Adaptive Learning via Off-Model Training and Importance Sampling for Fully Non-Markovian Optimal Stochastic Control.

Alberto Ohashi

Joint work with Simone Scotti, Dorival Leão, Adolfo da Silva

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Issues to be addressed

Let $\xi : C([0,T];\mathbb{R}^n) \to \mathbb{R}$ be a Borel functional, let $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ be a reference filtration generated by a multi-dimensional Brownian motion. Let $U_t^T; 0 \leq t < T$ be the set of $\mathbb{F}$-predictable controls defined over $(t, T]$ and taking values on a compact subset $\mathbb{A}$.

Let $\{X^u; u \in U_0^T\}$ be a family of $\mathbb{F}$-adapted controlled processes.

QUESTION 1: For a given error bound $\epsilon > 0$, how to design a **numerical scheme to compute** ϵ-optimal controls $\phi^{*,\epsilon}$, i.e.,

$$\mathbb{E}\Big[\xi\big(X^{\phi^{*,\epsilon}}\big)\Big] \leq \inf_{\phi \in U_0^T} \mathbb{E}\Big[\xi\big(X^{\phi}\big)\Big] + \epsilon.$$

This is a classical and (at some extent) well-understood question in case X^ϕ is a controlled Markov process. **In this talk, we will discuss about the fully non-Markovian case.**

Beyond Markovian case: Path-dependent SDEs driven by Brownian motion

$$dX^u(t) = \overbrace{\alpha(t, X_t^u, u(t))}^{\text{path-dependent functional}} dt + \overbrace{\sigma(t, X_t^u, u(t))}^{\text{path-dependent functional}} dB(t),$$

where B is a Brownian motion and $X_t^u := \{X^u(s); 0 \leq s \leq t\}$.

Controlled systems with non-trivial memory

In the fully non-Markovian case, feasible numerical approximation schemes are very challenging !

Typical non-trivial example:

$$dX^u(t) = \overbrace{\alpha(t, X^u_t, u(t))}^{\text{non-anticipative functional}} dt + \overbrace{\sigma(t, X^u_t, u(t)) dB^H(t)}^{\text{fully non-Markovian}}$$

where B^H is a fractional Brownian motion with exponent $H \in (0,1)$ given by

$$B \mapsto B^H(\cdot) = \int_0^{\cdot} K_H(\cdot, u) dB(u)$$

Controlled systems with non-trivial memory

$$B \mapsto B^H(\cdot) = \int_0^{\cdot} K_H(\cdot, u) dB(u)$$

- Highly singular infinite-dimensional map for $0 < H < \frac{1}{2}$
- Regular infinite-dimensional map for $\frac{1}{2} < H < 1$
- If $H \neq \frac{1}{2}$, one cannot reduce it to a Markovian situation without adding infinitely many degrees of freedom.

Partial hedging with Rough volatility

$$dX^u(t) = u(t)dS(t)$$

$$dS(t) = \mu S(t)dt + S(t)\vartheta(V(t))dB^{(1)}(t)$$

Here, $V(t) = \exp(Z(t))$ and for instance,

$$dZ(t) = \zeta dW^H(t) - \beta(Z(t) - \varkappa)dt$$

is the fractional Ornstein-Uhlenbeck process for $0 < H < \frac{1}{2}$, $\zeta, \beta > 0$, $\mu \in \mathbb{R}$ and $\varkappa \in \mathbb{R}$. Here

$$W^H(t) = \int_0^t K_H(t,s)dW(s), \quad W = \rho B^{(1)} + \bar{\rho}B^{(2)}$$

for a two-dimensional Brownian motion $B = (B^{(1)}, B^{(2)})$ and $\bar{\rho} := \sqrt{1-\rho^2}$ for $-1 \leq \rho \leq 1$.

Typical examples we have in mind

In this talk, we will present a **concrete numerical scheme for computing near optimal controls** for controlled processes adapted to the Brownian filtration: The typical examples are

$$dX^u(t) = \overbrace{\alpha(t, X^u_t, u(t))}^{\text{path-dependent functional}} dt + \overbrace{\sigma(t, X^u_t, u(t))}^{\text{path-dependent functional}} dB(t),$$

$$dX^u(t) = \overbrace{\alpha(t, X^u_t, u(t))}^{\text{non-linear functional}} dt + \overbrace{\sigma dB^H(t)}^{\text{fully non-Markovian}}$$

and

$$dX^u(t) = u(t)dS(t)$$

$$dS(t) = \mu S(t)dt + \underbrace{S(t)\vartheta(V(t))}_{\text{fully non-Markovian}} dB^{(1)}(t)$$

Here, $\log V(t) = Z(t)$ and

$$dZ(t) = \zeta dW^H(t) - \beta(Z(t) - \varkappa)dt$$

where B^H and W^H are fractional Brownian motion with $H \in (0, \frac{1}{2})$.

Issues to be addressed

QUESTION 2: Let $\xi : \mathbb{R}^n \to \mathbb{R}$ and a terminal time $0 < T < \infty$. For concreteness, take a concrete path-dependent SDE subject to risk on parameters

$$dX^{u,\theta}(t) = \alpha_\theta(t, X_t^{u,\theta}, u(t))dt + \sigma(t, X_t^{u,\theta}, u(t))dB(t),$$

for $\theta \in \Theta$. **Assume that the true parameter** $\theta_{\text{true}} \in \Theta$. How to design a scalable adaptive learning algorithm for computing optimal strategies for

$$\inf_u \mathbb{E}[\xi(X^{u,\theta}(T))]$$

along $\theta \in \Theta$ as $\theta \to \theta_{\text{true}}$?

Main References

- Leão, D. O-A. Scotti, S. Silva, A. M. (2026). Adaptive Learning via Off-Model Training and Importance Sampling for Fully Non-Markovian Optimal Stochastic Control. Arxiv.

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- Leão, D. O-A. Scotti, S. Silva, A. M. (2026). Adaptive Learning via Off-Model Training and Importance Sampling for Fully Non-Markovian Optimal Stochastic Control. Arxiv.
- Leão, D., O-A. and Souza, F. (2024). Solving non-Markovian Stochastic Control Problems driven by Wiener Functionals. *AAP*.
- Leão,D. O-A. and Simas, A. B. (2018). A weak version of path-dependent functional Itô calculus. *AOP*.
- O-A and Souza, F.A. (2020) L^p uniform random walk-type approximation for fractional Brownian motion with Hurst exponent $0 < H < \frac{1}{2}$. *EJP*.

Basic discretization scheme

We are going to fix a d-dimensional Brownian motion $B = \{B^1, \dots, B^d\}$ on $(\Omega, \mathbb{F}, \mathbb{P})$, where Ω is the space $C(\mathbb{R}_+; \mathbb{R}^d) := \{f : \mathbb{R}_+ \to \mathbb{R}^d \text{ continuous}\}$, $\mathbb{P}$ is the Wiener measure on Ω such that $\mathbb{P}\{B(0) = 0\} = 1$ and $\mathbb{F} := (\mathcal{F}_t)_{t \geq 0}$ is the usual $\mathbb{P}$-augmentation of the natural filtration generated by the Brownian motion.
In the sequel,

$$u \mapsto X^u$$

is a controlled $\mathbb{F}$-adapted continuous process defined on U_0^T.

The underlying discrete skeleton $\mathscr{D}$

We start by constructing a sequence $\mathcal{T} := \{T^k_n; n \geq 0\}$ of hitting times which will be the basis for our discretization scheme. Fix a sequence $\epsilon_k \downarrow 0$ as $k \to +\infty$. We set $T^k_0 := 0$ and

$$T^k_n := \inf\{T^k_{n-1} < t < \infty; |B(t) - B(T^k_{n-1})| = \epsilon_k\}, \quad n \geq 1.$$

Then, we define $A^k := (A^{k,1}, \cdots, A^{k,d})$ by

$$A^{k,j}(t) := \sum_{n=1}^{\infty} \Big(B^j(T^k_n) - B^j(T^k_{n-1})\Big) \mathbb{1}_{\{T^k_n \leq t\}}; \;\; t \geq 0, \; j = 1, \ldots, d,$$

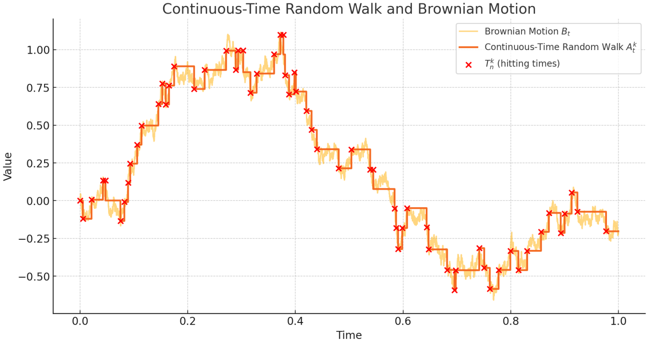
for integers $k \geq 1$.

The underlying discrete skeleton

In the one dimensional case, we have

$$T_n^k = \inf\left\{T_{n-1}^k < t < \infty; |B(t) - B(T_{n-1}^k)| = \varepsilon_k\right\}, \quad n \geq 1.$$

$\{A^k(T_n^k) - A^k(T_{n-1}^k); n \geq 1\}$ is an iid sequence of Bernoulli variables.



The underlying discrete skeleton

Let $\mathbb{F}^k = (\mathcal{F}^k_t)_{t\geq 0}$ be the filtration generated by A^k. One can check

$$\mathcal{F}^k_{T^k_n} = \sigma\Big(\Delta T^k_i, \Delta A^k(T^k_i); 1 \leq i \leq n\Big),$$

where $\Delta T^k_n := \underbrace{T^k_n - T^k_{n-1}}_{\text{Burq-Jones algorithm (2008)}} \stackrel{d}{=} T^k_1$ and

$$\Delta A^k(T^k_n) := A^k(T^k_n) - A^k(T^k_{n-1}) \stackrel{(d)}{=} \Delta A^k(T^k_1),$$

$\Delta A^k(T^k_1) \stackrel{(d)}{=}$ Bernoulli (for 1-dim Brownian motion) or approximately truncated Gaussian (for a $1 < d$-dim Brownian motion) conditioned on the Brownian coordinate which hits ϵ_k as $k \uparrow +\infty$.

Definition

The structure $\mathscr{D} = \{\mathcal{T}, A^k; k \geq 1\}$ is called a **discrete-type skeleton** for the Brownian motion.

The number of steps

The compact interval $[0, T]$ is replaced by a suitable number of steps:

$$e(k, T) := \left\lceil \frac{\epsilon_k^{-2} T}{\chi_d} \right\rceil,$$

where $\lceil x \rceil$ is the smallest integer greater or equal to $x \geq 0$ and

$$\chi_d := \mathbb{E} \min\{\tau^1, \ldots, \tau^d\},$$

where $(\tau^j)_{j=1}^d$ is an iid sequence of random variables with distribution $\inf\{t > 0; |W(t)| = 1\}$ for a real-valued standard Brownian motion W.

Discretizing the set of controls

Let $U_0^{k,e(k,T)}$ be the set of $\mathbb{F}^k$-predictable processes of the form

$$v^k(t) = \sum_{j=1}^{e(k,T)} v_{j-1}^k \mathbb{1}_{\{T_{j-1}^k < t \leq T_j^k\}},$$

where for each $j = 1, \ldots, e(k, T)$, v_{j-1}^k is an $\mathbb{A}$-valued $\mathcal{F}^k_{T^k_{j-1}}$-measurable random variable.

Controlled imbedded discrete structures

The structure $\mathscr{D}$ is dense in the Wiener space in the following sense:

Theorem Leão, O-A (2018, 2024)

For a given controlled process $u \mapsto X^u$ satisfying a rather weak continuity assumption, we can associate a discrete-type structure $\mathcal{X} = \big((X^k)_{k\geq 1}, \mathscr{D}\big)$ of the following form: For each $\phi \in U_0^{k,e(k,T)}$,

$$X^{k,\phi}(t) = \sum_{n=0}^{\infty} X^{k,\phi}(T_n^k)\mathbb{1}_{\{T_n^k \leq t\wedge T^k_{e(k,T)} < T^k_{n+1}\}},$$

where $X^{k,\phi}(T_n^k)$ is $\mathcal{F}^k_{T_n^k}$-measurable for every $n \geq 0$ and $k \geq 1$. Moreover, there exists a positive sequence $h_k \downarrow 0$ such that

$$\sup_{\phi\in U_0^{k,e(k,T)}} \mathbb{E}\|X^{k,\phi} - X^{\phi}\|_\infty \lesssim h_k,$$

for $k \geq 1$. A pair $(X, \mathcal{X})$ is called an **imbedded discrete structure** for X.

Typical examples of controlled imbedded discrete structures

(Controlled path-dependent SDEs)

$$
\begin{aligned}
X^{k,v^k}(T^k_n) &= X^{k,v^k}(T^k_{n-1}) + \alpha\Big(T^k_{n-1}, X^{k,v^k}_{T^k_{n-1}}, v^k_{n-1}\Big)\Delta T^k_n \\
&+ \sigma\Big(T^k_{n-1}, X^{k,v^k}_{T^k_{n-1}}, v^k_{n-1}\Big)\Delta A^k(T^k_n),
\end{aligned}
$$

for $n \geq 1$.

(Controlled path-dependent SDEs driven by FBM)

$$
\begin{aligned}
X^{k,v^k}(T^k_n) &= X^{k,v^k}(T^k_{n-1}) + \alpha\Big(T^k_{n-1}, X^{k,v^k}_{T^k_{n-1}}, v^k_{n-1}\Big)\Delta T^k_n \\
&+ \sigma\Delta B^k_H(T^k_n),
\end{aligned}
$$

B^k_H is a $\mathscr{D}$-discretization of B^H.

The FBM imbedded discrete structure

Let $K_H(t,s) = K_{H,1}(t,s) + K_{H,1}(t,s)$ be the classical Volterra-kernel of FBM,

$$K_{H,1}(t,s) := c_{H,1} t^{H-\frac{1}{2}} s^{\frac{1}{2}-H} (t-s)^{H-\frac{1}{2}},$$

$$K_{H,2}(t,s) := c_{H,2} s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{3}{2}} (u-s)^{H-\frac{1}{2}} du$$

for $s < t$, constants $c_{H,1}$ and $c_{H,2}$ and $0 < H < \frac{1}{2}$.

Let C_0^λ be the space of Hölder continuous functions f, $\frac{1}{2} - H < \lambda < \frac{1}{2}$ such that $f(0) = 0$. For each $f \in C_0^\lambda$, we set

$$\begin{aligned}(\Lambda_H f)(t) &:= \int_0^t \partial_s K_{H,1}(t,s)[f(t) - f(s)] ds \\ &- \int_0^t \partial_s K_{H,2}(t,s) f(s) ds.\end{aligned}$$

The FBM imbedded discrete structure

Theorem O-A, Souza (2020)

Any FBM with exponent $0 < H < \frac{1}{2}$ on a time interval $[0, T]$ can be represented by $\Lambda_H B$ for a real-valued standard Brownian motion B.

The FBM imbedded discrete structure

Let $\bar{t}_k := \max\{T^k_n; T^k_n \le t\}$ and $\bar{t}^+_k := \min\{T^k_n; \bar{t}_k < T^k_n\} \wedge T$.

$$\begin{aligned} B^k_H(t) &:= \int_0^{\bar{t}_k} \partial_s K_{H,1}(\bar{t}_k, s)[A^k(\bar{t}_k) - A^k(\bar{s}^+_k)]ds \\ &- \int_0^{\bar{t}_k} \partial_s K_{H,2}(\bar{t}_k, s) A^k(s) ds \\ &= \sum_{j=2}^{n} \Delta A^k(T^k_j)\, K_{H,1}(T^k_n, T^k_{j-1}) + \sum_{j=1}^{n-1} \Delta A^k(T^k_j)\, K_{H,2}(T^k_n, T^k_j) \end{aligned}$$

for $t = T^k_n$ with $n \ge 2$.

Theorem O-A, Souza (2020)

Fix $0 < H < \frac{1}{2}$ and $p \ge 1$. For every pair (δ, λ) such that $\max\{0, 1 - \frac{pH}{2}\} < \delta < 1$, $\lambda \in \left(\frac{1-H}{2}, \frac{1}{2} + \frac{\delta - 1}{2}\right)$, we have

$$\mathbb{E}\|B^k_H - B_H\|^p_\infty \lesssim_{p,\delta,\lambda,H,T} \epsilon_k^{p(1-2\lambda)+2(\delta-1)} \to 0$$

as $k \to +\infty$.

The approximated value process

Notation: $\xi_{X^k}(u) := \xi(X^{k,u})$ for a given controlled imbedded discrete structure $u \mapsto X^{k,u}$.
We set

$$V^k(T_n^k, u) := \operatorname*{ess\,inf}_{\phi \in U_n^{k,e(k,T)}} \mathbb{E}\Big[\xi_{X^k}\ \overbrace{(u \otimes_n \phi)}^{\text{concatenation}}\ \big|\mathcal{F}_{T_n^k}\Big],$$

for $n = 1, \ldots, e(k,T) - 1$, with

$$V^k(0) := V^k(0,u) := \inf_{\phi \in U_0^{k,e(k,T)}} \mathbb{E}\big[\xi_{X^k}(\phi)\big]$$

and

$$V^k(T_{e(k,T)}^k, u) := \xi_{X^k}(u).$$

Intuition of aggregation

For a given control $u^k \in U_0^{k,e(k,T)}$ and a controlled structure $\mathcal{X} = \big((X^k)_{k\geq 1}, \mathscr{D}\big)$, we set the history of the controlled system as

$$\Xi_j^{k,u^k} := \underbrace{\Big(\mathcal{W}_1^k, \Delta X^{k,u^k}(T_1^k)), \ldots, \mathcal{W}_j^k, \Delta X^{k,u^k}(T_j^k))\Big)}_{\text{later this will be transformed into a training data}},$$

for $1 \leq j \leq e(k,T)$. Here $\mathcal{W}_j^k := \big(\Delta T_j^k, \Delta A^k(T_j^k)\big)$. The value functionals can be represented by a big functional (starting backwards with the loss functional $\mathbb{V}_{e(k,T)}^k := \xi$)

$$\begin{aligned} V^k(T_n^k, u^k) &= \inf_{\theta \in \mathbb{A}} \int_{\mathbb{W}_k} \mathbb{V}_{n+1}^k\Big(\Xi_n^{k,u^k}, \overbrace{\mathfrak{X}_{n+1}^k\big(\theta, \Xi_n^{k,u^k}, w^k\big)}^{\text{typical Euler-type increment}}\Big)\nu^k(dw^k) \\ &= \mathbb{V}_n^k(\Xi_n^{k,u^k}) \ (\text{aggregation}) \end{aligned}$$

where $\mathfrak{X}_n^k$ is the jump of the $\mathscr{D}$-controlled state as step n and ν^k is the law of $(\Delta T_1^k, \Delta A^k(T_1^k))$ taking values on a set $\mathbb{W}_k$.

The Dynamic Programming Principle

Theorem Leão, O-A (2024)-Pathwise Dynamic Programming Equation

Let ν^k be the law of $(\Delta T_1^k, \Delta A^k(T_1^k))$. Starting from a given controlled state (standard terminal condition)

$$\mathbb{V}^k_{e(k,T)}(\cdot) = \xi(\cdot),$$

the value functionals ($\mathscr{D}$-version of the original value process) satisfy

$$V^k(T_j^k, u) = \mathbb{V}_j^k(\Xi_j^{k,u})$$

where

$$\begin{aligned}
\mathbf{U}_j^k(\mathbf{o}_j^k, \theta) &:= \int_{\mathbb{W}_k} \mathbb{V}_{j+1}^k\Big(\mathbf{o}_j^k, \mathfrak{X}_{j+1}^k(\theta, \mathbf{o}_j^k, w^k)\Big)\nu^k(dw^k) \\
\mathbb{V}_j^k(\mathbf{o}_j^k) &:= \inf_{\theta \in \mathbb{A}} \mathbf{U}_j^k(\mathbf{o}_j^k, \theta),\ j = e(k, T) - 1, \ldots, 0.
\end{aligned}$$

Optimal Strategies

Let $\mathbb{H}_k = \mathbb{W}_k \times \mathbb{R}^q$. For a given $\epsilon > 0$, compute $C^{\epsilon}_{k,j} : \mathbb{H}^j_k \to \mathbb{A}$ (**How ?**) such that

$$\mathbb{V}^k_j(\mathbf{o}^k_j) + \epsilon \geq \int_{\mathbb{W}_k} \mathbb{V}^k_{j+1}\Big(\mathbf{o}^k_j, \mathfrak{X}^k_{j+1}(C^{\epsilon}_{k,j}(\mathbf{o}^k_j), \mathbf{o}^k_j, w^k)\Big)\nu^k(dw^k),$$

for every $\mathbf{o}^k_j$ training data, where $j = e(k,T) - 1, \ldots, 1$. Let $\eta_k(\epsilon) = \frac{\epsilon}{e(k,T)}$ and $u^k \in U_0^{k,e(k,T)}$. Define $\phi^{k,\eta_k(\epsilon)}_j$ as follows

$$\phi^{k,\eta_k(\epsilon)}_j = C^{\eta_k(\epsilon)}_{k,j}(\Xi^{k,u^k}_j); j = e(k,T) - 1, \ldots, 0.$$

Theorem (Leão, O-A (2024))

The control

$$\phi^{*,k,\epsilon} := (\phi^{k,\eta_k(\epsilon)}_0, \phi^{k,\eta_k(\epsilon)}_1, \ldots, \phi^{k,\eta_k(\epsilon)}_{e(k,T)-1})$$

realizes

$$\mathbb{E}\big[\xi_X\big(\phi^{*,k,\epsilon}\big)\big] \leq \inf_{\phi \in U^T_0} \mathbb{E}\big[\xi_X(\phi)\big] + \epsilon,$$

for every k sufficiently large and we have convergence rates ...

Optimal control of drifts

Theorem Leão, O-A (2024)

Let X be the controlled SDEs driven by FBM with $0 < H < \frac{1}{2}$, where the controls affect only the drift coefficients. Assume the drift has convex range in $\mathbb{R}^n$. Then, for

$$\left| \sup_{\phi \in U_0^T} \mathbb{E}[\xi_X(\phi)] - V_k(0) \right| \lesssim \epsilon_k^{H^-} \to 0, \text{ as } k \to \infty.$$

Some notation: Suppressing the index k

For any stepwise constant process Z jumping at the hitting times, we denote

$$Z_n := Z(T_n^k), \quad \Delta Z_n := Z_n - Z_{n-1}, \quad \varepsilon = \varepsilon_k,$$

for simplicity. We denote

$$m = e(k, T).$$

Recall that the ℓ-fold Cartesian product is written as

$$\text{(History space)} \quad \mathbb{H}^\ell = \left(\mathbb{W} \times \mathbb{R}^q\right)^\ell; 1 \le \ell \le e(k, T).$$

The elements of $\mathbb{H}^\ell$ are denoted by

$$\mathbf{o}_\ell = (w_1, y_1, \dots, w_\ell, y_\ell), w_j = (s_i, \eta_j).$$

where we interpret $w_i = (s_i, \eta_i)$ as a realization of $(\Delta T_i, \Delta A_i)$.

The class of non-Markovian structure

We assume that the existence of Borel functions

$$\blacktriangle x_j : (\mathbb{R}_+ \times \mathbb{R}^q)^{j-1} \times \mathbb{A} \times (0,\infty) \times \mathbb{R}^d \to \mathbb{R}^q, \qquad \overbrace{\ell_j : \mathbb{W}^j \to \mathbb{R}^d}^{\text{extrinsic noise operator}} \quad ,$$

such that the transition of the controlled system at step j knowing a constant control $a \in \mathbb{A}$ and a history $\mathbf{o}_{j-1} = (w_1, y_1, \dots, w_{j-1}, y_{j-1})$ is

$$\Delta X_j^a = \blacktriangle x_j\Big(\pi_2(\mathbf{o}_{j-1}), a, \Delta T_j, \ell_j(\pi_3(\mathbf{o}_{j-1}), \mathcal{W}_j)\Big),$$

where

$$\pi_2(\mathbf{o}_i) := ((s_1, y_1), \dots, (s_i, y_i)) \quad \pi_3(\mathbf{o}_i) := (w_1, \dots, w_i),$$

for $i \geq 1$. Recall $w_i = (s_i, \eta_i)$ is a realization of $(\Delta T_i, \Delta A_i)$.

Typical example: Controlled SDEs driven by FBM

The law of ΔX^u_n conditioned on $\Xi_{n-1} = \mathbf{o}_{n-1}$ is

$$\begin{aligned} \text{Law}\left[\Delta X^u_n | \Xi_{n-1} = \mathbf{o}_{n-1}\right] &\stackrel{(d)}{=} \varrho\big(\pi_2(\mathbf{o}_{n-1}), u\big)\Delta T_n \\ &+ \sigma \ell_n(\pi_3(\mathbf{o}_{n-1}), \mathcal{W}_n) \\ &= \blacktriangle x_n\Big(\pi_2(\mathbf{o}_{n-1}), u, \Delta T_n, \ell_n(\pi_3(\mathbf{o}_{n-1}), \mathcal{W}_n)\Big) \end{aligned}$$

where for $\mathcal{A}_n = (\overbrace{\Delta T_1, \Delta A_1, \ldots, \Delta T_{n-1}, \Delta A_{n-1}}^{\mathcal{A}_{n-1}}, \overbrace{\Delta T_n, \Delta A_n}^{\mathcal{W}_n})$, we set $\ell_n(\mathcal{A}_n) := \Delta B^H_n$ with

$$B^H_n = \sum_{j=2}^{n} \Delta A_j K_{H,1}(T_n, T_{j-1}) + \sum_{j=1}^{n-1} \Delta A_j K_{H,2}(T_n, T_j).$$

Solving by Deep Learning

By definition, the value functions are

$$\mathbb{V}_j(\mathbf{o}_j) = \mathbf{U}_j(\mathbf{o}_j, g_j(\mathbf{o}_j)); j = m-1, \ldots, 0.$$

The class of functions which contains $\{g_j\}_{j=0}^{m-1}$ is unknown. For this reason, we postulate two neural network spaces (here $\mathcal{B}$ and Θ are suitable parameter sets).

$$\mathcal{C}_j = \{\mathbb{H}^j \ni x \mapsto C(x, \theta) \in \mathbb{A}; \theta \in \Theta\}$$

and

$$\mathcal{V}_j = \{\mathbb{H}^j \ni x \mapsto \Phi(x, \beta); \beta \in \mathcal{B}\}.$$

Solving by Deep Learning: Training data

In order to design a Monte Carlo scheme for $(\mathbb{V}_j)_{j=0}^{m-1}$, we will make use of a training data denoted by

$$O_n = (\mathcal{W}_1, Y_1, \ldots, \mathcal{W}_n, Y_n); 1 \le n \le m.$$

Assumption H0: We will assume the pair $(\mathcal{W}_i, Y_i)_{i=1}^m$ is i.i.d and generated by a product probability measure $\nu \otimes \mu$ on $\mathbb{H} = \mathbb{W} \times \mathbb{R}^q$, where ν is the law of $(\Delta T_1, \Delta A_1)$. The sequence $\{Y_i; i \ge 1\}$ is iid generated by a probability measure μ and Y_i is independent of $\mathcal{W}_i$ for every $i \ge 1$.

Importance Sampling Weights

Assumption H1: We assume there exists a transition function $r_j : \mathbb{A} \times \mathbb{R}^q \times \mathbb{H}^{j-1} \to \mathbb{R}_+$ such that

$$\mathbb{P}[\Delta X_j^a \in dx | \Xi_{j-1} = b] = \underbrace{r_j(a, x; b)}_{\text{IS weight}} \mu(dx)$$

where

$$\|r\|_\infty := \max_{1 \le j \le m} \sup_{x' \in \mathbb{R}^q, a \in \mathbb{A}, b \in \mathbb{H}^{j-1}} |r_j(a, x'; b)| < \infty.$$

Moreover, there exists $\|r\|$ such that

$$|r_j(a, x; b) - r_j(a', x; b')| \le \|r\| \{|a - a'| + |b - b'|\}$$

for every $x \in \text{supp}\mu$, $a, a' \in \mathbb{A}$, $b, b' \in \mathbb{H}^{j-1}$ and $1 \le j \le m$.

Importance Sampling Weights

Why this property is important ? Starting with $\mathbb{V}_m(\mathbf{o}_m) := \xi\left(x_0 + \sum_{i=1}^m y_i\right)$, we can write

$$\begin{aligned} \mathbf{U}_j(\mathbf{o}_j, a) &= \int_{\mathbb{H}} \mathbb{V}_{j+1}(\mathbf{o}_j, x', x)\, r_j(a, x; \mathbf{o}_j)\mu(dx)\nu(dx') \\ \mathbb{V}_j(\mathbf{o}_j) &= \inf_{a \in \mathbb{A}} \mathbf{U}_j(\mathbf{o}_j, a) \end{aligned} \tag{1}$$

for $j = m-1, \ldots, 0$.

In our approach, IS weights play a key role for what we call **adaptive Neural Network MC scheme with parametric model risk**

IS weight for path-dependent SDEs

For simplicity, we present the one-dimensional case. Fix $J \overset{(d)}{=} \Delta T_1$ and $\mathbf{B} \overset{(d)}{=}$ Bernoulli$(\pm\epsilon)$. We assume that

$$J \text{ is truncated at a given region } [\underline{M}, \overline{M}]$$

for $0 < \underline{M} < \overline{M} < \infty$.

In the sequel, for a given history $\Xi_{n-1} = \mathbf{o}_{n-1} = (w_0, y_0, \ldots, w_{n-1}, y_{n-1})$ with $\Xi_0 = \mathbf{o}_0 = (0, 0, x_0)$ and $s_0 = 0$, we denote

$$t_{n-1} := \sum_{\ell=0}^{n-1} s_\ell,$$

for $n \geq 1$ and a fixed initial condition $y_0 = x_0 \in \mathbb{R}$. The dynamics of ΔX_n^a conditioned on $\Xi_{n-1} = \mathbf{o}_{n-1} = (w_0, y_0, \ldots, w_{n-1}, y_{n-1})$ is given by

$$\Delta X_n^a = \alpha\big(t_{n-1}, \{y_i\}_{i=0}^{n-1}, a\big) J + \sigma\big(t_{n-1}, \{y_i\}_{i=0}^{n-1}, a\big)\mathbf{B},$$

where J is independent of $(\mathbf{B}, \Xi_{n-1})$, for $n \geq 1$.

IS weight for path-dependent SDEs

We set

$$R_{c,v} := cJ + v\mathbf{B},$$

for $(c, v) \in \mathbb{R}^2$. We denote

$$\mathcal{R}(c, v; \cdot) := \text{density function of } R_{c,v}.$$

for $(c, v) \in \mathbb{R}^2$ with $c \neq 0, v \neq 0$. One can check

$$\mathcal{R}(c, v; x) = \frac{1}{2|c|} \sum_{s=\pm 1} f_J\Big(\frac{x - sv\varepsilon}{c}\Big) \mathbb{1}_{\{\frac{x - sv\varepsilon}{c} \in [\underline{M}, \overline{M}]\}},$$

where f_J is the density of J (see e.g. the classical book of Borodin and Saminen (2002))

IS weight for path-dependent SDEs

(B1) (*Lipschitz property*) The non-anticipative coefficients (α, σ) are globally Lipschitz.

(B2) (*Compactness*) There exist $0 < c_{\min} < c_{\max} < \infty$ and $v_{\max}$ such that

$$c_{\min} \;\le\; |\,\alpha(t, \text{path}, a)\,| \;\le\; c_{\max} \quad \text{and} \quad |\sigma(t, \text{path}, a)| \le v_{\max},$$

for every (t, path, a).

For a discretization level ε, let

$$K = \Big[- v_{\max}\epsilon - c_{\max}\overline{M}, v_{\max}\varepsilon + c_{\max}\overline{M} \Big].$$

Let $\mathcal{M}_K$ be the set of probability densities q with support on K such that the following condition is fulfilled: There exists a positive finite constant Q_K such that

$$\inf_{x \in K} q(x) \ge Q_K > 0.$$

IS weight for path-dependent SDEs

Theorem

Suppose (B1-B2) are fulfilled. Then, for a given history $\mathbf{o}_{j-1} = (w_1, y_1, \ldots, w_{j-1}, y_{j-1})$ and a control value a, the conditional law of $(\mathcal{W}_j, \Delta X_j^a)$ is given by

$$\mathbb{P}[\Delta X_j^a \in dx | \Xi_{j-1} = \mathbf{o}_{j-1}] = r_j(a, x; \mathbf{o}_{j-1})\mu(dx),$$

where $\mu \in \mathcal{M}_K$ with density q and

$$r_j(a, x; \mathbf{o}_{j-1}) = \frac{\mathcal{R}(c(\mathbf{o}_{j-1}, a), v(\mathbf{o}_{j-1}, a); x)}{q(x)},$$

for $v(\mathbf{o}_{j-1}, a) = \sigma(t_{j-1}, \{y_i\}_{i=0}^{j-1}, a)$ and $c(\mathbf{o}_{j-1}, a) = \alpha(t_{j-1}, \{y_i\}_{i=0}^{j-1}, a)$. Moreover, assumptions H1 is fulfilled.

IS weight for SDEs driven by R-L FBM

Fix a smooth $\varrho : \mathbb{R} \times \mathbb{A} \to \mathbb{R}$, where $\mathbb{A}$ is compact. The dynamics of ΔX_n^a conditioned on $\Xi_{n-1} = (w_1, y_1, \ldots, w_{n-1}, y_{n-1})$ is given by

$$\Delta X_n^a = \varrho(\mathbf{y}_{n-1}, a) J + \sigma \ell_n(\mathcal{A}_{n-1}, J), \quad \mathbf{y}_{n-1} = \sum_{i=1}^{n-1} y_i,$$

where we shall represent the discretization of the increment of the R-L FBM as

$$\ell_n(\mathcal{A}_{n-1}, J) = \sum_{i=1}^{n-1} \Delta A_i [K(T_{n-1}, T_i) - K(T_{n-1} + J, T_i)],$$

for $n \geq 2$ and we set $\ell_1(\cdot) = 0$. Here, K is the Riemann-Liouville fractional Brownian motion kernel with $H \neq \frac{1}{2}$,

$$K(t, s) = \sqrt{2H}(t - s)^{H - \frac{1}{2}}; s < t.$$

IS weight for SDEs driven by FBM

For a realized history $b = \Xi_{j-1}$, $c \in \mathbb{R}$ and a control value a, we denote

$$\psi_b(u) := \sum_{i=0}^{j-1} \Delta A_i \left[K(T_{j-1}, T_i) - K(T_{j-1} + u, T_i) \right],$$

$$\phi_{b,c}(u) := cu + \psi_b(u),$$

for $u \geq 0$. One can check

$$\mathrm{Law}[\Delta X_j^a | \Xi_{j-1} = b] \overset{(d)}{=} \phi_{b_{j-1}, c_{j-1}}(J),$$

for each realized history $b = \Xi_{j-1}$, a control slope $c_{j-1} := \varrho(\mathbf{y}_{j-1}, a)$ and a control value a.

Let us define the following constants

$$C_1 := \sqrt{2H}(H - \frac{1}{2})\varepsilon m \underline{M}^{H-\frac{3}{2}}, \quad C_0 := \sqrt{2H}\varepsilon m \overline{M}^{H-\frac{1}{2}}.$$

(A1) The drift ϱ is globally Lipschitz.

(A2) (*control separation*) There exist $0 < c_{\min} < c_{\max} < \infty$ such that

$$0 < c_{\min} \;\leq\; |\,\varrho(y,a)\,| \;\leq\; c_{\max} \qquad \text{for all } (y,a) \in \mathbb{R} \times \mathbb{A}.$$

(A3) (*large control separation*) $C_1 \leq \frac{1}{2} c_{min}$.

IS weight for SDEs driven by FBM

Under (A1-A2-A3), one can prove $u \mapsto \phi_{b,c}(u)$ is C^1 and strictly increasing and $\mathrm{Law}[\Delta X_j^a | \Xi_{j-1} = b]$ has a density

$$f_{\phi_{b,c}}(y) = \frac{f_J(\phi_{b,c}^{-1}(y))}{|\phi'_{b,c}(\phi_{b,c}^{-1}(y))|}; \quad y \in \phi_{b,c}\left([\underline{M}, \overline{M}]\right),$$

where we shall set $f_{\phi_{b,c}}(y) = 0$ whenever $y \notin \phi_{b,c}\left([\underline{M}, \overline{M}]\right)$. Here,

$$\phi'_{b,c}(u) = c + \psi'_b(u),$$

$$\psi'_b(u) = -c_H \sum_{i=1}^{j-1} \Delta A_i \left(T_{j-1} + u - T_i\right)^{H-\frac{3}{2}}$$

IS weight for SDEs driven by FBM

Theorem

Assume (A1-A2-A3) are fulfilled. Let μ be the two-sided Laplace probability distribution of the form

$$\mu(dx') = q_\beta(x')\,dx', \qquad q_\beta(x') = \frac{\beta}{2} e^{-\beta|x'|},$$

where $0 < \beta \leq \frac{\gamma_\Delta}{c_{\max}}$. Then, for a given history $b_{j-1} = \Xi_{j-1} = (w_1, y_1, \dots, w_{j-1}, y_{j-1})$ and a control value a, the conditional law of ΔX_j^a is given by

$$\mathbb{P}[\Delta X_j^a \in dx | \Xi_{j-1} = b_{j-1}] = r_j(a, x; b_{j-1})\mu(dx),$$

where

$$r_j(a, x; b_{j-1}) := \frac{f_{\phi_{b_{j-1},c_{j-1}}}(x)}{q_\beta(x)}$$

and $f_{\phi_{b_{j-1},c_{j-1}}}$ is the density of $\phi_{b_{j-1},c_{j-1}}(J)$ for $c_{j-1} = \varrho(\mathbf{y}_{j-1}, a)$. Moreover, assumption (H1) holds true.

Solving by Deep Learning

For a training data $O_n = \big(\mathcal{W}_1, Y_1, \dots, \mathcal{W}_n, Y_n\big)$ and a $C \in \mathbb{A}^{\mathbb{H}^{n-1}}$, let us denote

$$\mathcal{X}_n^C := (\mathcal{W}_n, \Delta X_n^C),$$

$$\Delta X_n^C := \blacktriangle x_n\big(\pi_2(O_{n-1}), C(O_{n-1}), \Delta T_n, \ell_n(\pi_3(O_{n-1}), \mathcal{W}_n)\big),$$

for $n = m, \dots, 1$.

Terminal condition: $\widehat{\mathbb{V}}_m := \mathbb{V}_m$. For $n = m-1, \dots, 1, 0$,

1. Compute the approximated control at time n

$$\overbrace{\hat{a}_n^M \in \underset{C \in \mathcal{C}_M(n)}{\arg\min}}^{\text{stochastic gradient descent(ADAM)}} \quad \mathbb{E}\Big[\widehat{\mathbb{V}}_{n+1}^M\Big(O_n, \mathcal{X}_{n+1}^C\Big)\Big]$$

2. compute the estimation of the value function at time n

$$\overbrace{\widehat{\mathbb{V}}_n^M \in \underset{\Phi \in \mathcal{V}_M(n)}{\arg\min}}^{\text{stochastic gradient descent (ADAM)}} \quad \mathbb{E}\Big[\widehat{\mathbb{V}}_{n+1}^M\big(O_n, \mathcal{X}_{n+1}^{\hat{a}_n^M}\big) - \Phi(O_n)\Big]^2.$$

Monte Carlo convergence rate

Theorem

Assume hypotheses (H0-H1). Assume there exists an optimal feedback control $(\phi_\ell^{opt})_{\ell=n}^{m-1}$ for the control problem with value functionals $\mathbb{V}_n$, for $n = 0, \ldots, m$. Then

$$\begin{aligned}
\mathbb{E}_M|\widehat{\mathbb{V}}_n^M(O_n) - \mathbb{V}_n(O_n)| &= \mathcal{O}_{\mathbb{P}}\Bigg(\left(\delta_M^4 N_M \frac{\log(M)}{M}\right)^{2^{-(m-n)}} \\
&+ \left(\rho_M^2 \delta_M^8 \eta_M^6 \|\psi\|^2 M^{-1}\right)^{2^{-(m-n+1)}} \\
&+ \left(\max_{n\le\ell\le m} \inf_{\Phi\in\mathcal{V}_M(\ell)} \|\Phi(O_\ell) - \mathbb{V}_\ell(O_\ell)\|_{M,2}\right)^{2^{-(m-n+1)}} \\
&+ \left(\max_{n\le\ell\le m} \inf_{G\in\mathcal{C}_M(\ell)} \sqrt{\|G(O_\ell) - \phi_\ell^{opt}(O_\ell)\|_{M,1}}\right)^{2^{-(m-n-1)}}\Bigg)
\end{aligned}$$

Adaptive model learning

Suppose we have a family of controlled path-dependent SDEs of the form

$$dX^\theta(t) = \alpha_\theta(t, X_t^{\theta,u}, u(t))dt + \sigma_\theta(t, X_t^{\theta,u}, u(t))dB(t)$$

where θ lies in a compact set $\Theta \subset \mathbb{R}$. Let $\mathbb{V}_j^\theta$ be the family of value functionals parameterized by θ: $\mathbb{V}_m^\theta(\mathbf{o}_m) = \mathbb{V}_m(\mathbf{o}_m) = \xi(x_0 + \sum_{i=1}^m y_i)$,

$$\mathbb{V}_j^\theta(\mathbf{o}_j) := \min_{a \in \mathbb{A}} \mathbf{U}_j^\theta(\mathbf{o}_j, a),$$

$$\mathbf{U}_j^\theta(\mathbf{o}_j, a) := \int_{\mathbb{H}} \mathbb{V}_{j+1}^\theta(\pi_2(\mathbf{o}_j), x, y) r_j^\theta(a, y, \mathbf{o}_j) q(y) dy \nu(dx),$$

for $j = m-1, \ldots, 0$. Recall the IS weight is

$$r_j^\theta(a, y, \mathbf{o}_j) := \frac{\mathcal{R}(c^\theta(\mathbf{o}_j, a), v^\theta(\mathbf{o}_j, a); y)}{q(y)}; y \in K,$$

where $c^\theta(\mathbf{o}_j, a) := \alpha_\theta(t_j, \{y_i\}_{i=0}^{j-1}, a)$, $v^\theta(\mathbf{o}_j, a) := \sigma_\theta(t_j, \{y_i\}_{i=0}^{j-1}, a)$ and the parameterized functionals $(\alpha_\theta, \sigma_\theta)$ satisfy assumptions (B1-B2).

Monte Carlo convergence rate

Theorem

Let $\mathbb{V}^{\theta}$ be the value function under a parameter model θ. If $\theta_n \to \theta^\star$, then

$$\begin{aligned} \mathbb{E}_M|\widehat{\mathbb{V}}^{\theta_n}_{j,M}(O_j) - \mathbb{V}^{\theta^\star}_j(O_j)| \quad &\lesssim \quad \underbrace{\mathbb{E}_M|\widehat{\mathbb{V}}^{\theta_n}_{j,M}(O_j) - \mathbb{V}^{\theta_n}_j(O_j)|}_{\textit{Sampling error controlled by } M} \\ &+ \quad \underbrace{|\theta_n - \theta^\star|}_{\textit{model risk}}, \end{aligned}$$

for $j = m-1, \ldots, 0$.

Numerical experiment: Quadratic hedging

Table: Diagnostic statistics for the P&L of an ATM European put. Off-policy rough-volatility experiment with $r_{\mathrm{train}} = 0.5$.

k	m	Mean P&L	Var(P&L)	$q_{5\%}$	$q_{1\%}$
3	6	0.036333	12.305737	-4.622572	-5.780918
4	22	0.071766	6.220595	-4.087614	-5.981536
5	86	-0.010717	2.463002	-2.620666	-3.595436
6	342	0.019073	0.537016	-1.107734	-1.628304

Training data is generated by

$$\mu = \mathcal{Z}_{\#}\beta, \quad \beta = U[-r_{\mathrm{train}}, r_{\mathrm{train}}] \times U[-dS_{\max}, +dS_{\max}],$$

where $\mathcal{Z}(a, z) := (z, az)^{\top}$.

Numerical experiment: Quadratic hedging

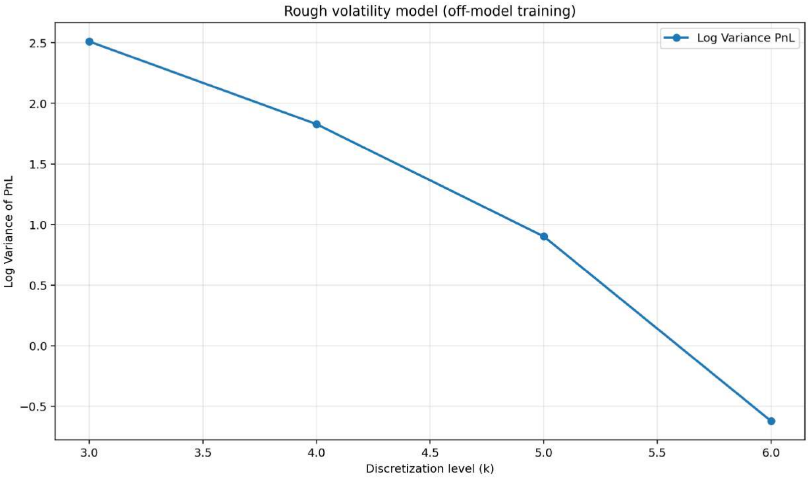


Figure: Empirical $\mathrm{Var}(\mathrm{P\&L})$ as a function of the discretization level (Rough SV model, $r_{\mathrm{train}} = 0.5$).

Numerical experiment: Quadratic hedging

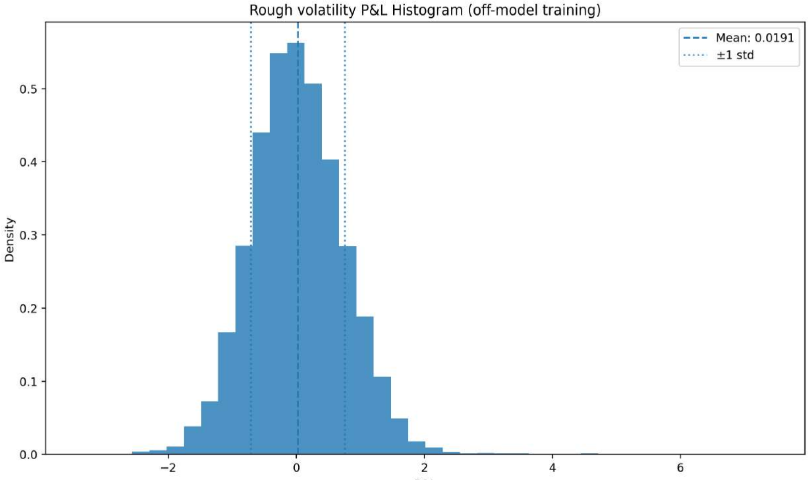


Figure: Histogram of P&L. Rough volatility. ATM Put with $S_0 = 100 = K$. Number of Monte Carlo samples $= 7000$. $r_{\text{train}} = 0.5$

THANK YOU VERY MUCH FOR YOUR ATTENTION !!