Fluctuations Around the Mean-Field Limit for Attractive Riesz Interaction Kernels in the Moderate Regime

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joint work with Li Chen (Univ. Mannheim) and Ansgar Jüngel (TU Wien)

Workshop on Stochastic Analysis: Irregularity, Mean-Fields and Related Topics

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Structure of the Talk

1. Moderately Interacting Particles
   - General introduction
   - Diffusion-aggregation model

2. Fluctuations around the mean-field limit
   - Main concept
   - Literature for moderate interaction

3. Fluctuations for moderately interacting particles with **attraction**
   - Quantitative mean-field limit in L^2-norm with proof ideas
   - Focus on the technique for **attractive settings**

Mean-field derivations of PDEs

- △ Goal: **Microscopic particle approximation** of partial differential equations
  - Particles can be molecules, bacteria, birds, humans, neurons
  - Idea goes back to Boltzmann (late 19th century), Hilbert (beginning of 20th century), Kac (1950s)
- Interacting particle system of size N modelled by a system of SDEs on $\mathbb{R}^d$:

$$\begin{cases} \mathrm{d}X_i^N(t) = \frac{1}{N}\sum_{k=1}^N \nabla V(X_i^N - X_k^N)\mathrm{d}t + \sqrt{2\sigma}\mathrm{d}W_i(t) \\ X_i^N(0) = \zeta_i \qquad i = 1, ..., N \end{cases}$$

  - Interaction is captured by a *mean-value* of all interactions (∇V = interaction kernel)
  - $X_i^N(t)$ denotes position of i-th particle

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△ Limit: Show that for $N \to \infty$ the particle dynamics can be captured by

$$\partial_t u = -\mathrm{div}(u(\nabla V * u)) + \sigma \Delta u \quad t > 0, \quad u(0) = u_0$$

- Solution u models the density function of a typical particle (modelled by a SDE)

△ Empirical measure: $\mu_N(t) := \frac{1}{N}\sum_{i=1}^N \delta_{X_i^N(t)} \to u(t)$ in distribution

Moderate Models

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- △ Can we derivate also local PDEs (without integral operators)?

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△ Can we derivate also local PDEs (without integral operators)?

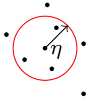
▶ Change the strength of interaction to moderate interaction

△ **Interaction kernel coupled to the number of particles** N in a certain way:

- New variable $\eta > 0$

$$V^\eta(x) := \eta^{-d} V(|x|/\eta) \to \kappa\delta_0 \quad \text{if } \eta \to 0$$

- If V has compact support on $B_1(0)$, then $\eta =$ *interaction radius*



△ Connect η and N (for example $\eta = N^{-\beta}$ or $\eta = \log(N)^{-\alpha}$) then for $N \to \infty$:

- Strength of interaction is stronger $\sim \frac{\eta^{-(d+1)}}{N} = N^{\beta(d+1)-1}$
- motivation for *moderate regime*

Moderate Models

- How does the limiting PDE change?
- Since $V^\eta \to \kappa\delta_0$ for $N \to \infty, \eta \to 0$ instead of

$$\partial_t u = -\mathsf{div}(u(\nabla V^\eta * u)) + \sigma\Delta u$$

we get

$$\to \partial_t u = -\kappa\mathsf{div}(u\nabla u) + \sigma\Delta u$$

  - limiting equation is a local PDE
- Oelschläger ('85)[1], Méléard and Roelly-Coppoletta ('87)[2]
  - $\kappa = -1$: propagation of chaos for viscous porous media equation
  - **only** repulsive case !

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[1] K. Oelschläger. A law of large numbers for moderately interacting diffusion processes, 1985.

[2] S. M and S. R. A propagation of chaos result for a system of particles with moderate interaction, 1987.

Strategy for moderate models (via coupling)

△ Define an *intermediate* stochastic particle system for fixed $\eta > 0$:

- Hierarchy of stochastic models (N particles, $\eta > 0$):
  I. Interacting particles: $\mathrm{d}X_i^N(t) = \frac{-1}{N}\sum_{k=1}^{N} \nabla V^\eta(X_i^N - X_k^N)\mathrm{d}t + \sqrt{2\sigma}\mathrm{d}W_i(t)$

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  II. Intermediate with $\eta > 0$: $\mathrm{d}\overline{X}_i^\eta(t) = -(\nabla V^\eta * u^\eta)(\overline{X}_i^\eta)\mathrm{d}t \quad + \quad \sqrt{2\sigma}\mathrm{d}W_i(t)$
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△ Step I: Compare particles X_i^N and $\overline{X}_i^\eta$ (classical theory) with tracking dependence of $\eta > 0$

△ Step II: Compare process $\overline{X}_i^\eta$ and $\hat{X}_i$ (already independent) using PDE estimates

$$u^\eta \to u \quad \text{if } \eta \to 0 \quad (N \to \infty) \quad \text{in suitable norm}$$

△ Step III: Combine two steps to estimate $X_i^N - \hat{X}_i$

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- Taking $\mathbb{E} \to \log$-connection for η and N via Gronwall estimates
  - more refined methods e.g. **relative entropy** overcome this issue

In which model are we interested in?

Aggregating Behaviour in Nature and Mathematics

- Aggregation effects in self-organizing behaviour

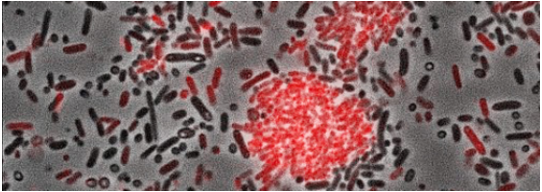

Figure: Aggregation of bacteria

PDE for density: $\partial_t u - \nabla \cdot (\sigma \nabla u - u \nabla V * u) = 0$

- **Singularity:** $V(x) = C(d)/|x|^\lambda$ (Keller-Segel-type) or $V(x) = \delta_0(x)$

Diffusion-aggregation model

△ Consider a system of N interacting particles with positions $X_i^\eta(t) \in \mathbb{R}^d$

$$\begin{aligned} \mathrm{d}X_i^\eta(t) &= \frac{\kappa}{N} \sum_{j=1}^{N} \nabla V(X_i^\eta(t) - X_j^\eta(t))\mathrm{d}t + \sqrt{2\sigma}\mathrm{d}W_i(t), \quad t > 0, \\ X_i^\eta(0) &= \zeta_i \quad \text{in } \mathbb{R}^d,\ i = 1, \dots, N. \end{aligned} \tag{1}$$

- $\kappa = 1$ (aggregating case), $\kappa = -1$ (repulsive case)
- $V = |x|^{-\lambda}$ for $\lambda > 0$, e.g. Keller-Segel, neural networks

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- $\kappa = 1$ (aggregating case), $\kappa = -1$ (repulsive case)
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  - ⋆ well-defined? (especially for attractive systems difficult)

Diffusion-aggregation model

overcome: moderate interaction models!

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- $\kappa = 1$ (aggregating case), $\kappa = -1$ (repulsive case)
- $V^\eta = \chi^\eta * \Phi, \quad \Phi(x) = |x|^{-\lambda}$, where for $\eta > 0$

$$\chi^\eta(x) = \eta^{-d}\chi(|x|/\eta)$$

  is a smooth mollification kernel

Moderate Regime for Riesz kernels - recent work

.. diffusion-aggregation system in the moderate regime:

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Theorem (Olivera-Richard-Tomašević '23)

Let $0 < \lambda < d - 1$, then for $\eta = N^{-\alpha}$ for $\alpha > 0$ and $\mathfrak{p} > 0$ small enough

$$\mathbb{P}\Big(\|\chi^\eta * \mu_N - u\|_X > C\Big) \lesssim C^{-m}(C(u_0) + N^{-\mathfrak{p}+\varepsilon})^m \quad \forall m \in \mathbb{N}$$

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- $\|\cdot\|_X = \|\cdot\|_{L^\infty(0,T;L^1(\mathbb{R}^d)\cap L^r(\mathbb{R}^d))}$ for suitable $r > 1$,
- u solves

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- first **quantitative propagation of chaos result** in this setting (allows for attractive regime)
- **However, for fluctuations around the mean-field limit we need a different notion**

Prove a different type of convergence result

A Sneak Peek

... together with Li Chen & Ansgar Jüngel

- Instead of studying

$$\chi^\eta * \mu_N \longrightarrow u,$$

we use convergence at the level of **trajectories**

$$(X_1^\eta, \dots, X_N^\eta)$$

- Allows us to study *next order correction* (fluctuations)

- We see the concrete structure of the convergence result later

Literature Context - Moderate regime for singular kernels

Recently, there has been a lot of progress in this direction:

Disclaimer: only a short selection! *literature on singular kernels with/without cut-off not included*

Semigroup approach based on seminal works of Flandoli et al:

- **Olivera–Richard–Tomašević '23, '25**: First quantitative PoC for (attractive/repulsive) singular kernels, including
$$\Phi(x) = |x|^{-\lambda}, 0 < \lambda < d - 1$$
on $\mathbb{R}^d$, '25 includes Burgers equation in $d = 1$
- Knorst-Olivera-de Souza '24: Application to Mean Field Games
- Cavallazzi-Richard-Tomašević '26: Keller-Segel model with birth and death mechanisms
- Hao-Jabir-Menozzi-Röckner-Zhang '26: PoC for kinetic McKean-Vlasov setting with singular kernels (non-smooth anisotropic Besov space)

Relative entropy and related approaches

- Carrillo-Guo-Holzinger '25: Multi-species extension up to Newtonian case on $\mathbb{R}^d$
- Chen-Nikolaev-Prömel '24: PoC for bounded kernels,
- Bol-Chen-Li '26: Signal-dependent Keller-Segel PoC
- Olivera-de Souza '26: stochastic Fokker-Planck equation via relative entropy on torus

End of part I

Moderate mean-field scaling...

- ✓ ... changes the strength of interaction
- ✓ ... leads to **local** PDEs
- ✓ ... can be also used for other types of singular potentials, e.g. Riesz-type kernels

Fluctuation around the mean-field limit – Motivation

Mean-field limits help us understand micro/macro-connections, but...

- △ ... **stochastic** particle system $\rightarrow$ **deterministic** PDE
  - Information gets lost - is there a stochastic correction?
- △ ... particle system is **not unique**

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$$\partial_t u(x,t) = \Delta u(x,t) + \Delta u^2(x,t) \quad x \in \mathbb{R}^d,\ t > 0$$

    - ⋆ different from well-known work by Figalli and Philipowski[1] '08
  - difference from a modelling point of view?

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  - ▶ difference from a modelling point of view?
- ▷ Need a correction E_N of the form

$$\mu_N \sim u + E_N$$

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Definition of the fluctuation process

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- Fluctuation process $\eta_N \sim$ 'Next-order correction':
- Define $\eta_N(t) := \sqrt{N}(\mu_N(t) - u(t))$

$$\mu_N(t) = u(t) + \frac{1}{\sqrt{N}}\Big(\underbrace{\sqrt{N}(\mu_N(t) - u(t))}_{=\eta_N}\Big) \quad \text{for fixed } N.$$

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- If S is a Gaussian process $\to$ **Central Limit Theorem**

Fluctuations in mean-field setting – (old & recent results)

In the 1980s... (not gradient case $\to \nabla V$ changes to K)

△ Beginning clear that CLT holds in the classical mean-field setting

- e.g. Tanaka-Hitsuda '81 on $\mathbb{R}$ with $K(x-y) = -\lambda(x-y)$

$$\sqrt{N}(\mu_N - u) \to S$$

  - ⋆ u solves corresponding McKean-Vlasov equation
  - ⋆ S solves the Gaussian linearized Dean–Kawasaki SPDE (ξ_t is vector-valued space-time white noise)

$$\partial_t S = \mathcal{L}_u(S) + \sqrt{2\sigma}\nabla \cdot (\sqrt{u}\xi_t)$$

[1] Always under suitable assumptions at $t = 0$

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...**but recently, there are results for**

- △ Singular interaction kernels: (Wang-Zhao-Zhu '21) on $\mathbb{T}^d$, $d \geq 2$
  - K bounded or $|x|K$ bounded

[1] Always under suitable assumptions at $t = 0$

Fluctuations in mean-field setting – (old & recent results)

In the 1980s... (not gradient case $\to \nabla V$ changes to K)

- △ Beginning clear that CLT holds in the classical mean-field setting
  - e.g. Tanaka-Hitsuda '81 on $\mathbb{R}$ with $K(x-y) = -\lambda(x-y)$

$$\sqrt{N}(\mu_N - u) \to S$$

  - ★ u solves corresponding McKean-Vlasov equation
  - ★ S solves the Gaussian linearized Dean–Kawasaki SPDE (ξ_t is vector-valued space-time white noise)

$$\partial_t S = \mathcal{L}_u(S) + \sqrt{2\sigma}\nabla \cdot (\sqrt{u}\xi_t)$$

...**but recently, there are results for**

- △ Singular interaction kernels: (Wang-Zhao-Zhu '21) on $\mathbb{T}^d$, $d \geq 2$
  - K bounded or $|x|K$ bounded
  - S solves D-K SPDE[1], equivalently for $\phi(x)$ smooth

$$\langle S_t, \phi\rangle \sim \mathcal{N}\left(0, \langle u_0, (T^t_\phi(0))^2\rangle - \langle u_0, T^t_\phi(0)\rangle^2 + 2\sigma \int_0^t \langle u(s), |\nabla T^t_\phi(s)|^2\rangle \mathrm{d}s\right)$$

[1] Always under suitable assumptions at $t = 0$

What is T_ϕ^t?

Limiting behaviour in Wang-Zhao-Zhu:

$$\langle S_t, \phi \rangle \sim \mathcal{N}\left(0, \langle u_0, (T_\phi^t(0))^2 \rangle - \langle u_0, T_\phi^t(0) \rangle^2 + 2\sigma \int_0^t \langle u(s), |\nabla T_\phi^t(s)|^2 \rangle \mathrm{d}s\right)$$

... where $T_\phi^t(s)$ is the unique solution to the

- **dual backward** problem of the **linearized** version of the mean-field PDE on $\mathbb{T}^d$, i.e.,

$$T_\phi^t(t) = \phi, \quad -\partial_s T_\phi^t(s) - \sigma \Delta T_\phi^t(s) = (K * u) \cdot \nabla T_\phi^t(s) - K * (u \nabla T_\phi^t(s)) \quad s \in (0, t).$$

Fluctuation for moderate interaction (local PDEs)

△ Literature (few results)

- Results by Oelschläger '87 [3] for repulsive particles
- Jourdain and Méléard '98 [4], but not with $\sqrt{N}$ scaling
  - $\eta = \log(N)^{-\alpha}$ **not algebraic**
  - deterministic correction, no **Central Limit Theorem**

[3] K. Oelschläger. A fluctuation theorem for moderately interacting diffusion processes, 1987.
[4] J. and M. Propagation of chaos and fluctuations for a moderate model with smooth initial data, 1998.

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  - $\eta = \log(N)^{-\alpha}$ **not algebraic**
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△ Goal: Generalise result by Oelschläger such that it includes

- **aggregating particles** and
- **singular interaction kernels**
- motivation: common phenomena in biology, physics, opinion formation

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Structure of the Talk

✓ Diffusion-aggregation model with singular Riesz kernel

- Moderately Interacting Particles ✓
- Existing literature ✓

✓ Fluctuations around the mean-field limit

- Why is it relevant? ✓
- Literature for singular potentials ✓

☞ Fluctuations for attractive Riesz kernels

- Quantitative mean-field limit in L^2-norm with proof ideas
- Focus on the technique for attractive settings

A closer look to Oelschläger's technique 1/2

...what is the right notion of convergence ?

- △ Oelschläger '87 [5] for viscous porous media equation

$$\partial_t u = \Delta u + \frac{1}{2}\mathrm{div}(u\nabla u) \quad x \in \mathbb{R}^d, t > 0$$

- △ Convergence to local PDE solution seems to slow for fluctuations with $\sqrt{N}$ scaling

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$$\sqrt{N}(\mu_N - u - c_N) \to \text{Gaussian} \quad c_N \to 0$$

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$$\sqrt{N}(\mu_N - u - c_N) \to \text{Gaussian} \quad c_N \to 0$$

- $c_N \sim (u_N - u)$ where u_N is a solution to the non-local PDE approximation

Non-local porous-media-type equation , $\eta = N^{-\beta} \to 0$ for $\alpha > 0$

$$\partial_t u_N = \Delta u_N + \frac{1}{2}\text{div}(u_N \nabla \chi^\eta * u_N) \quad x \in \mathbb{R}^d, t > 0$$

- **Where:** $\chi^\eta \to \delta_0$ interaction potential;

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A closer look to Oelschläger's technique 1/2

....remember what we want

- △ Diffusion-aggregation equation

$$\partial_t u = \Delta u + \frac{-\kappa}{2}\mathrm{div}(u\nabla\Phi * u) \quad x \in \mathbb{R}^d, t > 0$$

- △ Convergence to local PDE solution seems to slow for fluctuations with $\sqrt{N}$ scaling

$$\sqrt{N}(\mu_N - u - c_N) \to \text{Gaussian} \quad c_N \to 0$$

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Non-local diffusion-aggregation, $\eta = N^{-\beta} \to 0$ for $\alpha > 0$

$$\partial_t u_N = \Delta u_N + \frac{-\kappa}{2}\mathrm{div}(u_N\nabla V^\eta * u_N) \quad x \in \mathbb{R}^d, t > 0$$

- **Where:** $\chi^\eta \to \delta_0$ **we want:** $V^\eta = \chi^\eta * |x|^{-\lambda} \to |x|^{-\lambda} = \Phi(x)$

A closer look to Oelschläger's technique 2/2

- $\mathcal{F}_N(t) := \sqrt{N}(\mu_N - u_N)(t)$, ψ a suitable test function, **use that**

$$\chi^\eta = Z^\eta * Z^\eta$$

... to get the following SDE for $\mathcal{F}_N$

$$\begin{aligned} \mathrm{d}\langle \mathcal{F}_N(t), \psi(t)\rangle = \frac{C(\sigma)}{\sqrt{N}} \sum_{i=1}^{N} \nabla\psi(X_i^N(t))\mathrm{d}W_i(t) - \underbrace{\langle \mathcal{F}_N(t), (\mathcal{L}_t)^* \psi\rangle}\,\mathrm{d}t \\ + \sqrt{N}\langle (Z^\eta * (\mu_N - u_N))^2(t), \Delta\psi\rangle \mathrm{d}t + \text{error term}, \end{aligned} \tag{2}$$

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Key observations: Remember the result of Wang-Zhao-Zhu

$$\langle S(t), \phi\rangle \sim \mathcal{N}\left(0, \langle u_0, (T_\phi^t(0))^2\rangle - \langle u_0, T_\phi^t(0)\rangle^2 + 2\sigma \int_0^t \langle u(s), |\nabla T_\phi^t(s)|^2\rangle \mathrm{d}s\right)$$

- If $\psi = T_\phi^t$ then $(\mathcal{L}_t)^*\psi = 0$ and $\psi(t) = \phi$
- Need convergence of $Z^\eta * (\mu_N - u_N) \to 0$ in L^2-norm faster than $1/\sqrt{N}$

A closer look to Oelschläger's technique 2/2

...same strategy!

- $\mathcal{F}_N(t) := \sqrt{N}(\mu_N - u_N)(t)$, ψ a suitable test function, **use that** $V^\eta = Z^\eta * Z^\eta$ since

$$\Phi = \Psi * \Psi \quad \text{with} \quad \Psi = c_{\lambda,d}|x|^{-(\lambda+d)/2} \qquad (\chi = \zeta * \zeta)$$

... **to get** the following SDE for $\mathcal{F}_N$

$$\mathrm{d}\langle \mathcal{F}_N(t), \psi(t)\rangle = \frac{C(\sigma)}{\sqrt{N}} \sum_{i=1}^{N} \nabla\psi(X_i^N(t))\mathrm{d}W_i(t) - \underbrace{\langle \mathcal{F}_N(t), (\mathcal{L}_t)^*\psi\rangle}_{=0}\mathrm{d}t \\ + \sqrt{N}\langle (Z^\eta * (\mu_N - u_N))^2(t), \Delta\psi\rangle \mathrm{d}t + \text{error term}, \tag{2}$$

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- Need convergence of $Z^\eta * (\mu_N - u_N) \to 0$ in L^2-norm faster than $1/\sqrt{N}$
- Convergence of the error term is technical!

Oelschläger's technique for Riesz kernels – strategy

△ so far: **follow the approach of Oelschläger**

△ what are the important quantities?

- $\mathcal{F}_N(t) := \sqrt{N}(\mu_N - u_N)(t)$, where remember $\eta = N^{-\beta}$

$$\partial_t u_N = \sigma \Delta u_N - \kappa \mathrm{div}(u_N \nabla V^\eta * u_N), \quad t > 0, \quad u_N(0) = u_0 \quad \text{in } \mathbb{R}^d. \qquad (3)$$

- we compare to the **non-local PDE solution with smooth potential**!

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- However, the limiting dynamics $\rightarrow$ **non-local PDE solution with Riesz potential**, through $T^t_\phi(s)$, which solves

$$-\partial_s v - \sigma \Delta v = \kappa(\nabla \Phi * u) \cdot \nabla v - \kappa \nabla \Phi * (u \nabla v) \quad \text{in } \mathbb{R}^d, \; s \in (0,t), \quad v(t) = \phi,$$

where u solves

$$\partial_t u = \sigma \Delta u - \kappa \mathrm{div}(u \nabla \Phi * u), \quad t > 0, \quad u(0) = u_0 \quad \text{in } \mathbb{R}^d \tag{4}$$

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$$\partial_t u = \sigma \Delta u - \kappa \mathrm{div}(u \nabla \Phi * u), \quad t > 0, \quad u(0) = u_0 \quad \text{in } \mathbb{R}^d \qquad (4)$$

△ From before: L^2 is crucial $\sqrt{N}\langle (Z^\eta * (\mu_N - u_N))^2(t), \Delta\psi \rangle$

$$f^\eta := Z^\eta * \mu_N, \quad g^\eta := Z^\eta * u_N,$$

L^2 mean-field result in the subcoulomb regime

...we can formulate the L^2 **result** (**aggregating particles**):

- △ **Potential:** Let $0 < \lambda < d - 2$ and $\Phi = |x|^{-\lambda}$ for $d \geq 3$,
- △ **Moderate Regime:** $\eta = N^{-\beta}$ with $0 < \beta < \frac{1}{8\lambda+12}$ and $V^\eta = Z^\eta * Z^\eta$ with

$$V^\eta \to \Phi \quad \text{for } \eta \to 0 \quad (N \to \infty)$$

and

$$f^\eta := Z^\eta * \mu_N, \quad g^\eta := Z^\eta * u_N,$$

Theorem (Chen-H-Jüngel, 2026)

For any $T > 0$ there exists an $\varepsilon > 0$ and a $C > 0$, which depend on β, T, σ and d, such that

$$\mathbb{E}\left(\sup_{0<t<T} \|(f^\eta - g^\eta)(t)\|^2_{L^2(\mathbb{R}^d)} + \sigma \int_0^T \|\nabla(f^\eta - g^\eta)(t)\|^2_{L^2(\mathbb{R}^d)} \mathrm{d}t \right) \leq C N^{-1/2-\varepsilon}.$$

- Crucial **Theorem** for CLT (new technique for aggregation $\kappa = 1$)
- Was known to hold for viscous porous media equation (Oelschläger '87).

Idea of the proof 1/3

- Use structure of **regularized equation, Itô's formula and symmetry** of V^η to get

$$\|f^\eta(t)-g^\eta(t)\|_{L^2}^2-\|f^\eta(0)-g^\eta(0)\|_{L^2}^2+2\sigma\int_0^t\|\nabla(f^\eta(s)-g^\eta(s))\|_{L^2}^2\mathrm{d}s$$

$$=\frac{2\sqrt{2\sigma}}{N}\sum_{i=1}^N\int_0^t\nabla V^\eta*\mu_N(X_i^N(s)(s))-\nabla V^\eta*u_N(s,X_i^N(s))\mathrm{d}W_i(s)$$

$$+2C\sigma TN^{-1+\beta(d+2)}+2\kappa\int_0^t\langle\mu_N(s),\left|\nabla Z^\eta*f^\eta(s)-\nabla Z^\eta*g^\eta(s)\right|^2\rangle\mathrm{d}s$$

$$-2\int_0^t\langle u_N(s)-\mu_N(s),\nabla V^\eta*u_N(\nabla Z^\eta*f^\eta(s)-\nabla Z^\eta*g^\eta(s))\rangle\mathrm{d}s.$$

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- Use structure of **regularized equation, Itô's formula and symmetry** of V^η to get

$$
\begin{aligned}
&\|f^\eta(t) - g^\eta(t)\|_{L^2}^2 - \|f^\eta(0) - g^\eta(0)\|_{L^2}^2 + 2\sigma \int_0^t \|\nabla(f^\eta(s) - g^\eta(s))\|_{L^2}^2 \mathrm{d}s \\
&= \frac{2\sqrt{2\sigma}}{N} \sum_{i=1}^N \int_0^t \nabla V^\eta * \mu_N(X_i^N(s)(s)) - \nabla V^\eta * u_N(s, X_i^N(s))\mathrm{d}W_i(s) \\
&+ 2C\sigma T N^{-1+\beta(d+2)} + 2\kappa \int_0^t \langle \mu_N(s), \left|\nabla Z^\eta * f^\eta(s) - \nabla Z^\eta * g^\eta(s)\right|^2 \rangle \mathrm{d}s \\
&- 2 \int_0^t \langle u_N(s) - \mu_N(s), \nabla V^\eta * u_N(\nabla Z^\eta * f^\eta(s) - \nabla Z^\eta * g^\eta(s)) \rangle \mathrm{d}s.
\end{aligned}
$$

- $\mu_N \sim u_N$ can we use tricks from PDE theory?

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- Use structure of **regularized equation, Itô's formula and symmetry** of V^η to get

$$\|f^\eta(t) - g^\eta(t)\|_{L^2}^2 - \|f^\eta(0) - g^\eta(0)\|_{L^2}^2 + 2\sigma \int_0^t \|\nabla(f^\eta(s) - g^\eta(s))\|_{L^2}^2 \mathrm{d}s$$

$$= \frac{2\sqrt{2\sigma}}{N} \sum_{i=1}^N \int_0^t \nabla V^\eta * \mu_N(X_i^N(s)(s)) - \nabla V^\eta * u_N(s, X_i^N(s)) \mathrm{d}W_i(s)$$

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- $\mu_N \sim u_N$ can we use tricks from PDE theory?

$$\int_0^t \langle \mu_N(s), \left|\nabla Z^\eta * f^\eta(s) - \nabla Z^\eta * g^\eta(s)\right|^2 \rangle \mathrm{d}s$$

$$\lesssim \int_0^t \|Z^\eta * \mu_N\|_{L^\infty} \|\nabla(f^\eta(s) - g^\eta(s))\|_{L^2}^2 \mathrm{d}s$$

Idea of the proof 1/3

- Use structure of **regularized equation, Itô's formula and symmetry** of V^η to get

$$\|f^\eta(t)-g^\eta(t)\|_{L^2}^2 - \|f^\eta(0)-g^\eta(0)\|_{L^2}^2 + 2\sigma\int_0^t \|\nabla(f^\eta(s)-g^\eta(s))\|_{L^2}^2 \mathrm{d}s$$

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- $\mu_N \sim u_N$ can we use tricks from PDE theory?

$$\int_0^t \langle \mu_N(s), \left|\nabla Z^\eta * f^\eta(s) - \nabla Z^\eta * g^\eta(s)\right|^2\rangle \mathrm{d}s$$

$$\lesssim \int_0^t \cancel{\|Z^\eta * \mu_N\|_{L^\infty}} \|\nabla(f^\eta(s)-g^\eta(s))\|_{L^2}^2 \mathrm{d}s$$

- Need to estimate it directly! **Idea: Use convergence in Probability!**

Idea of the proof 2/3

- Introduce the *intermediate particle level* (corresponds to u_N)

$$\mathrm{d}\bar{X}_i^\eta(t) = \kappa\big(u_N * \nabla V^\eta(t, \bar{X}_i^\eta(t))\big)\mathrm{d}t + \sqrt{2\sigma}\mathrm{d}W_i(t), \quad t > 0, \tag{5}$$

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- Pivoting argument

$$\mathbb{E}\bigg(\int_0^T \langle \mu_N(s), \big|\nabla Z^\eta * f^\eta(s) - \nabla Z^\eta * g^\eta(s)\big|^2\rangle \mathrm{d}s\bigg) \le I_1 + I_2 + I_3,$$

where

- $I_3 := \mathbb{E}\bigg(\int_0^T \frac{1}{N} \sum_{i=1}^N \bigg| \frac{1}{N} \sum_{j=1}^N \big(\nabla V^\eta(\bar{X}_i^\eta - \bar{X}_j^\eta) - \nabla V^\eta * u_N(\bar{X}_i^\eta)\big)\bigg|^2 \mathrm{d}s \bigg)$ LLN ✓

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- $I_2 := \mathbb{E}\bigg(\int_0^T \frac{1}{N}\sum_{i=1}^N \bigg| \frac{1}{N}\sum_{j=1}^N \left(\nabla V^\eta(X_i^\eta - X_j^\eta) - \nabla V^\eta(\bar{X}_i^\eta - \bar{X}_j^\eta)\right)\bigg|^2 \mathrm{d}s\bigg)$
- Basic Idea (inspired by P. Pickl): Splitting $\Omega = C_\alpha \cup C_\alpha^c$
  1. $\omega \in C_\alpha$, **then** $\max_{i=1,\dots,N} |X_i^\eta(t,\omega) - \bar{X}_i^\eta(t,\omega)| \le N^{-\alpha}$, MVT ✓
  2. the **complement**: then show that

$$\|\nabla V^\eta\|_{L^\infty}^2 \mathbb{P}(C_\alpha^c) \le N^{-1/2-\varepsilon}$$

Idea of the proof 3/3

- Goal: $\mathbb{P}(C_\alpha^c)$ **small enough** to compensate for $\|\nabla V^\eta\|_{L^\infty}$

Theorem (Mean-field in Probability, Chen-H.-Jüngel, 2026)

Let $0 < \beta < \frac{1}{4\lambda+12}$ and $\beta(\lambda+3) < \alpha < \frac{1}{2} - \beta(\lambda+1)$ then it holds that for every $\gamma > 0$ there exists a $C(\gamma, T)$ such that

$$\mathbb{P}\Big(\max_{i=1,\dots,N} |X_i^\eta(t) - \bar{X}_i^\eta(t)| > N^{-\alpha} \Big) \leq C(\gamma, T) N^{-\gamma}.$$

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- 'Weaker' form of convergence, but general algebraic rate!
- **Basic idea:** Stopping time argument + Gronwall's lemma

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- 'Weaker' form of convergence, but general algebraic rate!
- **Basic idea:** Stopping time argument + Gronwall's lemma
- Reason for sub-coulomb setting $0 < \lambda < d-2$: **Taylor's expansion**

$$\||D^2 V^\eta| * u_N(t)\|_{L^\infty(\mathbb{R}^d)} \leq C$$

- **... otherwise this restriction is not needed!**

Asymptotic Central Limit Theorem

... and here comes the final result:

Theorem (CLT, Chen-H.-Jüngel, 2026)

For $0 < \lambda < d-2$, $\Phi(x) = |x|^{-\lambda}$, and under suitable assumptions on β, u_0, the conditional law fulfils

$$\mathrm{Law}(\langle \mathcal{F}_N(t), \phi\rangle | \mathscr{F}_0) \to \mathcal{N}\left(\langle \mathcal{F}_0, T^t_\phi(0)\rangle, 2\sigma \int_0^t \langle u(s), |\nabla T^t_\phi(s)|^2\rangle \mathrm{d}s\right).$$

This implies that asymptotically the fluctuations **become Gaussian**

$$\langle \mathcal{F}(t), \phi\rangle \sim \mathcal{N}\left(0, \langle u_0, (T^t_\phi(0))^2\rangle - \langle u_0, T^t_\phi(0)\rangle^2 + 2\sigma \int_0^t \langle u(s), |\nabla T^t_\phi(s)|^2\rangle \mathrm{d}s\right).$$

... by using the characteristic function

- compute $e^{i\theta\langle \mathcal{F}_N, \phi\rangle}$, use SDE fulfilled by $\mathcal{F}_N$
- iteration argument to overcome the error we make by approximation V^η

Conclusion & Outlook

⌛ We showed: Asymptotic Central Limit Theorem for attractive Riesz kernels (subcoulomb)

- generalized Oelschläger's proof in the **moderate regime**
- new technique for attractive case
- **main ingredients:** Convergence in Probability & L^2 estimates

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- I would be interested in:
  - the **original moderate regime**, i.e. $V^\eta \to \delta_0$ in the attractive case
  - can we show it also in the weak regime (subcoulomb particle system is well-defined in attractive case)?
  - higher order estimates, non-constant diffusion ...

Junior Trimester Program 2027 in Bonn - Winter school

- **Title:** Renewal Equations, PDEs and non-equilibrium systems in biological modelling
- Application deadline for Winter School is **September 27, 2026, 11:59 pm (CEST)**. (Requirement: viva at most 6 years ago)

Winter School: Differential Equations in Mathematical Biology: Analysis, Stochasticity, and Modelling

January 18 - 22, 2027

Venue: HIM lecture hall, Poppelsdorfer Allee 45, Bonn

Organizers: Daniel Morris, Nadia Loy, Manuel Esser, Oscar de Wit

Speakers:

- Murad Banaji (Lancaster University)
- Sylvie Méléard (Ecole polytechnique)
- Sepideh Mirrahimi (CNRS, Institut de mathématiques de Toulouse, Université Paul Sabatier)
- Michela Ottobre (Heriot Watt University)

Description: Biological systems are inherently multiscale and their mathemati cal treatment requires complementary perspectives attracting researchers from different scientific fields. From intracellular reaction networks to interacting populations and ecological communities, their dynamics are shaped by complex interactions, stochastic effects, and spatial structure. Capturing these features calls for a diverse mathematical toolbox ranging from stochastic processes and reaction networks to kinetic theory and nonlinear partial differential equations. This research school brings together various perspectives and techniques in mathematical biology. The lectures will introduce recent analytical and proba bilistic methods for studying biological systems across scales, emphasizing the interplay between deterministic and stochastic descriptions.
The aim of the school is to provide participants with a coherent view of mod ern mathematical techniques in mathematical biology, highlighting how PDE analysis, and stochastic techniques interact in the study of non-equilibrium bi ological systems. Particular attention will be paid to methodological bridges between frameworks and to emerging analytical challenges.
The school is intended to serve as a comprehensive introduction to the pro

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Schedule →

Abstract →

Video Recordings →

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Thank you for your attention! Obrigada!

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