Structure of the capacity region and its use in multiuser systems

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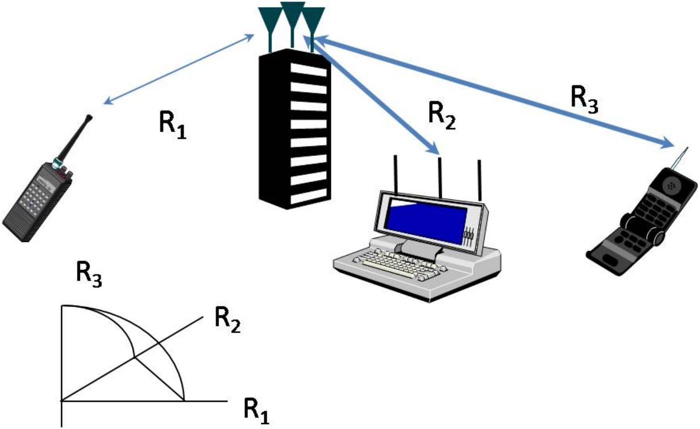
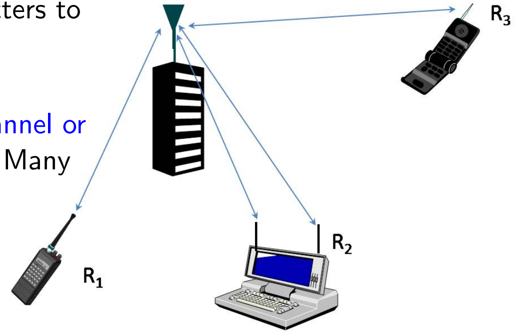
Capacity region

Capacity: set of simultaneously achievable rates $R_1, \ldots, R_n$ with arbitrarily small probability of error.



Uplink and downlink channels

- Uplink (Multiple Access Channel or MAC): Many Transmitters to One Receiver
- Downlink (Broadcast Channel or BC): One Transmitter to Many Receivers



Successive decoding

- The **asynchronous capacity region** of an M-user memoryless multiple-access channel (MAC) is union of certain M-dimensional polytopes
- **Successive decoding** is a method of canceling the interference from the already decoded users $\Rightarrow$ allows achieving optimal rates
- One user at a time can be decoded successively if a rate tuple lies on the vertex of the dominant face of such a polytope, using the codewords of already decoded users as side information

Group successive decoding

- Extension to group successive decoding for rate tuples that are on the boundary of the dominant face:
  – Each point on the boundary of the dominant face belongs to *a face of some dimension* $k \in \{0, 1, \ldots, M-2\}$
  – Forming $M - k$ *groups of users*
  – The users within a group are decoded *jointly* whereas groups are decoded *successively*
- The probability of error is smaller for joint decoding than for successive decoding

Group successive decoding

- **Group successive decoding** is extended to every face of the polytopes not only to the faces of the dominant face
- The **labeling technique** is extended in order to have label for every face
- **Non-degenerated polytopes** are considered for which the labels are unique
- The **group composition**, the decoding order and a number of structural properties for all rates on a face of interest are obtained from a label assigned to that face

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Main contribution

- Reviewing the multiple access capacity region
- Defining polytopes
- Labeling the faces of polytopes
- Deriving the necessary and sufficient conditions for faces to intersect

Discrete memoryless MAC

- An M - user **discrete memoryless multiple-access channel** is defined in terms of M discrete input-alphabets $\mathcal{X}_i$, $i \in \{1, \cdots, M\}$, an output alphabet $\mathcal{Y}$, and a stochastic matrix $W : \mathcal{X}_1 \times \mathcal{X}_2 \times \cdots \times \mathcal{X}_M \rightarrow \mathcal{Y}$ with entries $W_{Y|X_1,X_2,\cdots,X_M}(y|x_1, x_2, \cdots, x_M)$ describing the probability that the channel output is y when the inputs are $x_1, x_2, \cdots, x_M$

Region for a product input distribution I

- For any input distribution in product form $P_{X_1}, \cdots, P_{X_M}$, define $\mathcal{R}$ to be

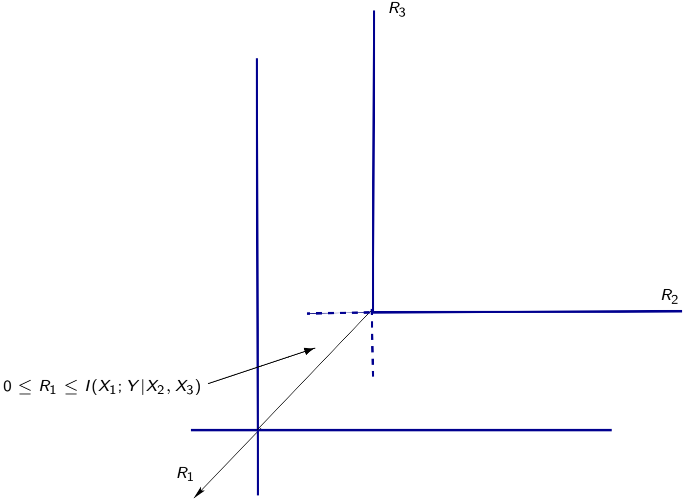
$$\mathcal{R} = \{R \in \mathbb{R}_+^M : R(\mathcal{S}) \leq I(X_\mathcal{S}; Y | X_{\mathcal{S}^c}), \quad \forall \mathcal{S} \subseteq [M]\},$$

  where $R(\mathcal{S}) \triangleq \sum_{i \in \mathcal{S}} R_i$, $X_\mathcal{S} \triangleq (X_i)_{i \in \mathcal{S}}$, $\mathcal{S}^c \triangleq [M] \setminus \mathcal{S}$, $[M] = \{1, 2, \ldots, M\}$, and $I(X_\mathcal{S}; Y | X_{\mathcal{S}^c})$ is the mutual information between $X_\mathcal{S}$ and Y given $X_{\mathcal{S}^c}$

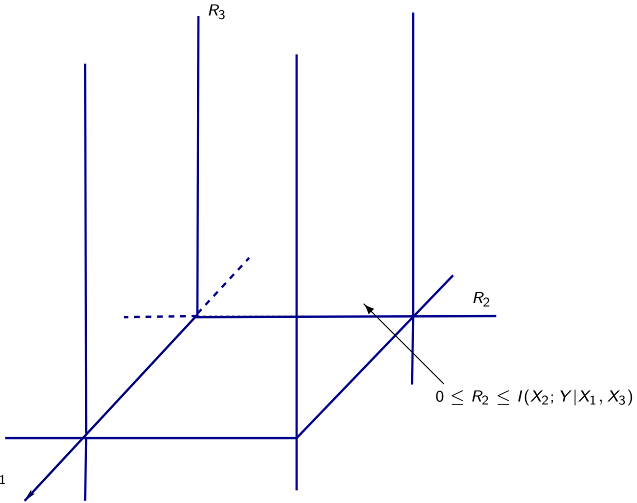
Region for a product input distribution II

$$
\begin{aligned}
\mathcal{R} = \{0 \leq R_1 &\leq I(X_1; Y|X_2, X_3), \\
0 \leq R_2 &\leq I(X_2; Y|X_1, X_3), \\
0 \leq R_3 &\leq I(X_3; Y|X_1, X_2), \\
R_1 + R_2 &\leq I(X_1, X_2; Y|X_3), \\
R_1 + R_3 &\leq I(X_1, X_3; Y|X_2), \\
R_2 + R_3 &\leq I(X_2, X_3; Y|X_1), \\
R_1 + R_2 + R_3 &\leq I(X_1, X_2, X_3; Y|X_1)\}
\end{aligned}
$$

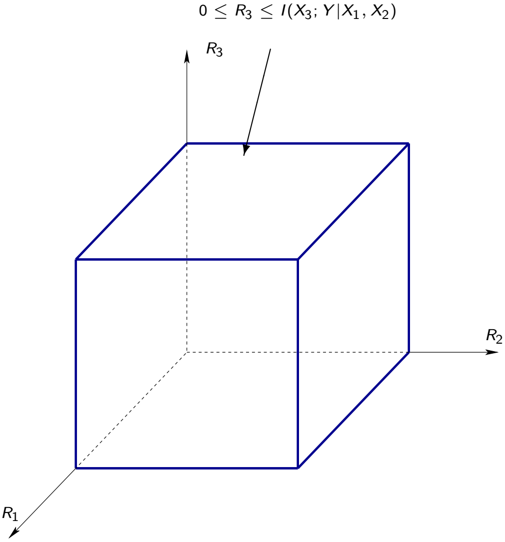
Capacity Region for $M = 3$



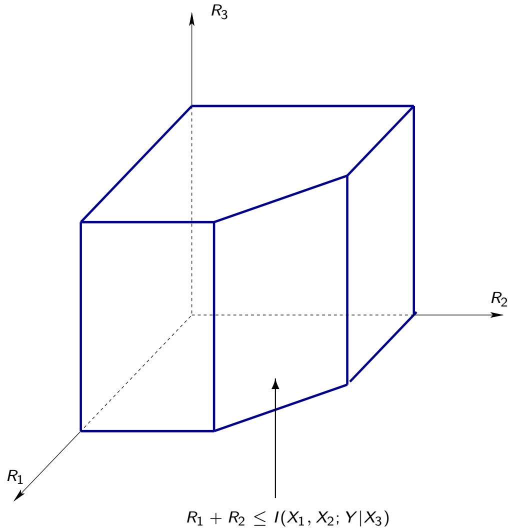
Capacity Region for $M = 3$



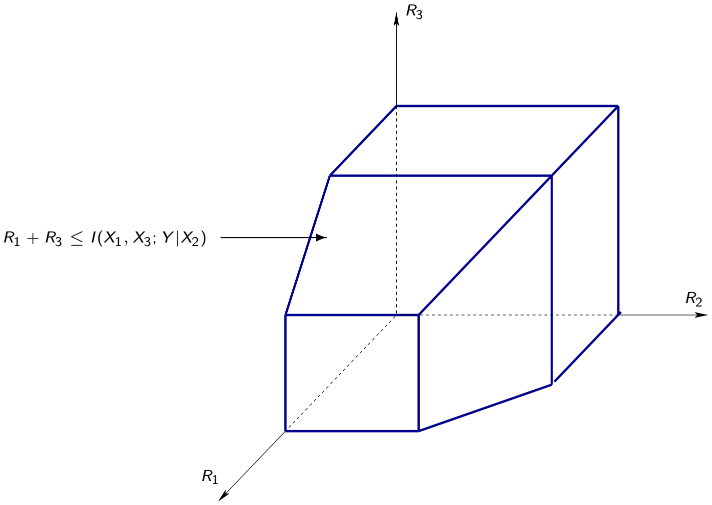
Capacity Region for $M = 3$



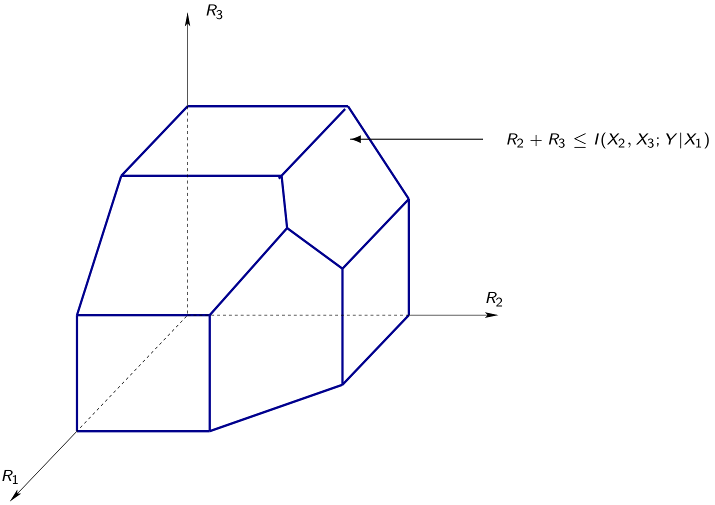
Capacity Region for $M = 3$



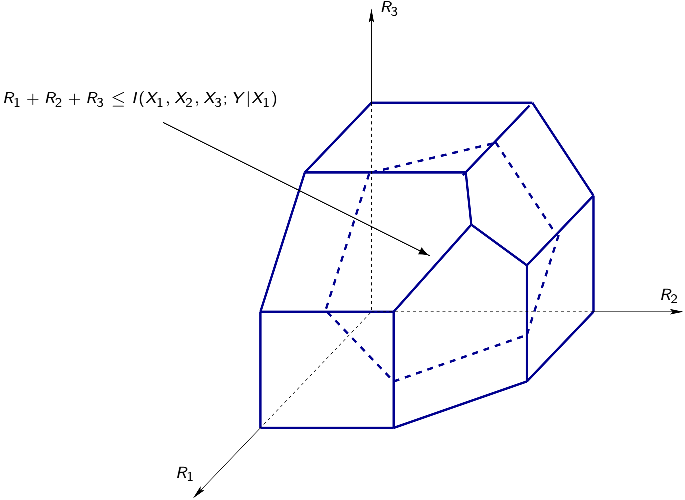
Capacity Region for $M = 3$



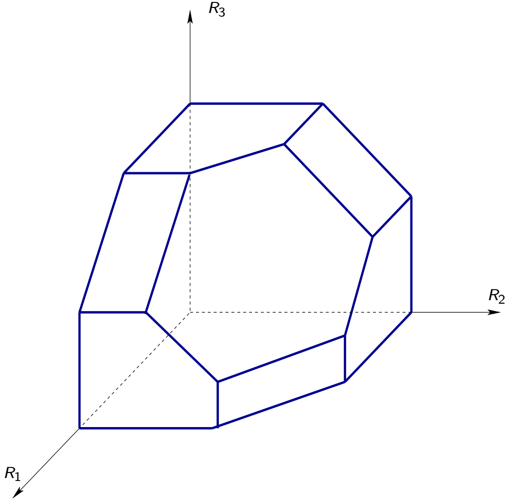
Capacity Region for $M = 3$



Capacity Region for $M = 3$



Capacity Region for $M = 3$



Capacity region

- The capacity region depends on whether the channel is *synchronous* or *asynchronous*
- A discrete-time channel is **synchronous** if the transmitters are able to index channel input sequences in such a way that all inputs with time index n enter the channel at the same time
- If there is an unknown shift between time indices, then the channel is said to be **asynchronous**
- The **capacity region** is described as a union of certain polytopes:

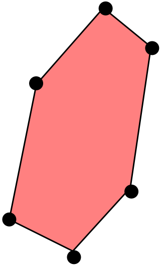
$$\mathcal{C}_{DMC} = \bigcup_{P_{X_1} P_{X_2} \cdots P_{X_M}} \mathcal{R}[W; P_{X_1} P_{X_2} \cdots P_{X_M}],$$

where the union is over all product input distributions

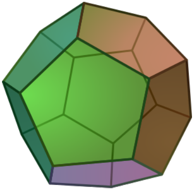
Polytopes

- **Polytope** is a geometric object with flat sides, which exists in any general number of dimensions
  - a polytope of n dimensions is an **n-polytope**

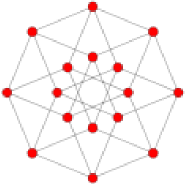
For example a two-dimensional polygon is a **2-polytope**



For example a three-dimensional polygon is a **3-polytope**



For example a four-dimensional polygon is a **4-polytope**



Non-degenerated region I

Definition 1

A region $\mathcal{R}$ is called non-degenerated if the following two conditions hold

1. $I(X_{\mathcal{S}}; Y) > 0$ for all non-empty sets $\mathcal{S} \subseteq [M]$,
2. $I(X_{\mathcal{S}}; Y|X_{\mathcal{A}}) < I(X_{\mathcal{S}}; Y|X_{\mathcal{B}})$ for all $\emptyset \subset \mathcal{S} \subset [M]$, $\mathcal{A} \subset \mathcal{B} \subset [M]$, and $\mathcal{S} \cap \mathcal{B} = \emptyset$.

- It is also true that for all $\mathcal{A} \subset [M]$, $\emptyset \subset \mathcal{S} \subset \mathcal{T} \subseteq [M]$, and $\mathcal{A} \cap \mathcal{T} = \emptyset$,

$$I(X_{\mathcal{S}}; Y|X_{\mathcal{A}}) < I(X_{\mathcal{T}}; Y|X_{\mathcal{A}})$$

Non-degenerated region II

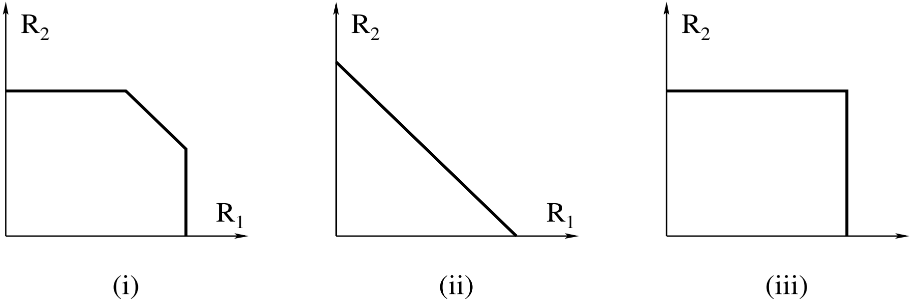
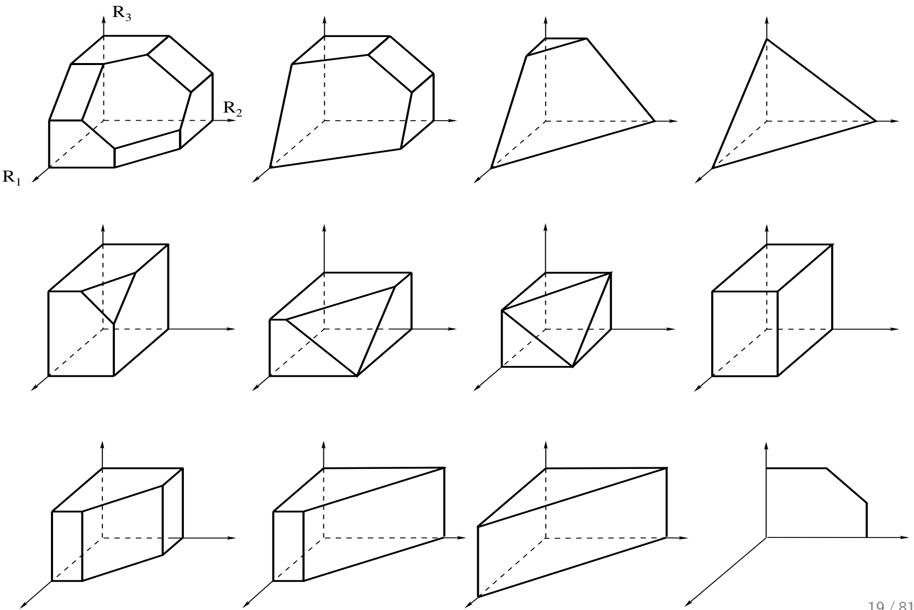


Figure : Shapes of $\mathcal{R}$ for a two-user channel: (i) is a non-degenerated case; (ii) and (iii) degenerated cases

Non-degenerated region III



Face of a region I

- A hyperplane of $\mathbb{R}_+^M$ of dimension $M-1$ is defined as:

$$\{R \in \mathbb{R}_+^M : R(\mathcal{S}) = c\}$$

  for some constant c

- The set:

$$\{R \in \mathbb{R}_+^M : R(\mathcal{S}) \leq c\}$$

  is one of the two half-spaces bounded by such a hyperplane

- $\mathcal{R}$ is a finite intersection of such half-spaces
- A **face** of $\mathcal{R}$ is defined as any set of the form

$$\mathcal{F} \;\; = \;\; \mathcal{R} \cap \{R \in \mathbb{R}_+^M : Ra = a_0\},$$

  where $Ra \leq a_0$ is a valid inequality for $\mathcal{R}$

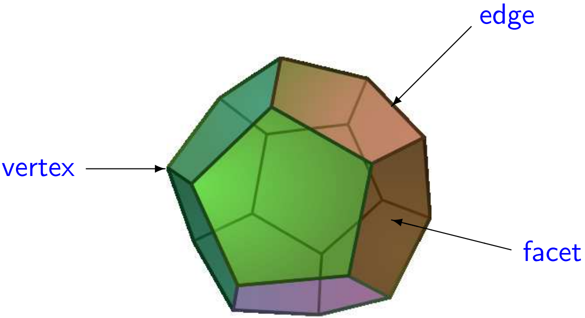
Face of a region II

- The dimension of a face is the dimension of its affine hull:

$$\dim(\mathcal{F}) := \dim(\text{aff}(\mathcal{F}))$$

- $\mathcal{R}$ itself and the empty set $\emptyset$ are called improper faces
- The number of faces of any dimension is maximal in non-degenerated cases
- The 0-faces are 0-dimensional **vertices**, the 1-faces are 1-dimensional **edges**, the 2-faces are 2-dimensional **faces** (polygonal faces), the 3-faces are 3-dimensional **cells** (polyhedral faces) and so on
- The faces of dimension $M - 2$ and $M - 1$ are called **ridges** and **facet** respectively

Face of a region III



Back and front facets

- A **Back facet** is defined as:

$$\mathcal{B}_i = \mathcal{R} \cap \{R \in \mathbb{R}_+^M : R_i = 0\}$$

  where $i \in [M]$
- A **front facet** is defined as:

$$\mathcal{F}_\mathcal{S} = \mathcal{R} \cap \{R \in \mathbb{R}_+^M : R(\mathcal{S}) = I(X_\mathcal{S}; Y | X_{\mathcal{S}^c})\}$$

  for every $\mathcal{S} \subseteq [M]$, $\mathcal{S} \neq \emptyset$

Labeling example

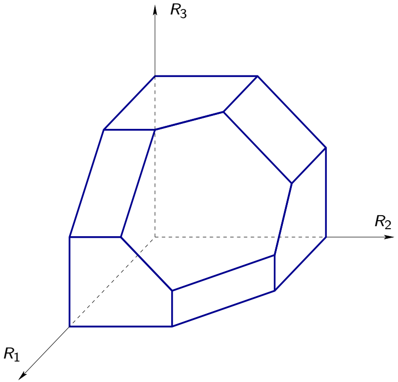


Figure : Region $\mathcal{R}$ with labels for a three-user MAC

Labeling example

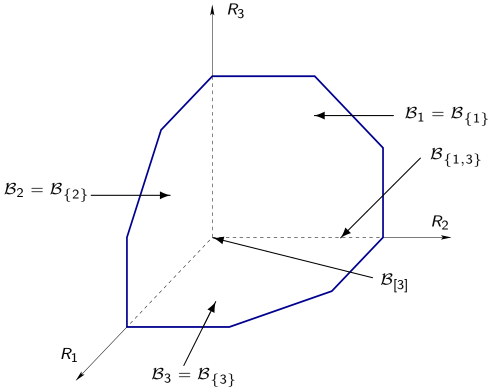


Figure : Back facets in a region $\mathcal{R}$ for a three-user MAC

Labeling example

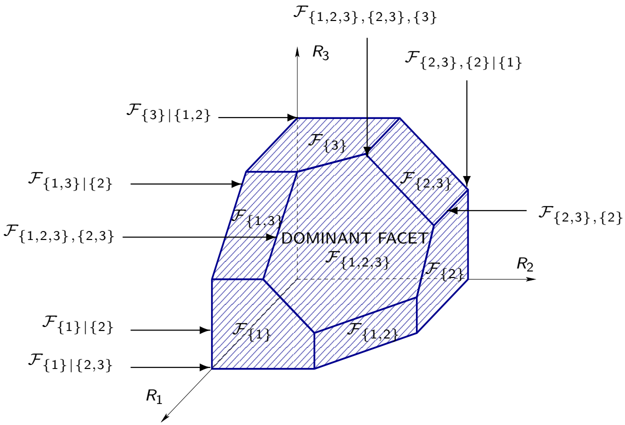


Figure : Front facets in a region $\mathcal{R}$ for a three-user MAC

Extending the notations

- Extending the notation:

$$\mathcal{B}_{\mathcal{A}} = \bigcap_{i \in \mathcal{A}} \mathcal{B}_i, \text{ with } \mathcal{B}_{\emptyset} = \mathcal{R} \text{ by convention,}$$

$$\mathcal{F}_{\mathcal{S}_1,\mathcal{S}_2,\ldots,\mathcal{S}_m} = \bigcap_{j=1}^{m} \mathcal{F}_{\mathcal{S}_j}, \text{ with } \mathcal{F}_{\emptyset} = \mathcal{R} \text{ by convention,}$$

$$\mathcal{F}_{\mathcal{S}_1,\mathcal{S}_2,\ldots,\mathcal{S}_m|\mathcal{A}} = \mathcal{F}_{\mathcal{S}_1,\mathcal{S}_2,\ldots,\mathcal{S}_m} \cap \mathcal{B}_{\mathcal{A}}$$

- $\mathcal{F}_{\mathcal{S}|\emptyset} = \mathcal{F}_{\mathcal{S}}$, $\mathcal{F}_{\emptyset|\mathcal{A}} = \mathcal{B}_{\mathcal{A}}$, $\mathcal{B}_i$ is a short-hand notation for $\mathcal{B}_{\{i\}}$, $\mathcal{F}_{\emptyset|\emptyset} = \mathcal{R}$, and the origin as a vertex is labeled by $\mathcal{B}_{[M]} = \mathcal{F}_{\emptyset|[M]}$

Front facets intersection I

Lemma 2

$\mathcal{F}_{\mathcal{S}_1} \cap \mathcal{F}_{\mathcal{S}_2}$ is not empty iff $\mathcal{S}_1 \subseteq \mathcal{S}_2$ or $\mathcal{S}_2 \subseteq \mathcal{S}_1$.

Proof:

(i) The "if" direction:

- True if $\mathcal{S}_1 = \mathcal{S}_2$
- Assume that $\mathcal{S}_1 \subset \mathcal{S}_2$
- Re-index users so that $\mathcal{S}_1 = [k]$ and $\mathcal{S}_2 = [\ell]$, where $\ell > k$
- $R = (R_1, \ldots, R_M)$ defined as follows:

$$R_i = \begin{cases} I(X_i; Y | X_{i+1}, \ldots, X_M), & i = 1, \ldots, M-1, \\ I(X_M; Y) & i = M \end{cases}$$

Front facets intersection II

- R is a vertex of the dominant face $\Rightarrow R \in \mathcal{R}$ and:

$$R([i]) = \sum_{j=1}^{i} R_j = \sum_{j=1}^{i} I(X_j; Y|X_{j+1}, \ldots, X_M) = I(X_{[i]}; Y|X_{[i]^c})$$

- For $i = k$, $R([k]) = I(X_{\mathcal{S}_1}; Y|X_{\mathcal{S}_1^c})$ and for $i = \ell$, $R([\ell]) = I(X_{\mathcal{S}_2}; Y|X_{\mathcal{S}_2^c}) \rightarrow R \in \mathcal{F}_{\mathcal{S}_1} \cap \mathcal{F}_{\mathcal{S}_2}$

Front facets intersection III

(ii) The "only if" direction:

- Let $R \in \mathcal{F}_{\mathcal{S}_1} \cap \mathcal{F}_{\mathcal{S}_2}$. Then:

$$\begin{aligned}
I(X_{\mathcal{S}_1 \cup \mathcal{S}_2}; Y | X_{(\mathcal{S}_1 \cup \mathcal{S}_2)^c}) &\overset{(a)}{\geq} R(\mathcal{S}_1 \cup \mathcal{S}_2) = R(\mathcal{S}_1) + R(\mathcal{S}_2) - R(\mathcal{S}_1 \cap \mathcal{S}_2) \\
&\overset{(b)}{=} I(X_{\mathcal{S}_1}; Y | X_{\mathcal{S}_1^c}) + I(X_{\mathcal{S}_2}; Y | X_{\mathcal{S}_2^c}) - R(\mathcal{S}_1 \cap \mathcal{S}_2) \\
&\overset{(c)}{\geq} I(X_{\mathcal{S}_1}; Y | X_{\mathcal{S}_1^c}) + I(X_{\mathcal{S}_2}; Y | X_{\mathcal{S}_2^c}) - I(X_{\mathcal{S}_1 \cap \mathcal{S}_2}; Y | X_{(\mathcal{S}_1 \cap \mathcal{S}_2)^c}) \\
&\overset{(d)}{=} I(X_{\mathcal{S}_1 \setminus \mathcal{S}_2}; Y | X_{\mathcal{S}_1^c}) + I(X_{\mathcal{S}_2}; Y | X_{\mathcal{S}_2^c}) \\
&\overset{(e)}{\geq} I(X_{\mathcal{S}_1 \setminus \mathcal{S}_2}; Y | X_{(\mathcal{S}_1 \cup \mathcal{S}_2)^c}) + I(X_{\mathcal{S}_2}; Y | X_{\mathcal{S}_2^c}) \\
&= I(X_{\mathcal{S}_1 \cup \mathcal{S}_2}; Y | X_{(\mathcal{S}_1 \cup \mathcal{S}_2)^c})
\end{aligned}$$

- (a) and (c) follow from $R \in \mathcal{R}$, (b) from the definition of $\mathcal{F}_{\mathcal{S}_i}, i = 1, 2$, (d) from the chain rule for mutual information, and (e) holds since the inputs are independent

Front facets intersection IV

- (a), (c), and (e) must be equalities. Equality in (e) means

$$I(X_{\mathcal{S}_1 \setminus \mathcal{S}_2}; Y | X_{\mathcal{S}_1^c}) = I(X_{\mathcal{S}_1 \setminus \mathcal{S}_2}; Y | X_{(\mathcal{S}_1 \cup \mathcal{S}_2)^c})$$

- $\mathcal{R}$ is non-degenerated $\Rightarrow$ either $\mathcal{S}_1 \setminus \mathcal{S}_2 = \emptyset$, i.e., $\mathcal{S}_1 \subseteq \mathcal{S}_2$ or $\mathcal{S}_1 = \mathcal{S}_1 \cup \mathcal{S}_2$, i.e., $\mathcal{S}_2 \subseteq \mathcal{S}_1$

Front and back facets intersection I

Lemma 3

Assume $R \in \mathcal{F}_{\mathcal{S}}$. Then for every $\mathcal{L} \subseteq \mathcal{S}$

$$I(X_{\mathcal{L}}; Y|X_{\mathcal{S}^c}) \leq R(\mathcal{L}) \leq I(X_{\mathcal{L}}; Y|X_{\mathcal{L}^c})$$

Proof:

- The second inequality is true for every $R \in \mathcal{R}$
- For the first inequality:

$$\begin{aligned} R(\mathcal{L}) &= R(\mathcal{S}) - R(\mathcal{S} \setminus \mathcal{L}) \\ &\overset{(a)}{=} I(X_{\mathcal{S}}; Y|X_{\mathcal{S}^c}) - R(\mathcal{S} \setminus \mathcal{L}) \\ &\overset{(b)}{\geq} I(X_{\mathcal{S}}; Y|X_{\mathcal{S}^c}) - I(X_{\mathcal{S}\setminus\mathcal{L}}; Y|X_{(\mathcal{S}\setminus\mathcal{L})^c}) \\ &\overset{(c)}{=} I(X_{\mathcal{L}}; Y|X_{\mathcal{S}^c}) \end{aligned}$$

Front and back facets intersection II

where (a) is true since $R \in \mathcal{F}_{\mathcal{S}}$, (b) since $R \in \mathcal{R}$ and (c) follows from the chain rule for mutual information

Lemma 4

$\mathcal{F}_{\mathcal{S}} \cap \mathcal{B}_{\mathcal{A}} \neq \emptyset$ iff $\mathcal{A} \cap \mathcal{S} = \emptyset$.

Proof:

- If $\mathcal{A} = \emptyset$ then the lemma is true
- Assume $\mathcal{A} \neq \emptyset$
  (i) Proving one direction:
  - Let $R \in \mathcal{F}_{\mathcal{S}} \cap \mathcal{B}_{\mathcal{A}}$
  - Then $0 = R(\mathcal{A}) = R(\mathcal{S} \cap \mathcal{A}) \geq I(X_{\mathcal{S} \cap \mathcal{A}}; Y | X_{\mathcal{S}^c}) \Rightarrow I(X_{\mathcal{S} \cap \mathcal{A}}; Y | X_{\mathcal{S}^c}) = 0$
  - R is non-degenerated $\Rightarrow \mathcal{A} \cap \mathcal{S} = \emptyset$

Front and back facets intersection III

(ii) Proving the other direction:

- Assume $\mathcal{A} \cap \mathcal{S} = \emptyset$
- Pick a rate $\tilde{R}$ such that $\tilde{R} \in \mathcal{F}_{\mathcal{S}}$
- Let R be obtained from $\tilde{R}$ by setting to 0 all coordinates with index in $\mathcal{A}$
- $R \in \mathcal{B}_{\mathcal{A}}$ but also $R \in \mathcal{F}_{\mathcal{S}}$ since $R(\mathcal{S}) = \tilde{R}(\mathcal{S})$

Unique label of all proper faces of $\mathcal{R}$ I

Proposition 5

The intersection $\mathcal{F}_{\mathcal{S}_1,\mathcal{S}_2,\ldots,\mathcal{S}_m|\mathcal{A}}$ is not empty, if and only if the following two conditions are satisfied
(i) The set sequence $\mathcal{S}_1,\mathcal{S}_2,\ldots,\mathcal{S}_m$ is telescopic, i.e., there is a permutation π on the index set $[m]$ such that
$\mathcal{S}_{\pi(1)} \supset \mathcal{S}_{\pi(2)} \supset \ldots \supset \mathcal{S}_{\pi(m)}$, and
(ii) $\mathcal{A} \cap \mathcal{S}_{\pi(1)} = \emptyset$.

Proof:
(i) Achievability:

- Assume that $\mathcal{S}_1 \supset \mathcal{S}_2 \supset \ldots \supset \mathcal{S}_m$ and $\mathcal{A} \cap \mathcal{S}_1 = \emptyset$
- "if" part of the proof of Lemma 2 leads to an $\tilde{R}$ in $\mathcal{F}_{\mathcal{S}_1,\mathcal{S}_2,\ldots,\mathcal{S}_m}$
- Let R be obtained from $\tilde{R}$ by setting to 0 all coordinates with index in $\mathcal{A}$

Unique label of all proper faces of $\mathcal{R}$ II

- Hence $R \in \mathcal{F}_{\mathcal{S}_i}$, $i = 1, \ldots, m$ and $R \in \mathcal{B}_{\mathcal{A}}$

(ii) Converse:

- if $\mathcal{S}_i$ is not contained in $\mathcal{S}_j$ or vice versa, then by Lemma 2, $\mathcal{F}_{\mathcal{S}_i} \cap \mathcal{F}_{\mathcal{S}_j} = \emptyset$
- if $\mathcal{S}_1 \cap \mathcal{A} \neq \emptyset$, then according to Lemma 4, $\mathcal{F}_{\mathcal{S}_1} \cap \mathcal{B}_{\mathcal{A}} = \emptyset$

Dominant facet I

- Points have *maximum sum-rate*
- Does *not intersect* with any back facet
- In the one-user case it is a vertex, in the two-user case it is an edge that has two vertices, in the three-user case a hexagon and so on
- Instead of writing the telescopic sequence can be written the sequence of *"decrements"*
- The sequence of decrements gives *the order* in which users are decoded and *count* vertices
- Since each permutation on the set $[M]$ is a vertex on the dominant face, it is clear that there are $M!$ *such vertices on the dominant face*

Dominant facet II

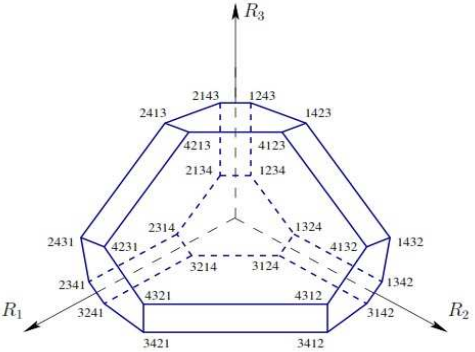


Figure : Dominant face of a four-user MAC. The fourth dimension, not shown here, has coordinate $R_4 = I(X_{\{1,2,3,4\}}; Y) - R_1 - R_2 - R_3$

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Main contribution

- Link between the label an group successive decoding
- Showing that the object formed by intersection of faces are Cartesian product fundamental region of channels related to the original channel
- Rates on the intersection of faces may be decoded successively in groups

Notations I

- Region $\mathcal{R}$ is completely specified by the channel $W_{Y|X_{[M]}}$ and by the input distribution $P_{X_{[M]}}$
  - With $W_{Y|X_{[M]}}$ is denoted the *original channel*
  - With $P_{X_{[M]}}$ is denoted the *input distribution*
- For any two sets $\mathcal{U}, \mathcal{V} \subset [M]$ such that $\mathcal{U} \cap \mathcal{V} = \emptyset$, there is a *channel* with inputs $X_{\mathcal{U}}$ and outputs $(Y, X_{\mathcal{V}}) \Rightarrow$

$$
\begin{aligned}
W_{YX_{\mathcal{V}}|X_{\mathcal{U}}}(y, x_{\mathcal{V}}|x_{\mathcal{U}}) &= P_{X_{\mathcal{V}}}(x_{\mathcal{V}}) W_{Y|X_{\mathcal{U}},X_{\mathcal{V}}}(y|x_{\mathcal{U}}, x_{\mathcal{V}})) \\
&= P_{X_{\mathcal{V}}}(x_{\mathcal{V}}) \sum_{x_{[M]\setminus(\mathcal{U}\cup\mathcal{V})}} W_{YX_{[M]\setminus(\mathcal{U}\cup\mathcal{V})}|X_{\mathcal{U}},X_{\mathcal{V}}}(y, x_{[M]\setminus(\mathcal{U}\cup\mathcal{V})}|x_{\mathcal{U}}, x_{\mathcal{V}}) \\
&= P_{X_{\mathcal{V}}}(x_{\mathcal{V}}) \sum_{x_{[M]\setminus(\mathcal{U}\cup\mathcal{V})}} P_{X_{[M]\setminus(\mathcal{U}\cup\mathcal{V})}}(x_{[M]\setminus(\mathcal{U}\cup\mathcal{V})}) W_{Y|X_{[M]}}(y|x_{[M]})
\end{aligned}
$$

Notations II

- A *rate tuple* for $W_{YX_{\mathcal{V}}|X_{\mathcal{U}}} \Rightarrow R_{\mathcal{U}} \triangleq (R_i)_{i \in \mathcal{U}}$
- The corresponding *region* is defined by

$$
\begin{aligned}
\mathcal{R}_{YX_{\mathcal{V}}|X_{\mathcal{U}}} &\triangleq \mathcal{R}[W_{YX_{\mathcal{V}}|X_{\mathcal{U}}}; P_{X_{\mathcal{U}}}] \\
&= \{R \in \mathbb{R}_+^{|\mathcal{U}|} : \quad R(\mathcal{L}) \leq I(X_{\mathcal{L}}; Y|X_{\mathcal{V} \cup (\mathcal{U} \setminus \mathcal{L})}), \quad \forall \mathcal{L} \subseteq \mathcal{U}\}
\end{aligned}
$$

- *Dimensionality* of a region $\mathcal{R}_{YX_{\mathcal{V}}|X_{\mathcal{U}}}$ is represent with $|\mathcal{U}|$
- *Dominant face* $(|\mathcal{U}| - 1)$ - dimensional subregion is obtained by adding the equality $R(\mathcal{U}) = I(X_{\mathcal{U}}; Y|X_{\mathcal{V}})$ i.e.,

$$
\begin{aligned}
\mathcal{D}_{YX_{\mathcal{V}}|X_{\mathcal{U}}} &\triangleq \mathcal{D}[W_{YX_{\mathcal{V}}|X_{\mathcal{U}}}; P_{X_{\mathcal{U}}}] \\
&= \{R \in \mathbb{R}_+^{|\mathcal{U}|} : R(\mathcal{L}) \leq I(X_{\mathcal{L}}; Y|X_{\mathcal{V} \cup (\mathcal{U} \setminus \mathcal{L})}), \forall \mathcal{L} \subset \mathcal{U}, R(\mathcal{U}) = I(X_{\mathcal{U}}; Y|X_{\mathcal{V}})\}
\end{aligned}
$$

Special cases used frequently

$$
\begin{aligned}
\mathcal{R}_{YX_{\mathcal{S}^c}|X_\mathcal{S}} &\triangleq \mathcal{R}[W_{YX_{\mathcal{S}^c}|X_\mathcal{S}}; P_{X_\mathcal{S}}] \\
&= \{R \in \mathbb{R}_+^{|\mathcal{S}|} : \quad R(\mathcal{L}) \leq I(X_\mathcal{L}; Y|X_{\mathcal{L}^c}), \quad \forall \mathcal{L} \subseteq \mathcal{S}\} \\
\mathcal{D}_{YX_{\mathcal{S}^c}|X_\mathcal{S}} &\triangleq \mathcal{D}[W_{YX_{\mathcal{S}^c}|X_\mathcal{S}}; P_{X_\mathcal{S}}] \\
&= \{R \in \mathbb{R}_+^{|\mathcal{S}|} : R(\mathcal{L}) \leq I(X_\mathcal{L}; Y|X_{\mathcal{L}^c}), \forall \mathcal{L} \subset \mathcal{S}, R(\mathcal{S}) = I(X_\mathcal{S}; Y|X_{\mathcal{S}^c})\} \\
\mathcal{R}_{Y|X_\mathcal{S}} &\triangleq \mathcal{R}[W_{Y|X_\mathcal{S}}; P_{X_\mathcal{S}}] \\
&= \{R \in \mathbb{R}_+^{|\mathcal{S}|} : \quad R(\mathcal{L}) \leq I(X_\mathcal{L}; Y|X_{\mathcal{S}\setminus\mathcal{L}}), \quad \forall \mathcal{L} \subseteq \mathcal{S}\} \\
\mathcal{D}_{Y|X_\mathcal{S}} &\triangleq \mathcal{D}[W_{Y|X_\mathcal{S}}; P_{X_\mathcal{S}}] \\
&= \{R \in \mathbb{R}_+^{|\mathcal{S}|} : R(\mathcal{L}) \leq I(X_\mathcal{L}; Y|X_{\mathcal{S}\setminus\mathcal{L}}), \forall \mathcal{L} \subset \mathcal{S}, R(\mathcal{S}) = I(X_\mathcal{S}; Y)\}
\end{aligned}
$$

Lemma 6

$R \in \mathcal{F}_{\mathcal{S}}$ if and only if $R_{\mathcal{S}^c} \in \mathcal{R}_{Y|X_{\mathcal{S}^c}}$ and $R_{\mathcal{S}} \in \mathcal{D}_{YX_{\mathcal{S}^c}|X_{\mathcal{S}}}$.

Proof:

(i) Achievability:

- $R \in \mathcal{F}_{\mathcal{S}}$
- From the definition of $\mathcal{F}_{\mathcal{S}}$, $\forall \mathcal{L} \subset \mathcal{S} \subseteq [M]$, $R(\mathcal{L}) \leq I(X_{\mathcal{L}}; Y|X_{\mathcal{L}^c})$ and $R(\mathcal{S}) = I(X_{\mathcal{S}}; Y|X_{\mathcal{S}^c}) \Rightarrow R_{\mathcal{S}} \in \mathcal{D}_{YX_{\mathcal{S}^c}|X_{\mathcal{S}}}$
- $\forall \mathcal{T} \subset \mathcal{S}^c$, $[M] = \mathcal{S} \cup \mathcal{T} \cup \mathcal{Q}$ as the union of disjoint sets

Cartesian product of region $\mathcal{R}$ and a dominant face $\mathcal{D}$ II

- Then,

$$\begin{aligned} R(\mathcal{T}) + R(\mathcal{S}) &\leq I(X_{\mathcal{T}\cup\mathcal{S}}; Y|X_{\mathcal{Q}}) \\ &= I(X_{\mathcal{T}}; Y|X_{\mathcal{Q}}) + I(X_{\mathcal{S}}; Y|X_{\mathcal{Q}\cup\mathcal{T}}) \\ &= I(X_{\mathcal{T}}; Y|X_{\mathcal{Q}}) + I(X_{\mathcal{S}}; Y|X_{\mathcal{S}^c}) \\ &= I(X_{\mathcal{T}}; Y|X_{\mathcal{S}^c\setminus\mathcal{T}}) + R(\mathcal{S}) \end{aligned}$$

- Knowing $R(\mathcal{T}) \leq I(X_{\mathcal{T}}; Y|X_{\mathcal{S}^c\setminus\mathcal{T}})$ and the corresponding region $\mathcal{R}_{YX_{\mathcal{V}}|X_{\mathcal{U}}} \Rightarrow R_{\mathcal{S}^c} \in \mathcal{R}_{Y|X_{\mathcal{S}^c}}$

Cartesian product of region $\mathcal{R}$ and a dominant face $\mathcal{D}$ III

(ii) Converse:

- $R_{\mathcal{S}} \in \mathcal{D}_{YX_{\mathcal{S}^c}|X_{\mathcal{S}}}$ and $R_{\mathcal{S}^c} \in \mathcal{R}_{Y|X_{\mathcal{S}^c}}$
- Prove that $R(\mathcal{S}) = I(X_{\mathcal{S}}; Y|X_{\mathcal{S}^c})$ and that for all $\mathcal{L} \subseteq [M]$, $R(\mathcal{L}) \leq I(X_{\mathcal{L}}; Y|X_{\mathcal{L}^c})$
- The former is true since $R_{\mathcal{S}} \in \mathcal{D}_{YX_{\mathcal{S}^c}|X_{\mathcal{S}}}$
- To prove the latter, let $\mathcal{T} = \mathcal{L} \cap \mathcal{S}$ and $\mathcal{Q} = \mathcal{L} \cap \mathcal{S}^c$
- $R_{\mathcal{S}} \in \mathcal{D}_{YX_{\mathcal{S}^c}|X_{\mathcal{S}}}$, $R(\mathcal{T}) \leq I(X_{\mathcal{T}}; Y|X_{\mathcal{S}^c \cup (\mathcal{S} \setminus \mathcal{T})}) = I(X_{\mathcal{T}}; Y|X_{\mathcal{T}^c})$ for all $\mathcal{T} \subseteq \mathcal{S}$
- $R_{\mathcal{S}^c} \in \mathcal{R}_{Y|X_{\mathcal{S}^c}}$, $R(\mathcal{Q}) \leq I(X_{\mathcal{Q}}; Y|X_{\mathcal{S}^c \setminus \mathcal{Q}})$ for all $\mathcal{Q} \subseteq \mathcal{S}^c$

- Follows,

$$\begin{aligned} R(\mathcal{L}) &= R(\mathcal{T} \cup \mathcal{Q}) = R(\mathcal{T}) + R(\mathcal{Q}) \\ &\leq I(X_{\mathcal{T}}; Y|X_{\mathcal{T}^c}) + I(X_{\mathcal{Q}}; Y|X_{\mathcal{S}^c \setminus \mathcal{Q}}) \\ &\leq I(X_{\mathcal{T}}; Y|X_{\mathcal{T}^c}) + I(X_{\mathcal{Q}}; Y|X_{(\mathcal{S}^c \setminus \mathcal{Q}) \cup (\mathcal{S} \setminus \mathcal{T})}) \\ &= I(X_{\mathcal{T} \cup \mathcal{Q}}; Y|X_{(\mathcal{T} \cup \mathcal{Q})^c}) \\ &= I(X_{\mathcal{L}}; Y|X_{\mathcal{L}^c}) \end{aligned}$$

for all $\mathcal{L} \subseteq [M]$ and this completes the proof

Cartesian product of region $\mathcal{R}$ and a dominant face $\mathcal{D}$ V

From **Lemma 6** the **dimension for a facet** $\mathcal{F}_{\mathcal{S}} \Rightarrow$

$$
\begin{aligned}
\dim(\mathcal{F}_{\mathcal{S}}) &= \dim(\mathcal{R}_{Y|X_{\mathcal{S}^c}}) + \dim(\mathcal{D}_{YX_{\mathcal{S}^c}|X_{\mathcal{S}}}) \\
&= |S^c| + |S| - 1 = M - 1
\end{aligned}
$$

The **contribution** is that *a rate point in* $\mathcal{F}_{\mathcal{S}}$ *may be approached via group successive decoding where groups are decoded in the order* $(\mathcal{S}^c, \mathcal{S})$

Generalization of cartesian product I

Theorem 7

Let $\mathcal{S}_1 \supset \mathcal{S}_2 \ldots \supset \mathcal{S}_m$ form a telescopic sequence. $R \in \mathcal{F}_{\mathcal{S}_1,\mathcal{S}_2,\ldots,\mathcal{S}_m}$ iff $R_{\mathcal{S}_1^c} \in \mathcal{R}_{Y|X_{\mathcal{S}_1^c}}$ and $R_{\mathcal{S}_i \backslash \mathcal{S}_{i+1}} \in \mathcal{D}_{YX_{\mathcal{S}_i^c}|X_{\mathcal{S}_i \backslash \mathcal{S}_{i+1}}}$ for $i = 1, \ldots, m$, where by way of convention we have defined $\mathcal{S}_{m+1} = \emptyset$.

- **Concludes** when $\mathcal{F}_{\mathcal{S}_1,\mathcal{S}_2,\ldots,\mathcal{S}_m}$ is not empty it is the Cartesian product of a region and m dominant faces

Proof:

(i) Achievability:

- From $R \in \mathcal{F}_{\mathcal{S}_1,\ldots,\mathcal{S}_m}$, $\mathcal{F}_{\mathcal{S}_1,\mathcal{S}_2,\ldots,\mathcal{S}_m} = \bigcap_{i=1}^m \mathcal{F}_{\mathcal{S}_i}$ and Lemma 6 $\Rightarrow$

$$R_{\mathcal{S}_i} \in \mathcal{D}_{YX_{\mathcal{S}_i^c}|X_{\mathcal{S}_i}} \quad \text{and} \quad R_{\mathcal{S}_i^c} \in \mathcal{R}_{Y|X_{\mathcal{S}_i^c}}, \quad i = 1, \ldots, m$$

Generalization of cartesian product II

- Which proves $R_{\mathcal{S}_1^c} \in \mathcal{R}_{Y|X_{\mathcal{S}_1^c}}$
- Next show that

$$R_{\mathcal{S}_i \setminus \mathcal{S}_{i+1}} \in \mathcal{D}_{YX_{\mathcal{S}_i^c}|X_{\mathcal{S}_i \setminus \mathcal{S}_{i+1}}}, \quad \forall i = 1, \ldots, m-1$$

- If $R(\mathcal{K}) \leq I(X_{\mathcal{K}}; Y|X_{\mathcal{S}_i^c \cup (\mathcal{S}_i \setminus (\mathcal{S}_{i+1} \cup \mathcal{K}))}) = I(X_{\mathcal{K}}; Y|X_{(\mathcal{S}_{i+1} \cup \mathcal{K})^c})$ holds for all $\mathcal{K} \subseteq \mathcal{S}_i \setminus \mathcal{S}_{i+1}$, $i = 1, \ldots, m$, with equality if $\mathcal{K} = \mathcal{S}_i \setminus \mathcal{S}_{i+1}$
- From Lemma 6, for any $\mathcal{K} \subseteq \mathcal{S}_i$, $R(\mathcal{K}) \leq I(X_{\mathcal{K}}; Y|X_{\mathcal{K}^c})$ with equality if $\mathcal{K} = \mathcal{S}_i$ and for any $\mathcal{L} \subseteq \mathcal{S}_{i+1}$, $R(\mathcal{L}) \leq I(X_{\mathcal{L}}; Y|X_{\mathcal{L}^c})$ with equality if $\mathcal{L} = \mathcal{S}_{i+1}$

Generalization of cartesian product III

- Then for $\mathcal{K} \subseteq \mathcal{S}_i \setminus \mathcal{S}_{i+1} \Rightarrow$

$$
\begin{aligned}
R(\mathcal{K}) &= R(\mathcal{S}_i) - R(\mathcal{S}_{i+1}) - R(\mathcal{S}_i \setminus (\mathcal{S}_{i+1} \cup \mathcal{K})) \\
&= I(X_{\mathcal{S}_i}; Y|X_{\mathcal{S}_i^c}) - I(X_{\mathcal{S}_{i+1}}; Y|X_{\mathcal{S}_{i+1}^c}) - R(\mathcal{S}_i \setminus (\mathcal{S}_{i+1} \cup \mathcal{K})) \\
&\overset{(a)}{\leq} I(X_{\mathcal{S}_i}; Y|X_{\mathcal{S}_i^c}) - I(X_{\mathcal{S}_{i+1}}; Y|X_{\mathcal{S}_{i+1}^c}) - I(X_{\mathcal{S}_i \setminus (\mathcal{S}_{i+1} \cup \mathcal{K})}; Y|X_{\mathcal{S}_i^c}) \\
&= I(X_{\mathcal{K}}; Y|X_{(\mathcal{S}_{i+1} \cup \mathcal{K})^c})
\end{aligned}
$$

where (a) follows from the fact that $\forall \mathcal{Q} \subset \mathcal{S}_i$, $R(\mathcal{Q}) \geq I(X_{\mathcal{Q}}; Y|X_{\mathcal{S}_i^c})$

  - The equality in (a) holds if $\mathcal{K} = \mathcal{S}_i \setminus \mathcal{S}_{i+1}$, that proves the direct part

Generalization of cartesian product IV

(ii) Converse:

- For $R_{\mathcal{S}_1^c} \in \mathcal{R}_{Y|X_{\mathcal{S}_1^c}}$, and $R_{\mathcal{S}_i \setminus \mathcal{S}_{i+1}} \in \mathcal{D}_{YX_{\mathcal{S}_i^c}|X_{\mathcal{S}_i \setminus \mathcal{S}_{i+1}}}$, for $i = 1, \ldots, m$ prove that $R(\mathcal{L}) \leq I(X_{\mathcal{L}}; Y|X_{\mathcal{L}^c})$ holds for all $\mathcal{L} \subseteq [M]$ with equality if $\mathcal{L} = \mathcal{S}_i$, $i = 1, \ldots, m$
- $R(\mathcal{S}_i) = I(X_{\mathcal{S}_i}; Y|X_{\mathcal{S}_i^c})$ is true since $R_{\mathcal{S}_i \setminus \mathcal{S}_{i+1}} \in \mathcal{D}_{YX_{\mathcal{S}_i^c}|X_{\mathcal{S}_i \setminus \mathcal{S}_{i+1}}}$ and

$$\begin{aligned} R(\mathcal{S}_i) &= \sum_{j=i}^{m} R(\mathcal{S}_j \setminus \mathcal{S}_{j+1}) \\ &= \sum_{j=1}^{m} I(X_{\mathcal{S}_j \setminus \mathcal{S}_{j+1}}; Y|X_{\mathcal{S}_j^c}) = I(X_{\mathcal{S}_i}; Y|X_{\mathcal{S}_i^c}), \quad i = 1, \ldots, m \end{aligned}$$

Generalization of cartesian product V

- Let $\mathcal{L} \subseteq [M]$ and $\mathcal{L}_i = \mathcal{L} \cap \mathcal{S}_i \setminus \mathcal{S}_{i+1}$, $i = 0, 1, \ldots, m$ with $\mathcal{S}_0 = [M]$ by convention
- Then $\mathcal{L} = \bigcup_{i=0}^{m} \mathcal{L}_i$ is a disjoint partition
- From $R_{\mathcal{S}_1^c} \in \mathcal{R}_{Y|X_{\mathcal{S}_1^c}}$ and $\mathcal{L}_0 \subseteq \mathcal{S}_1^c$ => $R(\mathcal{L}_0) \leq I(X_{\mathcal{L}_0}; Y|X_{\mathcal{S}_1^c \setminus \mathcal{L}_0})$
- Since $R_{\mathcal{S}_i \setminus \mathcal{S}_{i+1}} \in \mathcal{D}_{YX_{\mathcal{S}_i^c}|X_{\mathcal{S}_i \setminus \mathcal{S}_{i+1}}}$, and $R(\mathcal{L}_i) \leq I(X_{\mathcal{L}_i}; Y|X_{\mathcal{S}_i^c \cup (\mathcal{S}_i \setminus (\mathcal{S}_{i+1} \cup \mathcal{L}_i))}) = I(X_{\mathcal{L}_i}; Y|X_{\mathcal{S}_{i+1}^c \setminus \mathcal{L}_i})$ for all $\mathcal{L}_i \subseteq \mathcal{S}_i \setminus \mathcal{S}_{i+1} \Rightarrow$

Generalization of cartesian product VI

$$
\begin{aligned}
I(X_{\mathcal{L}}; Y|X_{\mathcal{L}^c}) &= \sum_{i=0}^{m} I(X_{\mathcal{L}_i}; Y|X_{\bigcup_{j=0}^{i-1} \mathcal{L}_j \cup \mathcal{L}^c}) \\
&\overset{(a)}{\geq} \sum_{i=0}^{m} I(X_{\mathcal{L}_i}; Y|X_{\mathcal{S}_{i+1}^c \setminus \mathcal{L}_i}) \\
&\geq \sum_{i=0}^{m} R(\mathcal{L}_i) \\
&= R(\mathcal{L})
\end{aligned}
$$

where (a) $\bigcup_{j=0}^{i-1} \mathcal{L}_j \cup \mathcal{L}^c = [M] \setminus \bigcup_{j=i}^{m} \mathcal{L}_j \supseteq \mathcal{S}_{i+1}^c \setminus \bigcup_{j=i}^{m} \mathcal{L}_j = \mathcal{S}_{i+1}^c \setminus \mathcal{L}_i$, and (b) for $j \geq i+1$, $\mathcal{L}_j \subseteq \mathcal{S}_{i+1}$ implies that $\mathcal{L}_j$ does not intersect with $\mathcal{S}_{i+1}^c$

Generalization of cartesian product VII

From the **Theorem 7** the **dimension** of $\mathcal{F}_{\mathcal{S}_1,\mathcal{S}_2,\ldots,\mathcal{S}_m} \Rightarrow$

$$\begin{aligned}
\dim(\mathcal{F}_{\mathcal{S}_1,\mathcal{S}_2,\ldots,\mathcal{S}_m}) &= \dim(\mathcal{R}_{Y|X_{\mathcal{S}_1^c}}) + \sum_{i=1}^{m} \dim(\mathcal{D}_{YX_{\mathcal{S}_i^c}|X_{\mathcal{S}_i \setminus \mathcal{S}_{i+1}}}) \\
&= M - |\mathcal{S}_1| + \sum_{i=1}^{m} (|\mathcal{S}_i| - |\mathcal{S}_{i+1}| - 1) \\
&= M - m
\end{aligned}$$

This **implies** that all points in $\mathcal{F}_{\mathcal{S}_1,\mathcal{S}_2,\ldots,\mathcal{S}_m}$ may be approached via *group successive decoding with groups of users decoded according to the following order:* $(\mathcal{S}_1^c, \mathcal{S}_1 \setminus \mathcal{S}_2, \mathcal{S}_2 \setminus \mathcal{S}_3, \ldots, \mathcal{S}_{m-1} \setminus \mathcal{S}_m, \mathcal{S}_m)$

Generalization of cartesian product VIII

Corollary 8

Let $\mathcal{S}_1 \supset \mathcal{S}_2 \supset \ldots \supset \mathcal{S}_m$ be a telescopic sequence. $R \in \mathcal{F}_{\mathcal{S}_1,\ldots,\mathcal{S}_m|\mathcal{A}}$ iff $R_{\mathcal{S}_1^c} \in \mathcal{R}_{Y|X_{\mathcal{S}_1^c}}$, $R_{\mathcal{S}_i \setminus \mathcal{S}_{i+1}} \in \mathcal{D}_{YX_{\mathcal{S}_i^c}|X_{\mathcal{S}_i \setminus \mathcal{S}_{i+1}}}$ for $i = 1, \ldots, m$, and $R_{\mathcal{A}} = \underline{0}$.

Concludes that

$$\begin{aligned} \dim(\mathcal{F}_{\mathcal{S}_1,\mathcal{S}_2,\ldots,\mathcal{S}_m|\mathcal{A}}) &= \dim(\mathcal{F}_{\mathcal{S}_1,\mathcal{S}_2,\ldots,\mathcal{S}_m}) - |\mathcal{A}| \\ &= M - |\mathcal{A}| - m \end{aligned}$$

And, $R \in \mathcal{F}_{\mathcal{S}_1,\ldots,\mathcal{S}_m|\mathcal{A}}$ may be approached by*decoding groups of users in the order* $([M] \setminus (\mathcal{A} \cup \mathcal{S}_1), \mathcal{S}_1 \setminus \mathcal{S}_2, \mathcal{S}_2 \setminus \mathcal{S}_3, \ldots, \mathcal{S}_{m-1} \setminus \mathcal{S}_m, \mathcal{S}_m)$

Table of Contents

Main contribution

- Derive the number of D - dimensional faces in $\mathcal{R}$ for any $D = 0, 1, \ldots, M$
  - Number of faces in $\mathcal{D}$
  - Number of front faces
  - Number of back faces
- Derive expressions for the total number of vertices and edges

Number of D - dimensional faces on the dominat face I

Proposition 9

The number of D - dimensional faces on the dominant face of $\mathcal{R}$ is

$$N_d(M, D) \ = \ \sum_{j=1}^{M-D} \binom{M-D}{j} (-1)^{M-D-j} j^M.$$

Proof:

- $\mathcal{F}_{[M],\mathcal{S}_2,\ldots,\mathcal{S}_{M-D}}$ is any D - dimensional face on the dominant face
- The difference sets $[M] \setminus \mathcal{S}_2, \mathcal{S}_2 \setminus \mathcal{S}_3, \ldots, \mathcal{S}_i \setminus \mathcal{S}_{i+1}, \ldots, \mathcal{S}_{M-D} \setminus \emptyset$ form an $(M-D)$ partition of $[M]$
- There is a one-to-one correspondence between a D - dimensional face and such a partition

Number of D - dimensional faces on the dominat face II

- The number of such ordered partitions is

$$N_d(M, D) = \sum_{\substack{m_1, m_2, \ldots, m_{M-D} \\ m_i \geq 1, \forall i \\ \sum_i m_i = M}} \binom{M}{m_1, m_2, \ldots, m_{M-D}} = \sum_{\substack{m_1, m_2, \ldots, m_{M-D} \\ m_i \geq 1, \forall i \\ \sum_i m_i = M}} \frac{M!}{\prod_i m_i!}$$

- By expanding the following polynomial

$$\left(\frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^M}{M!}\right)^{M-D} = \sum_{k=M-D}^{M(M-D)} x^k \sum_{\substack{m_1, \ldots, m_{M-D} \\ m_i \geq 1, \forall i \\ \sum_i m_i = k}} \frac{1}{m_1! m_2! \ldots m_{M-D}!},$$

and that the coefficient in front of x^M multiplied by $M!$ gives the number of ordered partitions

Number of D - dimensional faces on the dominat face III

- Therefore,

$$
\begin{aligned}
N_d(M,D) &= M!\left(\left(\sum_{i=1}^{M}\frac{x^i}{i!}\right)^{M-D}, x^M\right) \\
&\overset{(a)}{=} M!\left(\left(\sum_{i=1}^{\infty}\frac{x^i}{i!}\right)^{M-D}, x^M\right) \\
&\overset{(b)}{=} M!\left((e^x-1)^{M-D}, x^M\right) \\
&\overset{(c)}{=} \left.\frac{d^M}{dx^M}(e^x-1)^{M-D}\right|_{x=0},
\end{aligned}
$$

where $\text{coeff}(f(x), x^i)$ is the coefficient of x^i in the Taylor series expansion around zero of the function $f(x)$, (a) is true since taking all the terms up to M or up to infinity will not change the coefficient in front of x^M, (b) follows from the Taylor expansion of e^x, and (c) follows from the definition of the Taylor expansion

Number of D - dimensional faces on the dominat face IV

- Expand $(e^x - 1)^{M-D}$ with the Binomial formula $\Rightarrow$

$$(e^x - 1)^{M-D} = \sum_{j=0}^{M-D} \binom{M-D}{j} e^{jx} (-1)^{M-D-j}$$

- Taking the M - th derivative,

$$\frac{d^M}{dx^M}(e^x - 1)^{M-D} = \sum_{j=1}^{M-D} \binom{M-D}{j} e^{jx} (-1)^{M-D-j} j^M,$$

and setting $x = 0$ is obtained the Proposition 9

Number of faces on dominant face I

Proposition 10

The total number of front faces of dimension D, denoted by $N_f(M, D)$, equals

$$N_f(M, D) = N_d(M, D) + N_d(M, D - 1).$$

Proof:

- $\mathcal{F}_{\mathcal{S}_1,\mathcal{S}_2,\ldots,\mathcal{S}_{M-D}|\emptyset}$ is any D - dimensional front dominat face for some $\mathcal{S}_1 \subseteq [M]$. If $\mathcal{S}_1 = [M]$ and there are $N_d(M, D)$ such faces
- If $\mathcal{S}_1 \subset [M]$ the front face is not on the dominant face
- Since there is a one-to-one relationship between the subscripts of $\mathcal{F}_{\mathcal{S}_1,\mathcal{S}_2,\ldots,\mathcal{S}_{M-D}|\emptyset}$ and those of $\mathcal{F}_{[M],\mathcal{S}_1,\mathcal{S}_2,\ldots,\mathcal{S}_{M-D}|\emptyset}$ when $\mathcal{S}_1 \subset [M]$, it follows that the total number of front faces not on the dominant face is exactly $N_d(M, D - 1)$

Number of faces on dominant face II

- To obtain the total number of front D - faces we have to add $N_d(M, D-1)$ and the number $N_d(M, D)$ of D - faces on the dominant face

Number of faces on dominant face I

Propostion 11

The total number of D - dimensional back faces in $\mathcal{R}$ is given by

$$N_b(M, D) = \sum_{i=D}^{M-1} \binom{M}{i} N_f(i, D).$$

Proof:

- All back faces are front faces for some other channel with fewer users
- The dimension of this face is $M - m - |\mathcal{A}|$ and $\mathcal{A} \cap \mathcal{S}_1 = \emptyset$

Number of faces on dominant face II

- If we remove all users with index in $\mathcal{A}$, we obtain the front face $\mathcal{F}_{\mathcal{S}_1,\mathcal{S}_2,\ldots,\mathcal{S}_m|\emptyset}$ of an $(M-|\mathcal{A}|)$ - user MAC The dimensionality of this face is also $M-|\mathcal{A}|-m$. Running over all pertinent subsets $\mathcal{A}\subset[M]$ yields

$$N_b(M,D)=\sum_{\substack{\mathcal{A}\subset[M]\\ 0<|\mathcal{A}|\le M-D}} N_f(M-|\mathcal{A}|,D)$$

- Since there are $\binom{M}{|\mathcal{A}|}$ subsets of cardinality $|\mathcal{A}|\Rightarrow$

$$\begin{aligned} N_b(M,D) &= \sum_{|\mathcal{A}|=1}^{M-D}\binom{M}{|\mathcal{A}|}N_f(M-|\mathcal{A}|,D)=\sum_{i=1}^{M-D}\binom{M}{i}N_f(M-i,D)\\ &= \sum_{i=1}^{M-D}\binom{M}{M-i}N_f(M-i,D)=\sum_{i=D}^{M-1}\binom{M}{i}N_f(i,D) \end{aligned}$$

Number of D - dimensional faces in $\mathcal{R}$ I

Theorem 12

The total number of D - dimensional faces in $\mathcal{R}$, $0 \leq D \leq M$, is

$$N(M,D) = \sum_{i=D}^{M} \binom{M}{i} \left[(i+1-D)^i - \sum_{j=1}^{i-D} \binom{i-D}{j-1} (-1)^{i-D-j} j^i \right].$$

Proof:

- $N(M,D) = N_f(M,D) + N_b(M,D) = \sum_{i=D}^{M} \binom{M}{i} N_f(i,D)$
- Using the total number of front faces $\Rightarrow$

$$N(M,D) = \sum_{i=D}^{M} \binom{M}{i} [N_d(i,D) + N_d(i,D-1)],$$

where $N_d(D,D) = 0$, $N_d(D,D-1) = 1$ and, by convention, $N_d(i,-1) = 0$

- From the number od D - dimensional faces $\Rightarrow$

$$\begin{aligned}
& N_d(i,D) + N_d(i,D-1) \\
= \quad & (i-D+1)^i + \sum_{j=0}^{i-D} j^i (-1)^{i-D-j+1} \left[\binom{i-D+1}{j} - \binom{i-D}{j} \right] \\
= \quad & (i-D+1)^i - \sum_{j=0}^{i-D} \binom{i-D}{j} \frac{j^{i+1}(-1)^{i-D-j}}{i-D+1-j} \\
= \quad & (i-D+1)^i - \sum_{j=1}^{i-D} \binom{i-D}{j-1} (-1)^{i-D-j} j^i
\end{aligned}$$

Inserting this into previous $N(M,D)$ completes the proof

Number of vertices I

Lemma 13

The total number of vertices in $\mathcal{R}$ is $\lfloor eM! \rfloor$.

Proof:

- From $N(M, D)$ for $D = 0 \Rightarrow$

Number of vertices II

$$
\begin{aligned}
N(M,0) &= \sum_{i=0}^{M} \binom{M}{i} N_d(i,0) \\
&\overset{(a)}{=} \sum_{i=0}^{M} \binom{M}{i} i! = \sum_{i=0}^{M} \frac{M!}{(M-i)!} = \sum_{i=0}^{M} \frac{M!}{i!} \\
&= M! \sum_{i=0}^{\infty} \frac{1}{i!} - M! \sum_{i=M+1}^{\infty} \frac{1}{i!} \\
&\overset{(b)}{=} eM! - M! \sum_{i=M+1}^{\infty} \frac{1}{i!}
\end{aligned}
$$

where in (a) the number of vertices $N_d(i,0)$ on the dominant face of an i - user region is $i!$ and (b) follows from the Taylor series expansion of e

Number of vertices III

- Since $eM! - M!\sum_{i=M+1}^{\infty}\frac{1}{i!}$ is an integer, and

$$\begin{aligned}\sum_{i=M+1}^{\infty}\frac{M!}{i!} &= \sum_{i=1}^{\infty}\frac{M!}{(M+i)!} = \sum_{i=1}^{\infty}\frac{1}{\prod_{j=1}^{i}(M+j)}\\ &< \sum_{i=1}^{\infty}\prod_{j=1}^{i}\frac{1}{M+1} = \sum_{i=1}^{\infty}\left(\frac{1}{M+1}\right)^{i}\\ &= \frac{1/(M+1)}{1-1/(M+1)} = \frac{1}{M} \leq 1,\end{aligned}$$

it follows that

$$N(M,0) = \sum_{i=0}^{M}\frac{M!}{i!} = \left\lfloor N(M,0) + \sum_{i=M+1}^{\infty}\frac{M!}{i!}\right\rfloor = \lfloor eM! \rfloor$$

M	$\lfloor eM! \rfloor$
1	2
2	5
3	16
4	65
5	326
6	1957
7	13700
8	109601
9	986410
10	9864101

Stirling approximation

$$\ln(M!) = M \ln M - M + o(\ln M)$$

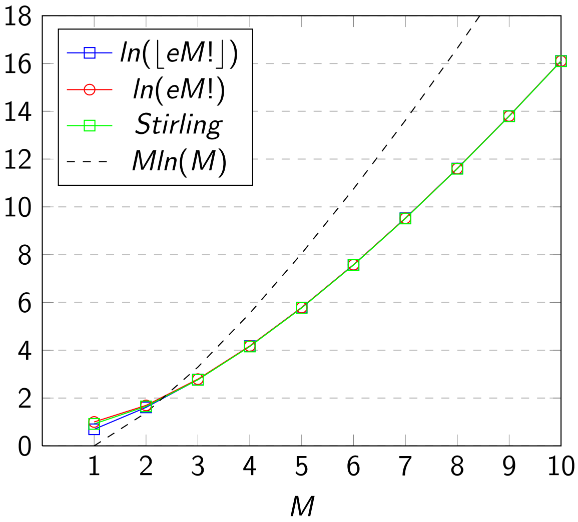
$$M! \sim \sqrt{2\pi M} \left(\frac{M}{e}\right)^M$$

$$\ln(M!) \sim M \ln M - M + \frac{1}{2} \ln M + \frac{1}{2} \ln(2\pi)$$

$$eM! \sim \sqrt{2\pi M} M^M e^{-M+1}$$

$$\begin{aligned} \ln(eM!) &\sim M \ln M - M + \frac{1}{2} \ln M + \frac{1}{2} \ln(2\pi) + 1 \\ &= \left(M + \frac{1}{2}\right) \ln M - M + \frac{1}{2} \ln(2\pi) + 1 \end{aligned}$$

$\lfloor eM! \rfloor$



Number of edges I

Lemma 14

The total number of edges in $\mathcal{R}$ is $\dfrac{M}{2}\lfloor eM! \rfloor$.

Homework:

- *Proof* that the number of edges is $\dfrac{M}{2}\lfloor eM! \rfloor$?
- *Proof* that the number of 2 dimensional faces is $\dfrac{(3M^2 - 5M + 3)\lfloor eM! \rfloor}{24} + \dfrac{M-3}{24}$?

Number of edges II

Proof:

- From $N(M, D)$ for $D = 1 \Rightarrow$

$$N(M,1) = \sum_{i=1}^{M} \binom{M}{i} (N_d(i,1) + N_d(i,0))$$

- Since $N_d(i,0) = i!$, from $N_d(M,D)$,

$$N_d(i,1) = \sum_{\substack{m_1,m_2,\ldots,m_{i-1} \\ m_j \geq 1, \forall j \\ \sum_j m_j = i}} \binom{i}{m_1,\ldots,m_{i-1}} = (i-1)\binom{i}{2,1,\ldots,1} = \frac{i!(i-1)}{2}$$

Number of edges III

- Therefore,

$$
\begin{aligned}
N(M,1) &= \sum_{i=1}^{M} \binom{M}{i} \left(i! + \frac{i-1}{2} i! \right) = \frac{1}{2} \sum_{i=1}^{M} \binom{M}{i} i!\,(i+1) \\
&= \frac{1}{2} \sum_{i=1}^{M} \frac{M!}{(M-i)!} (i+1) = \frac{1}{2} \sum_{j=0}^{M-1} \frac{M!}{j!} (M-j+1) \\
&= \frac{M+1}{2} \sum_{j=0}^{M-1} \frac{M!}{j!} - \frac{1}{2} \sum_{k=0}^{M-2} \frac{M!}{k!} \\
&\overset{(a)}{=} \frac{1}{2} [(M+1)(\lfloor eM! \rfloor - 1) - (\lfloor eM! \rfloor - M - 1)] = \frac{M}{2} \lfloor eM! \rfloor,
\end{aligned}
$$

where in (a) we use (1) to obtain $\sum_{j=0}^{M-1} M!/j! = \lfloor eM! \rfloor - 1$ and $\sum_{j=0}^{M-2} M!/j! = \lfloor eM! \rfloor - M - 1$

Number of faces of various dimensions for $\mathcal{R}$ and $\mathcal{D}$

Dimension D	Faces of $\mathcal{R}$ of dimension D	Faces of $\mathcal{D}$ of dimension D
$D = 0$ (vertices)	$\lfloor eM! \rfloor$	$M!$
$D = 1$ (edges)	$\frac{M}{2} \lfloor eM! \rfloor$	$M!(M-1)/2$
$D = 2$	$\frac{(3M^2-5M+3)\lfloor eM! \rfloor}{24} + \frac{M-3}{24}$	$\frac{M!(M-2)(3M-5)}{24}$
$\vdots$	$\vdots$	$\vdots$
D	$\sum_{i=D}^{M} \binom{M}{i} (i-D)! \left\{ {i+1 \atop i+1-D} \right\}$	$(M-D)! \left\{ {M \atop M-D} \right\}$
$\vdots$	$\vdots$	$\vdots$
$D = M-2$	$3^M + 2^{M-1}(M-4) + M(M-3)/2 + 1$	$2^M - 2$
$D = M-1$ (facets)	$M + 2^M - 1$	1
$D = M$	1	0

Total number of D - dimensional faces

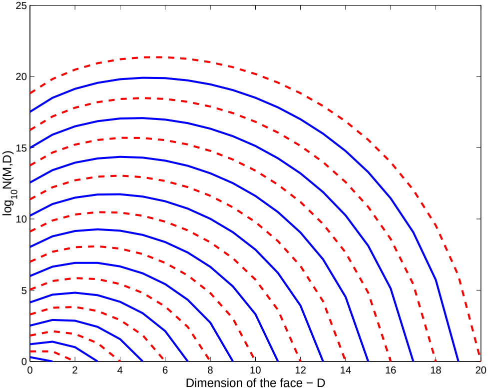


Figure : Total number of D - dimensional faces (expressed on a logarithmic scale) as a function of D. Each curve corresponds to a value of M. The curve that corresponds to $M = m$, $m = 1, 2, \ldots, 20$, is the one that hits the abscissa at $D = m$.

Table of Contents

Conclusion I

- The capacity region of an asynchronous memoryless multiple-access channel is the union of certain polytopes
- The points in those polytopes are the rate tuples that can be approached at an arbitrarily small error probability
- Operational and structural properties of such polytopes such as labels to tag their faces are of interest in information theory
- For non-degenerated cases, each face of dimension $M - m - |\mathcal{A}|$ has a unique label of the form $(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_m|\mathcal{A})$, where $\mathcal{A} \subseteq [M]$ and $([M] \setminus \mathcal{A}) \supset \mathcal{S}_1 \supset \mathcal{S}_2, \ldots, \supset \mathcal{S}_m$

Conclusion II

- A rate tuple on a face may be approached via successive decoding, in the following order:
  (1) The users with index in $[M] \setminus (\mathcal{A} \cup \mathcal{S}_1)$ are decoded first
  (2) The users with index in $\mathcal{S}_1 \setminus \mathcal{S}_2$
  (3) The users with index in $\mathcal{S}_2 \setminus \mathcal{S}_3$

  $$\vdots$$

  (m) The users with index in $\mathcal{S}_m$
  the users with index in $\mathcal{A}$ do not need to be decoded since they have vanishing rate

Muito obrigado!