right) \leq C_d N^{2\beta\gamma-\beta}$$

Proof: V_0 is not compactly supported

Then since that $|x-\cdot| \leq \sqrt{d}/2$, we obtain

$$\left|\left\langle S_s^N, V^N(x-\cdot)|x-\cdot|^\gamma \right\rangle\right| \leq C_d N^{-\theta_1},$$

with $\theta_1 \doteq \beta(1-2\gamma)$.

Joining those estimates, we get

$$\left|\left\langle S_s^N, V^N(x-\cdot)|x-\cdot|^\gamma \right\rangle\right| \leq C_d N^{-\theta_1} + N^{-\theta_2} \rho_s^N(x)$$

with $\theta_1 \doteq \beta(1-2\gamma)$ and $\theta_2 \doteq \frac{\beta\gamma^2}{d}$.

Proof: entropy estimates

For $\epsilon > 0$ and $\Lambda > 1$, such that $\rho \in (\Lambda^{-1}, \Lambda)$, a.s.,

$$
\begin{aligned}
\mathcal{H}(\rho_t^N|\rho_t) - \mathcal{H}(\rho_0^N|\rho_0) \leq {} & C_{\epsilon,K_0,\Lambda} \int_0^t \|\nabla^2 \rho_s\|_\infty^2 \mathcal{H}(\rho_s^N|\rho_s)\, ds \\
& + C_{\epsilon,K_0,\Lambda} \int_0^t \|\nabla \rho_s\|_\infty^2 \mathcal{H}(\rho_s^N|\rho_s)\, ds \\
& + 4\epsilon \int_0^t \mathcal{I}(\rho_s^N|\rho_s)\, ds \\
& + C_{\epsilon,K_0,\Lambda,d} \left[N^{-\theta_1} + N^{-\theta_2} \right] \int_0^t \|\nabla \rho_s\|_\infty^2\, ds \\
& - \frac{1}{2} \int_0^t \mathcal{I}(\rho_s^N|\rho_s)\, ds \\
& + tCN^{-1+\beta+2\frac{\beta}{d}} \\
& + MARTINGALE.
\end{aligned}
$$

Proof: MARTINGALE estimates

Let $\theta > 0$ and $C(m) \in [0, \infty)$ for $m \geq 1$. In addition, let Z_N, $N \in \mathbb{N}$, be a sequence of random variables such that

$$(\mathbb{E}\,|Z_N|^m)^{\frac{1}{m}} \leq C(m) N^{-\theta},$$

for all $m \geq 1$ and $N \in \mathbb{N}$. Then for all $\delta > 0$, there exists a random variable A_δ such that almost surely

$$|Z_N| \leq A_\delta N^{-\theta+\delta}.$$

Moreover,

$$\mathbb{E}\,|A_\delta|^m < \infty,$$

for all $m \geq 1$.

Proof: MARTINGALE estimates

By Borel-Cantelli Lemma, there exists a random variable, A_0

$$MARTINGALE \leq A_0 N^{-\theta_3+\delta},$$

which allows us to express the previous a priori estimates, as

$$\begin{aligned}
\mathcal{H}(\rho_t^N|\rho_t) - \mathcal{H}(\rho_0^N|\rho_0) \leq\ & C_{\epsilon,K_0,\Lambda} \int_0^t \|\nabla^2\rho_s\|_\infty^2 \mathcal{H}(\rho_s^N|\rho_s)\, ds \\
& + C_{\epsilon,K_0,\Lambda} \int_0^t \|\nabla\rho_s\|_\infty^2 \mathcal{H}(\rho_s^N|\rho_s)\, ds \\
& + C_{\epsilon,K_0,\Lambda,d} \left[N^{-\theta_1} + N^{-\theta_2}\right] \int_0^t \|\nabla\rho_s\|_\infty^2\, ds \\
& + tCN^{-1+\beta+2\frac{\beta}{d}} \\
& + A_0 N^{-\theta_3+\delta}
\end{aligned}$$

Proof: Gronwall's inequality

An application of the Gronwall's inequality implies that

$$\sup_{t\in[0,T]} \mathcal{H}(\rho_t^N|\rho_t) \lesssim \left(\mathcal{H}(\rho_0^N|\rho_0) + N^{-\theta_4}\int_0^T \|\nabla\rho_t\|_\infty^2 dt + A_0 N^{-\theta_4}\right)$$
$$\times \exp\left(\int_0^T \|\nabla\rho_t\|_\infty^2 dt + \int_0^T \|\nabla^2\rho_t\|_\infty^2 dt\right)$$

with

$$\theta_4 \doteq \min\left(\beta(1-2\gamma); \frac{\beta}{d}\gamma^2; \left(\frac{1}{2} - \beta\left(1+\frac{1}{d}\right)\right) - \delta\right).$$

Thank you for your attention.