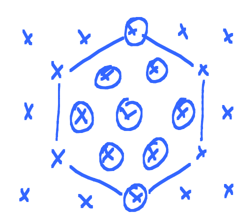
Lecture Plan

1. Definitions: Partition, Construction $\quad Vol(\Lambda)$ $\quad$ Modulo Λ
2. Figures of merit $\quad G(\Lambda)$
3. Dither & estimation $\quad noise(\Lambda)$
4. Entropy coding $\quad H(\Lambda)$
5. Infinite constellation $\quad P_e(\Lambda + noise)$
6. **Asymptotic goodness** $\quad (n \to \infty)$
7. Error exponents
8. Nested lattices $\quad \Lambda_2 \subset \Lambda_1$



9. Lattice (Voronoi-) shaping
10. Side-information problems $\quad Modulo^2(\Lambda)$
11. Gaussian networks $\quad Modulo^n(\Lambda)$

6. Asymptotic goodness

$$G(\Lambda_n) \xrightarrow{?} \frac{1}{2\pi e}, \quad \text{as } n \to \infty$$

$$\mu(\Lambda_n, P_e) \xrightarrow{?} 2\pi e, \quad \text{as } n \to \infty \quad \forall P_e > 0$$

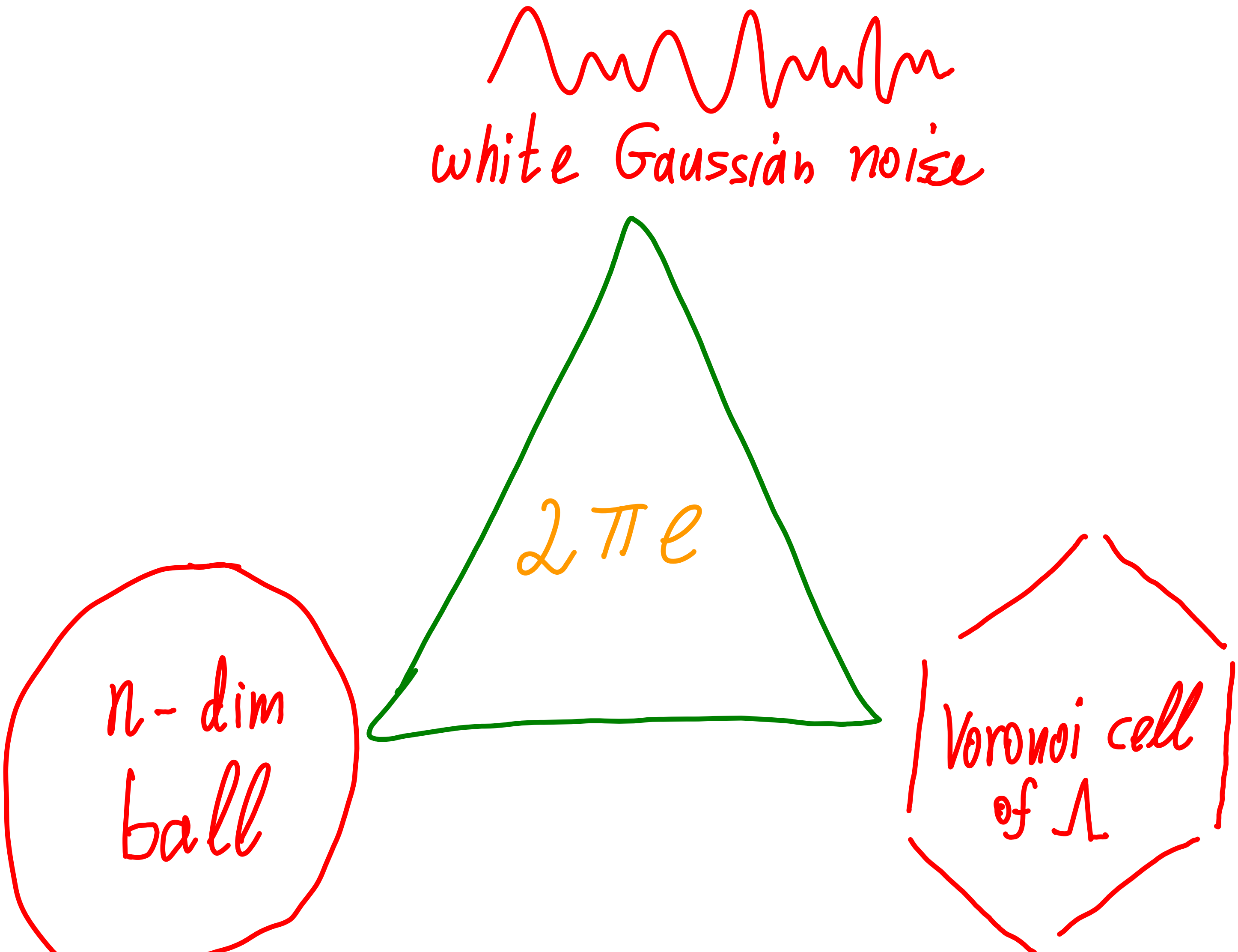
Vector Quantization Gain of Λ_n, for $n=1,2,3,\ldots$

Dimension	Lattice		Γ_q [dB]	Sphere Bound
1	$\mathbb{Z}$	integer	0	0
2	A_2	hexagonal	0.17	0.20
3	A_3	FCC	0.24	0.34
3	A_3^*	BCC	0.26	0.34
4	D_4	(Example 2.4.2)	0.36	0.45
5	D_5^*		0.42	0.54
6	E_6^*		0.50	0.61
7	E_7^*		0.57	0.67
8	E_8^*	Gosset*	0.65	0.72
12	K_{12}		0.75	0.87
16	BW_{16}	Barnes-Wall	0.86	0.97
24	Λ_{24}^*	Leech*	1.03	1.10
∞	?	?	1.53	1.53

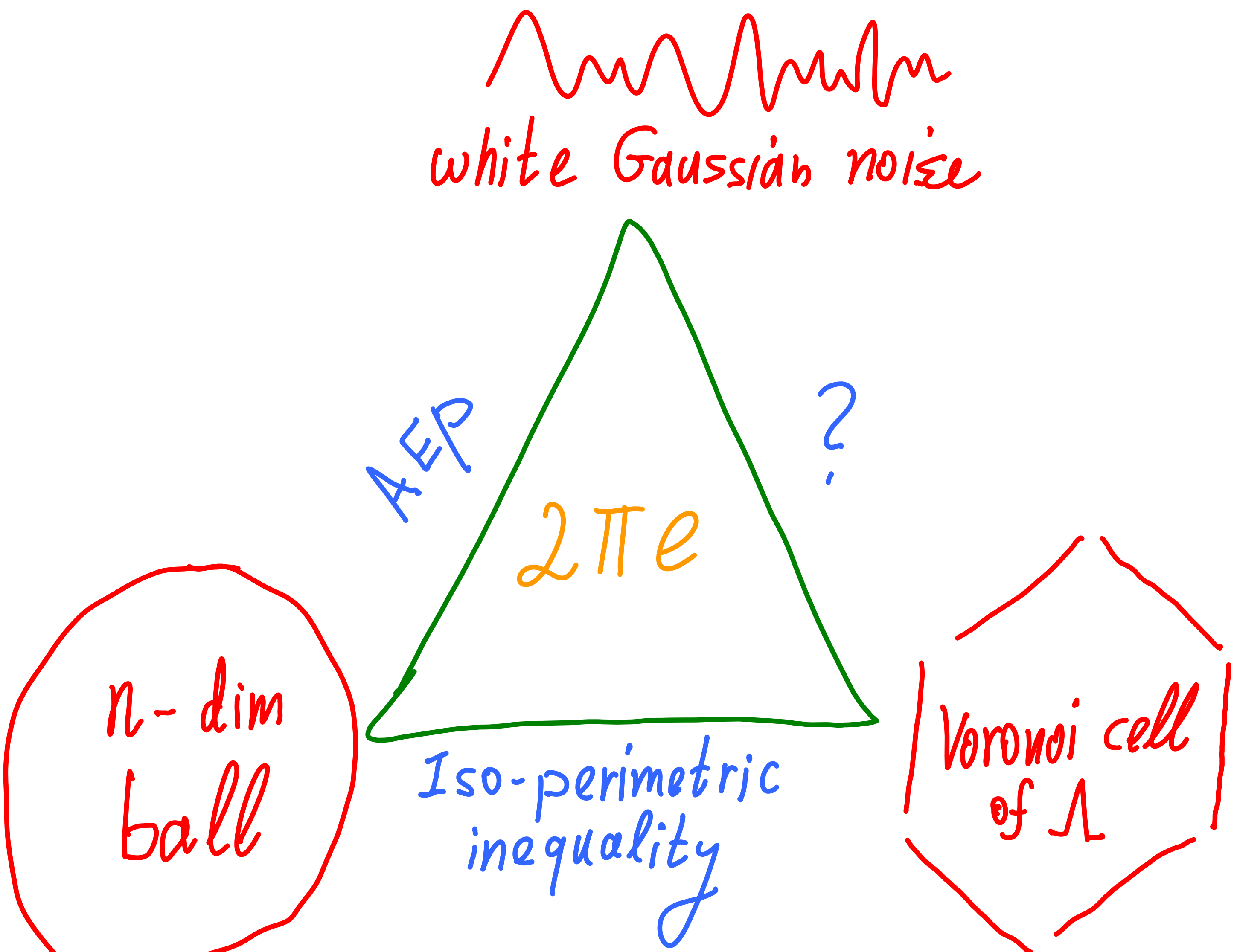
Coding Gain of Λ_n, for $n=1,2,3,\ldots$

SER		10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
Dim.	Lattice					
1	$\mathbb{Z}^1$	0	0	0	0	0
2	A_2	0.14 (0.16)	0.27 (0.33)	0.33 (0.45)	0.42 (0.54)	0.46 (0.6)
3	A_3	0.20 (0.27)	0.42 (0.56)	0.55 (0.78)	0.65 (0.93)	0.72 (1.05)
	A_3^*	0.20 (0.27)	0.40 (0.56)	0.52 (0.78)	0.59 (0.93)	0.61 (1.05)
4	D_4	0.29 (0.36)	0.60 (0.75)	0.82 (1.03)	0.95 (1.24)	1.00 (1.40)
8	E_8	0.50 (0.56)	1.08 (1.2)	1.49 (1.68)	1.80 (2.04)	2.00 (2.30)
16	BW_{16}	0.63 (0.75)	1.47 (1.63)	2.09 (2.32)	2.52 (2.83)	2.80 (3.22)
24	Λ_{24}	0.75 (0.84)	1.76 (1.85)	2.51 (2.65)	3.08 (3.25)	3.50 (3.71)
∞	?	-2.0	1.9	4.0	5.5	6.6

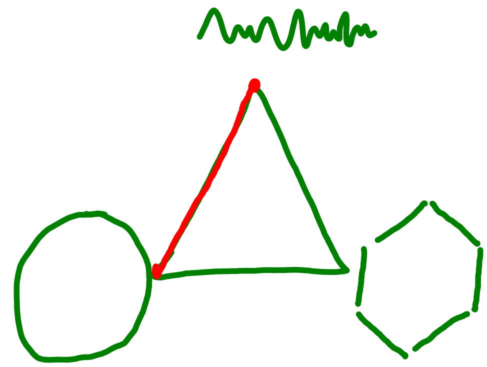
W.G.N. $\longleftrightarrow$ Ball $\longleftrightarrow$ Λ



W.G.N. $\longleftrightarrow$ Ball $\longleftrightarrow$ Λ



Shannon's AEP:



W.G.N. $\longrightarrow$ ball

$$Z_1 \dots Z_n \sim \text{AWGN} \quad N(0,\sigma^2)$$

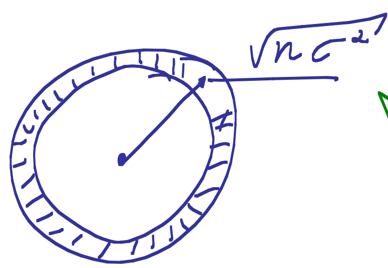
$$A_\varepsilon = \left\{ \underline{z} : \frac{1}{n}\log f_z(\underline{z}) = h \pm \varepsilon \right\}$$

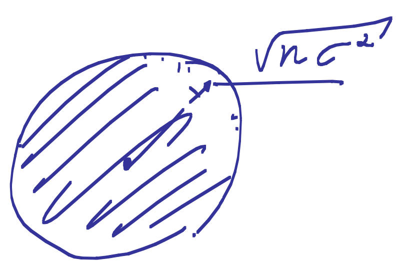
AWGN

$f_z \sim e^{-\frac{\|z\|^2}{2\sigma^2}}$

$h = \frac{1}{2}\log 2\pi e\sigma^2$

$$= \left\{ \underline{z} : \|\underline{z}\| = \sqrt{n(\sigma^2 \pm \varepsilon')} \right\}$$

$=$ 

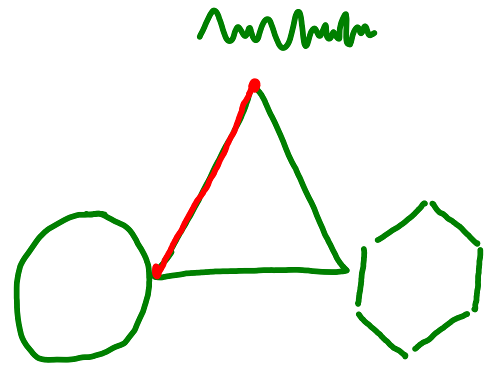
$\cong$ 

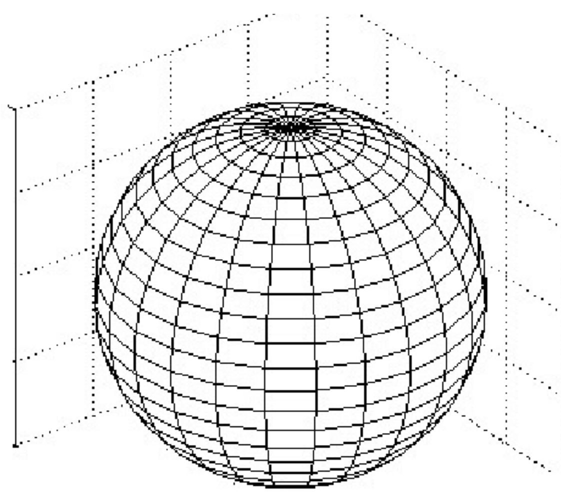
$\triangleq r_{noise}$

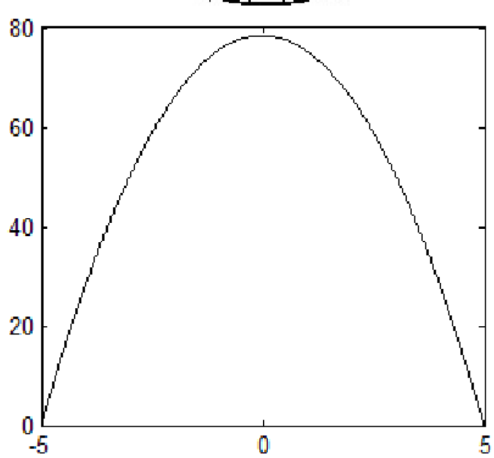
Thm. [AEP]: $\text{AWGN} \cong \text{Unif}\left(B(\underline{0}, \sqrt{n\sigma^2})\right)$

"Reverse" AEP:

W.G.N. $\leftarrow$ ball





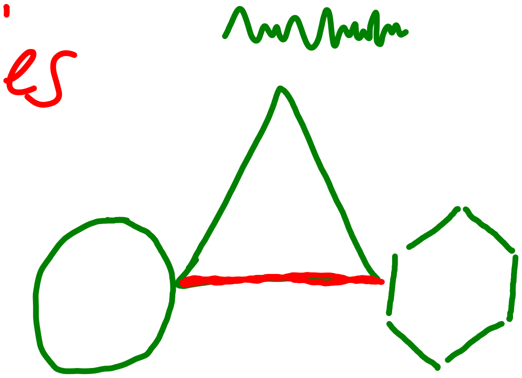


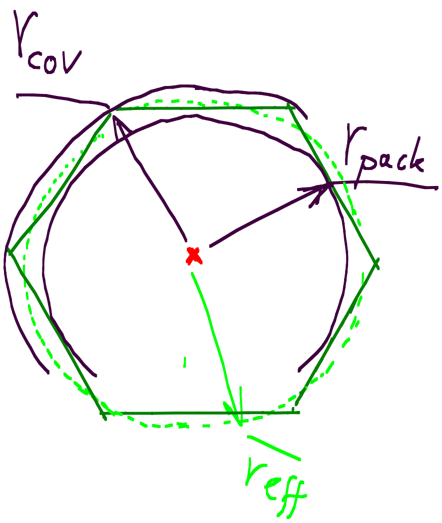
Thm. [Reverse AEP]:

If $(Z_1, \dots, Z_n) \sim \mathrm{Unif}\left(\mathrm{Ball}\left(0, \sqrt{n\sigma^2}\right)\right)$,

then $Z_1 \xrightarrow{dist} N(0, \sigma^2)$ as $n \to \infty$

Iso-perimetric Inequalities
(Sphere bounds)





Ball minimizes

* * *

over all bodies

of a fixed volume!

$$\sigma^2(\Lambda) \ge \sigma^2(\text{ball with radius } r_{eff})$$

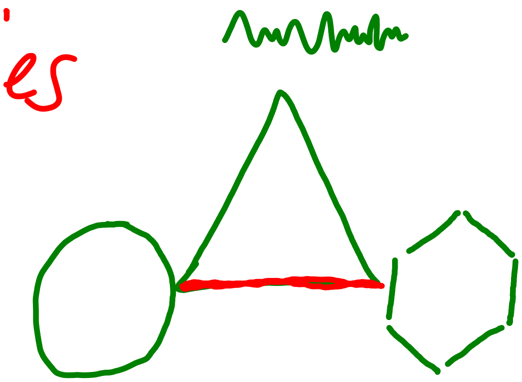
$$P_e(\Lambda) \ge P_e(\text{" \quad " \quad " \quad "})$$

$\Longrightarrow$

$$G(\Lambda) \ge \text{N.S.M. of } n\text{-dim ball}$$

$$\mu(\Lambda, P_e) \ge \text{V.N.R. " \quad " \quad "}$$

Iso-perimetric Inequalities



$$G(\Lambda) \geq G_n(\text{Ball})$$

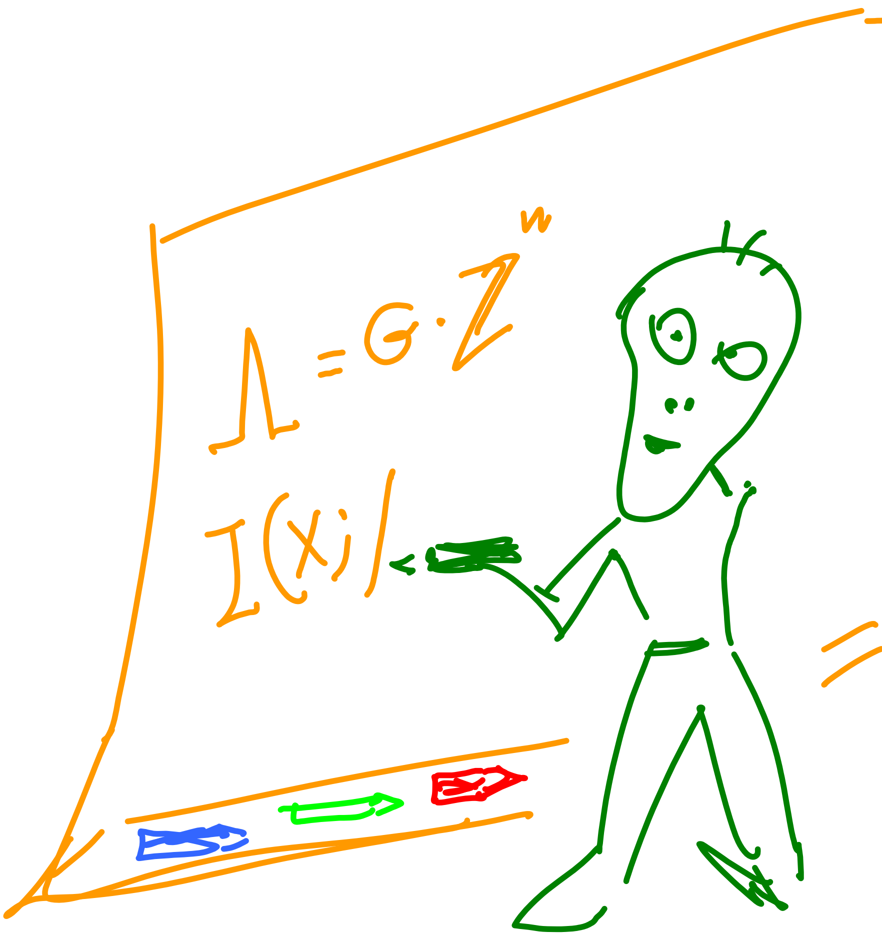
$$\mu(\Lambda, p_e) \geq \mu_n(\text{Ball}, p_e)$$

Asymptotic sphere bounds:

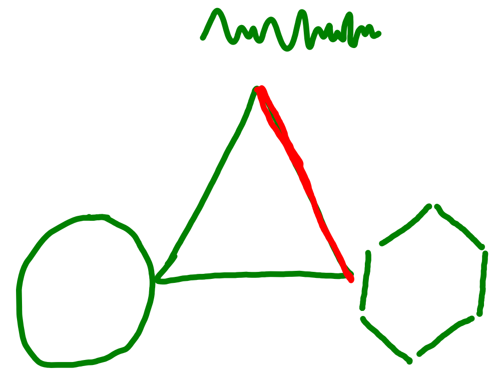
$$G_n(\text{Ball}) \longrightarrow \frac{1}{2\pi e} \quad \text{as} \quad n \to \infty$$

$$\mu_n(\text{Ball}, p_e) \longrightarrow 2\pi e \quad \text{as} \quad n \to \infty$$

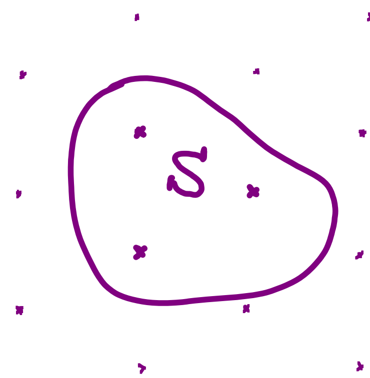
On-Board Calculation...



A Random Lattice Ensemble: Minkowski-Hlawka-Siegel



$N_\Lambda(S) \triangleq$ number of nonzero points of Λ inside a body S



Theorem: For every dimension n, there exists an ensemble $\{\Lambda\}$ of lattices with a constant point density $\boxed{\gamma = \frac{1}{V_\Lambda}}$ (= a prob. measure over all generator matrices G with determinant $1/\gamma$) such that for every bounded body S

$$E_{MHS}\left\{N_\Lambda(S)\right\} = \gamma \cdot Vol(S)$$

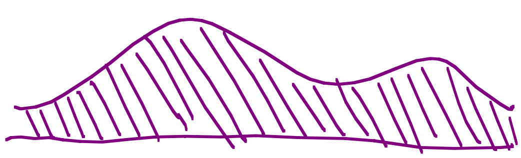
Just like a uniformly distributed random code!

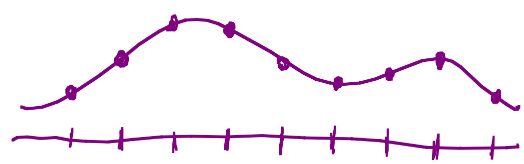
Minkowski - Hlawka - Siegel

$\Longrightarrow$

1. For any Riemann integrable function $f(\cdot)$

$$\underset{\displaystyle \int_{\mathbb{R}^n} f(x)\,dx}{\underset{\displaystyle \|}{\text{integral}}} = \frac{1}{\gamma}\cdot E_{MHS}\left\{\underset{\displaystyle \sum_{\lambda\in\Lambda} f(\lambda)}{\underset{\displaystyle \|}{\begin{matrix}\text{lattice-samples}\\ \text{sum}\end{matrix}}}\right\}$$



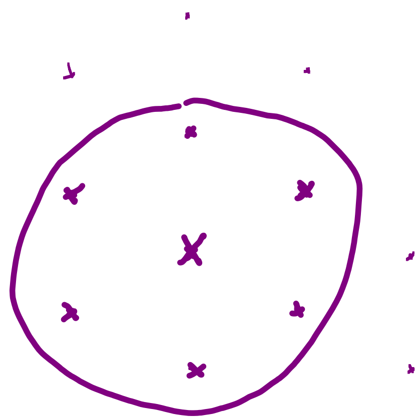


2. There exists (at least one) lattice which is (at least) as "good" as (1.)

Implication 1 : packing Goodness

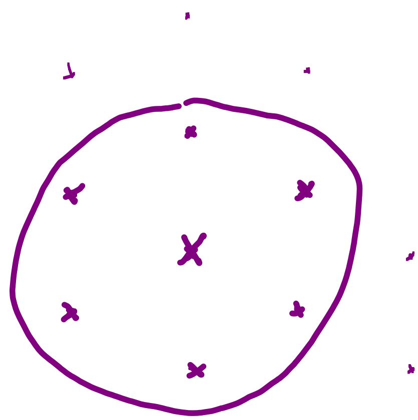
$$S = \text{Ball}(0, r)$$

$$\boxed{E_{MHS}\left\{ N_{\Lambda}(\text{Ball}) \right\} = \gamma \cdot V_n \cdot r^n}$$



Implication 1 : packing Goodness

$$S = \mathrm{Ball}(0, r)$$



$$\boxed{E_{MHS}\left\{N_\Lambda(\mathrm{Ball})\right\} = \gamma \cdot V_n \cdot r^n}$$

If $\mathrm{Vol}(\mathrm{Ball}) = V_n \cdot r^n < 1/\gamma$

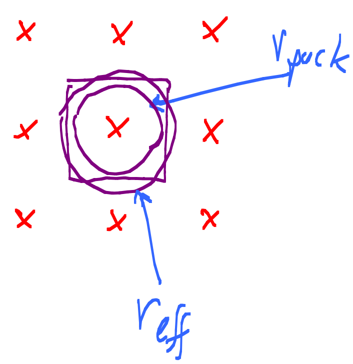
$\Longleftrightarrow \quad r < r_{eff} \qquad (*)$

$\Longrightarrow \quad E\{N_\Lambda\} < 1$

But $\quad N_\Lambda = \text{integer}$

$\Longrightarrow \quad N_\Lambda = 0$ for some $\Lambda \in MHS$

$\Longrightarrow \quad d_{min} = \|\text{shortest vector}\| > r$



$\Longrightarrow \quad r_{pack} > r/2 \qquad (**)$

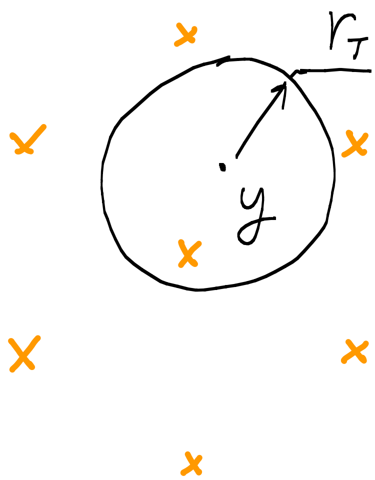
$(*) + (**) \Longrightarrow$ packing efficiency $= \dfrac{r_{pack}}{r_{eff}} \geq 1/2$

(for each dim n)

Implication 2: Modulation goodness

Theorem (achieving Poltyrev capacity):

$$\boxed{\exists\, \Lambda_n \text{ s.t. } \mu(\Lambda_n, P_e) \to 2\pi e, \quad n \to \infty.}$$



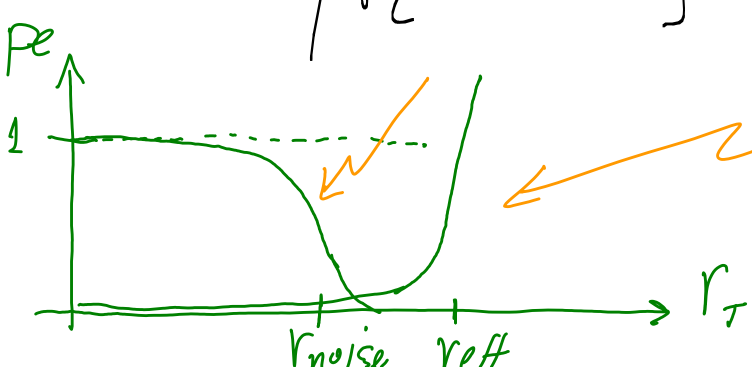
$$y = \lambda + Z$$

$$\hat{\lambda} = \begin{cases} \Lambda \cap \mathrm{Ball}(y, r_T), & \text{if unique} \\ ?, & \text{if empty or non-unique} \end{cases}$$

where r_T = search radius ("threshold")

$\Rightarrow$ $P_e = \Pr\{\text{true } \lambda \text{ outside Ball OR other } \lambda \text{ inside Ball}\}$

$$\leq \Pr\{\|Z\| > r_T\} + N_\Lambda(\mathrm{Ball}(Z, r_T))$$



$$= \gamma \cdot V_n \cdot r_T^{\,n}$$

$$= (r_T / r_{eff})^n$$

On-Board Calculation...

Implication 3: Asymptotic Goodness
~ for Covering & Quantization ~

$\exists \Lambda_n$:

$$\rho_{cov}(\Lambda_n) \triangleq \frac{r_{cov}}{r_{eff}}(\Lambda_n) \xrightarrow[n\to\infty]{} 1 \quad \text{[Rogers 195*]}$$

$$\Rightarrow \quad G(\Lambda_n) \to \frac{1}{2\pi e} \quad \text{[Poltyrev-Z.-Feder 1995]}$$

Note:

1. $\frac{1}{2\pi e}$ = Normalized Second Moment (Ball) $n \to \infty$

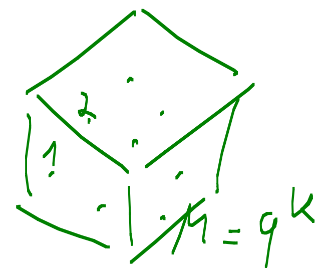
2. $\frac{1}{2}\log\left(2\pi e\, G(\Lambda)\right) = D\left(\mathrm{Unif}(\mathcal{V}_0) \,\|\, \mathrm{WGN}\right)$

$\Rightarrow$ Cell $\to$ Ball, unif(cell) $\to$ WGN

Alternative Ensemble:
Random Construction A (Loeliger 97, Erez et al 2005)

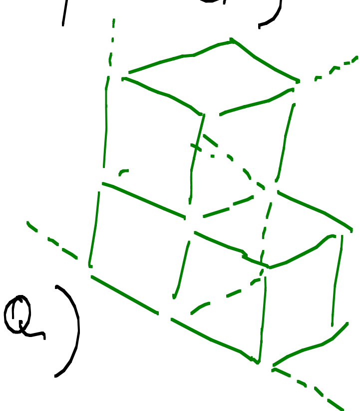
Let $\mathcal{C} =$ q-ary (n,k) liner code over $Q=\{0,\dots,q-1\}$

$$= \{ \underbrace{\underline{G}}_{n\times k}\cdot \underline{i} \; : \; \underline{i} \in Q^k \}$$



Let $\Lambda_{\mathcal{C}} =$ modulo-q lattice

$$= \{ \lambda \in \mathbb{R}^n : \lambda \bmod q \in \mathcal{C} \}$$



G random (iid uniform on Q)

$\Rightarrow \Lambda_{\mathcal{C}} =$ <u>random</u> <u>lattice</u>

$\therefore \; G(\Lambda_{\mathcal{C}}), \; \mu(\Lambda_{\mathcal{C}}, P_e) = \text{func}\{q, k, n\}$

Simultaneous Goodness

Thm. [Erez - Litsyn - Z 2004]

There exists a sequence of lattices Λ_n in dim. $n = 1, 2, \ldots$, such that as $n \to \infty$

$$\rho_{cov}(\Lambda_n) \longrightarrow 1$$

$$\underline{\lim}\ \rho_{pack}(\Lambda_n) \geq \frac{1}{2}$$

$$G(\Lambda_n) \longrightarrow \frac{1}{2\pi e}$$

$$\mu(\Lambda_n, p_e) \longrightarrow 2\pi e \qquad \forall\, p_e > 0$$

Random Construction A:
How Large q, k, n Should Be?

good packing: $\quad q, n \to \infty \quad , \quad 1 \le k \le n-1$

$\rho_{pack} \simeq 1/2$

good covering: $\quad q, k, n \to \infty$

$\rho_{cov} \simeq 1$

good modulation:

$\mu(\Lambda, p_e) \simeq 2\pi e$

complete lattice $\to$ $q, n \to \infty \quad , \quad 1 \le k \le n-1$

finite constellation (= cosets) $\to$ $n \to \infty \quad , \quad k \cdot \log(q) \propto n$ *

good quantization: $\quad n \to \infty \quad , \quad k \cdot \log(q) \propto n$ *

$G(\Lambda) \simeq 1/2\pi e$

* For example, modulo-2 lattices ($q=2$, $k \propto n$) are ok!

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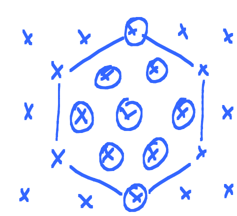
8. Nested lattices $\quad \Lambda_2 \subset \Lambda_1$



9. Lattice (Voronoi) shaping
10. Side-information problems $\quad Modulo^2(\Lambda)$
11. Gaussian networks $\quad Modulo^n(\Lambda)$

Tutorial - Part A Outline

7. Error exponents

$E(\Lambda)$

Implication 2: Asymptotic Goodness for Modulation

[Gallager 1962]

$$\boxed{P_e^{ML}(\Lambda) = \int_0^\infty p(\|\underline{z}\| = r)\cdot \underbrace{P\begin{pmatrix}\text{nonzero codeword}\\ \text{in } \mathrm{Ball}(\underline{z}, r)\end{pmatrix}}_{\le\, N_{\underline{\Lambda}}(\mathrm{Ball}(\underline{z},r))}\, dr}$$

Implication 2: Asymptotic Goodness for Modulation

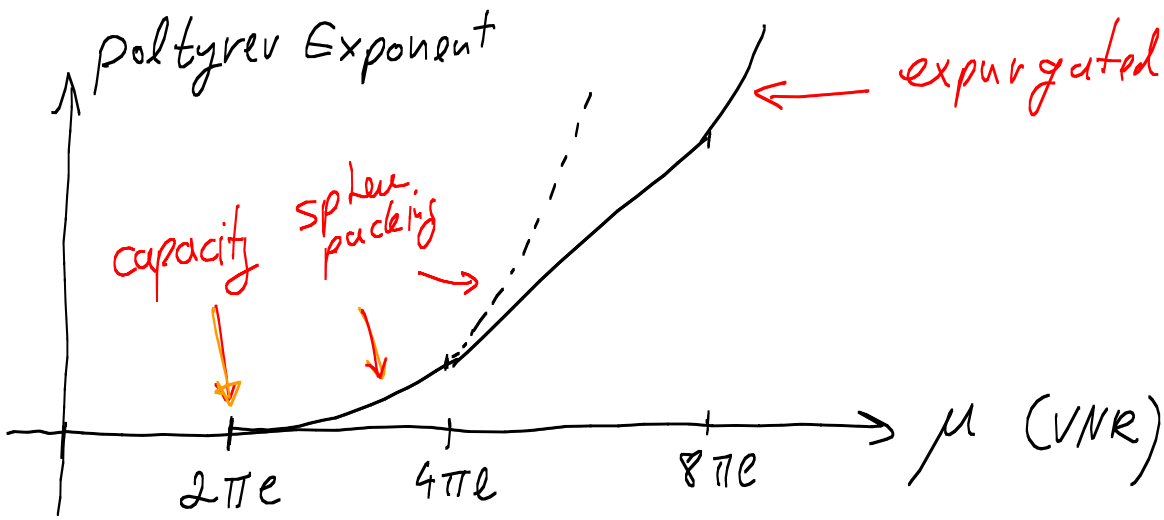
$$P_e^{ML}(\Lambda) = \int_0^\infty p(\|\underline{Z}\| = r)\cdot P\begin{pmatrix}\text{nonzero codeword} \\ \text{in } Ball(\underline{Z}, r)\end{pmatrix} dr$$

[Gallager 1962]

$$\wedge$$

$$N_\Lambda(Ball(\underline{Z}, r))$$

$$E_{MHS}\{\quad\} \Longrightarrow \quad = \quad \gamma \cdot V_n \cdot r^n$$



$$\therefore \exists \Lambda_n : \mu(\Lambda_n, P_e) \xrightarrow[n\to\infty]{} 2\pi e \quad \forall P_e > 0$$

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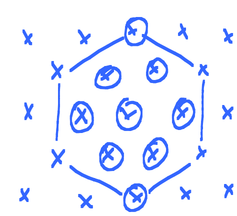
8. **Nested lattices** — $\Lambda_2 \subset \Lambda_1$



9. Lattice (Voronoi) shaping
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8. Nested Lattices

$$\Lambda_2 \subset \Lambda_1$$

Nested Lattices

$$\Lambda_2 \subset \Lambda_1 \Rightarrow \underline{\underline{G_2}} = \underline{\underline{G_1}} \cdot \underline{\underline{J}}$$

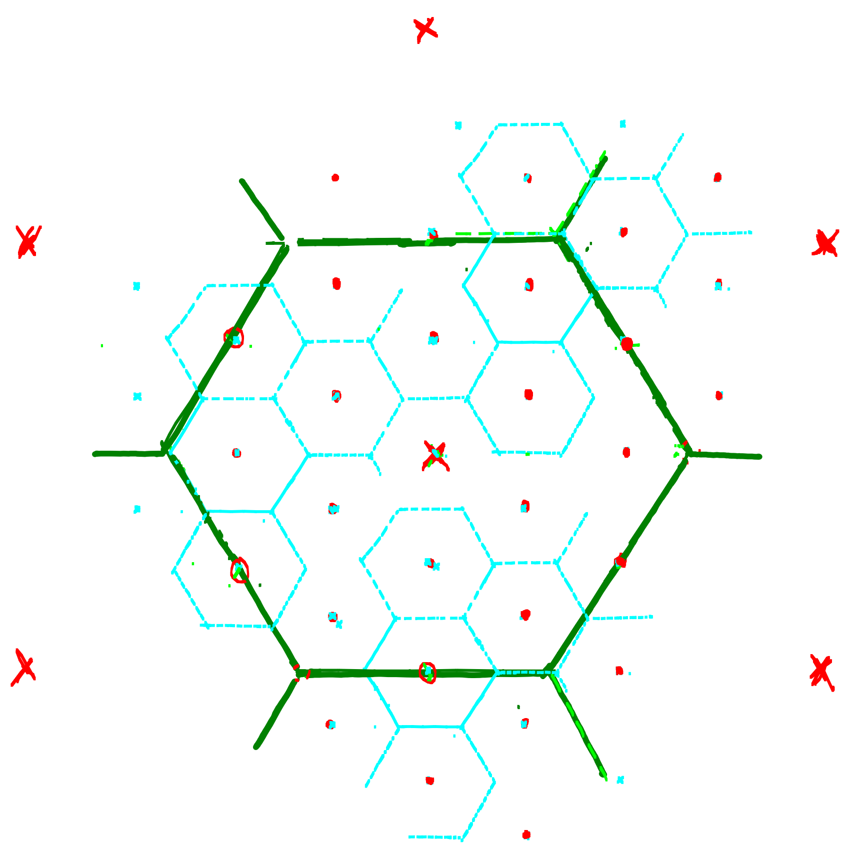
coarse lattice

fine lattice

integer matrix

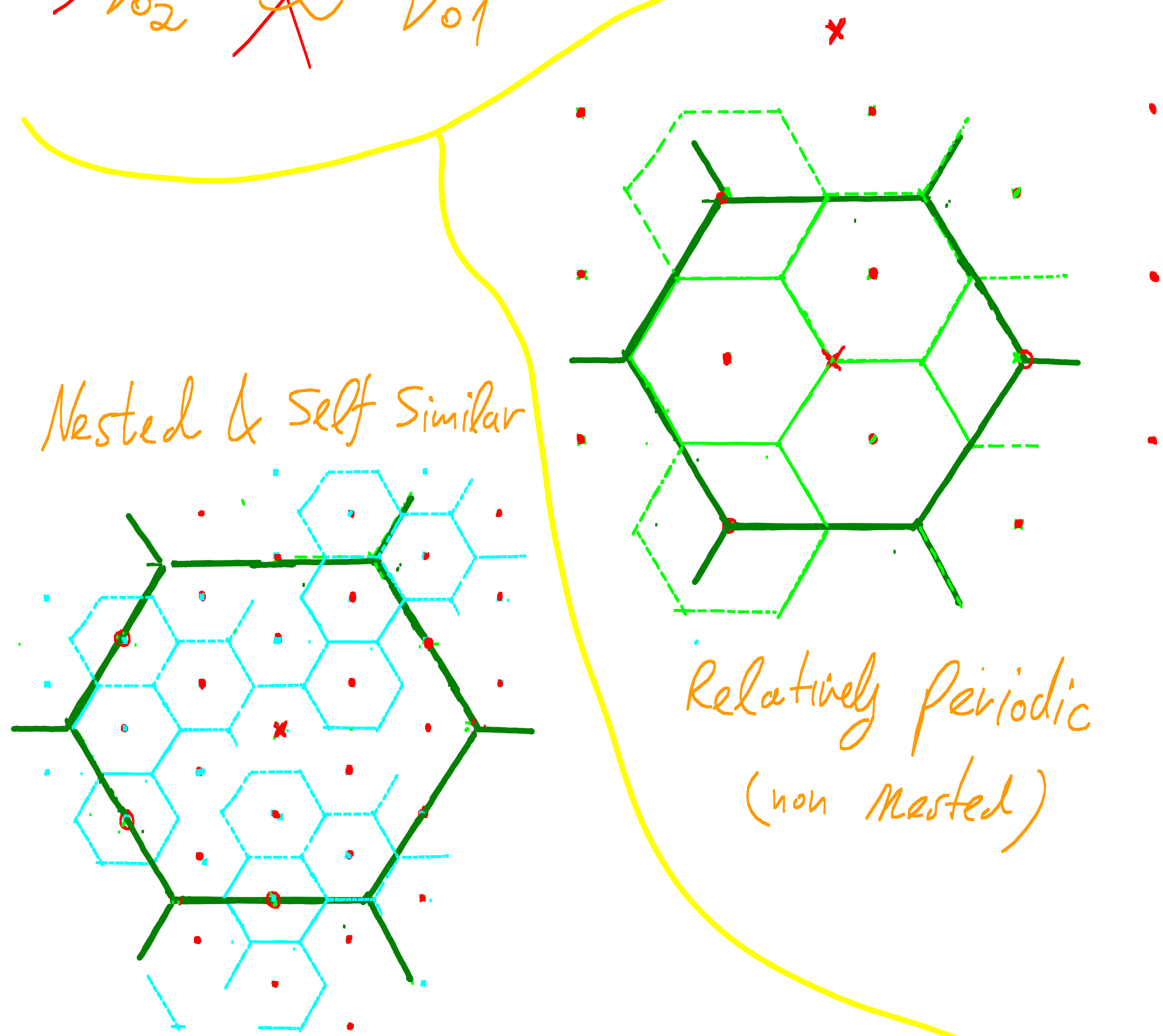
$$\text{Nesting Ratio} = \left(\frac{V_2}{V_1}\right)^{1/k} = \left|\det(J)\right|^{1/k}$$

4:1



Not necessarily "Self Similar"!

$\Rightarrow V_{02} \not\subset V_{01}$



Diagonal Form

If $\Lambda_2 \subset \Lambda_1$, then $\exists$ generator matrices G_1, G_2 s.t. the nesting matrix J is diagonal

$$\underline{\underline{J}} = \begin{pmatrix} j_1 & & 0 \\ & \ddots & \\ 0 & & j_n \end{pmatrix}$$

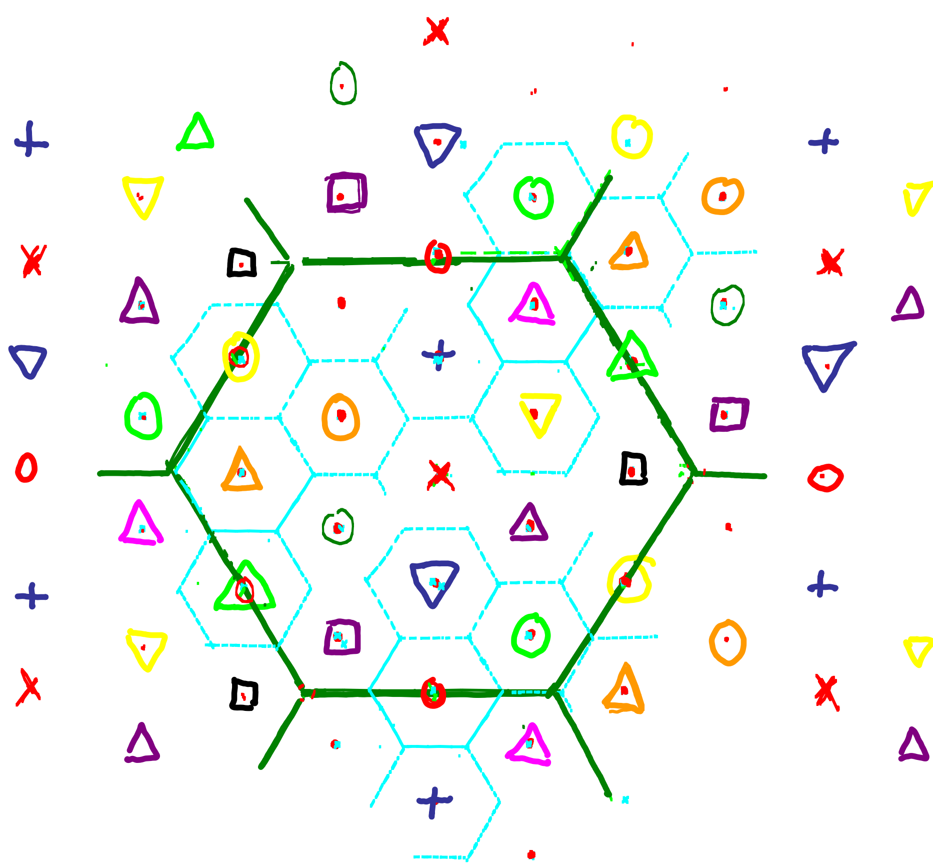
Nested Lattices

$\Lambda_2 \subset \Lambda_1 \Rightarrow \underline{\underline{G}}_2 = \underline{\underline{G}}_1 \cdot \underline{\underline{J}}$

Relative Cosets $= \Lambda_2 / \Lambda_1$

Coset $\triangleq l_1 + \Lambda_2$, for some $l_1 \in \Lambda_1$

$|\Lambda_2 / \Lambda_1| = V_2 / V_1 = |\det(J)|$



Open Question:

Good nested pairs ?

(low shaping & coding losses $\Longrightarrow$
low $G(\Lambda_2) * \mu(\Lambda_1, p_e)$)

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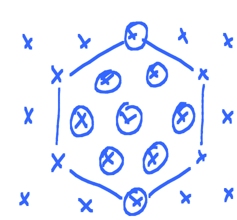
9. **Lattice (Voronoi-) shaping**



10. Side-information problems — $Modulo^2(\Lambda)$
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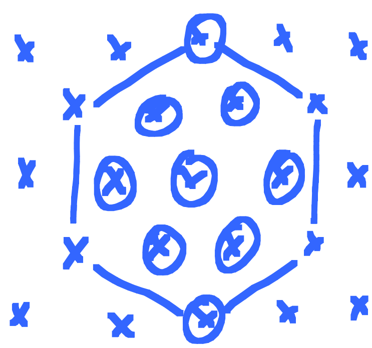
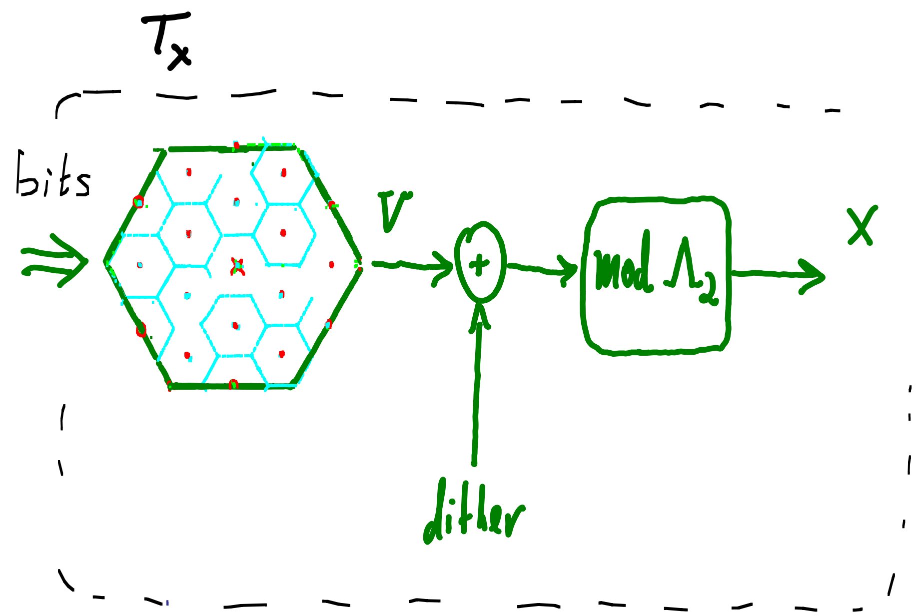
9. Lattice (Voronoi-) shaping



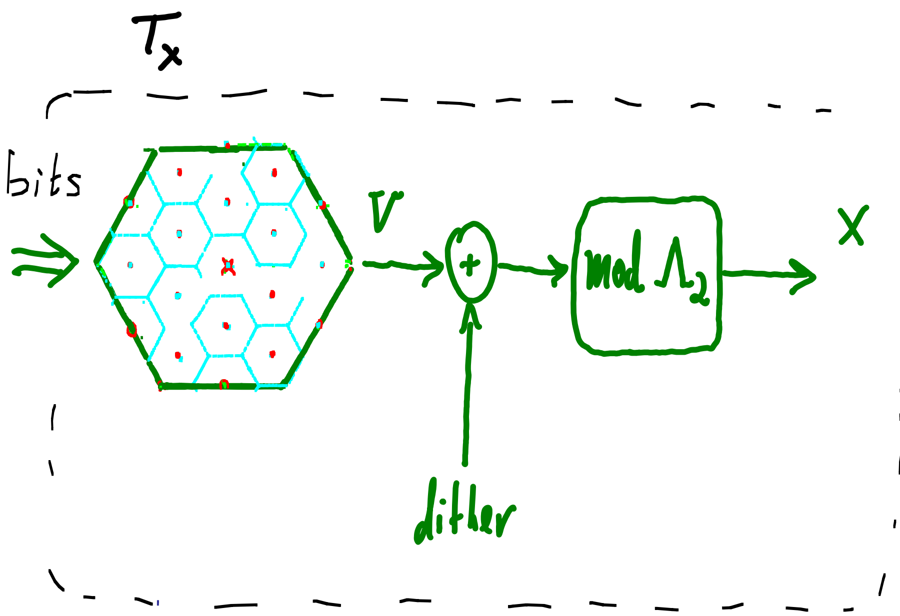
Dithered Voronoi Modulation

message = fine point in coarse fund. cell
$\iff$ coset in Λ_1 / Λ_2



transmitted vector = reduction to Voronoi cell
$\iff$ min energy coset point

Dithered Voronoi Modulation



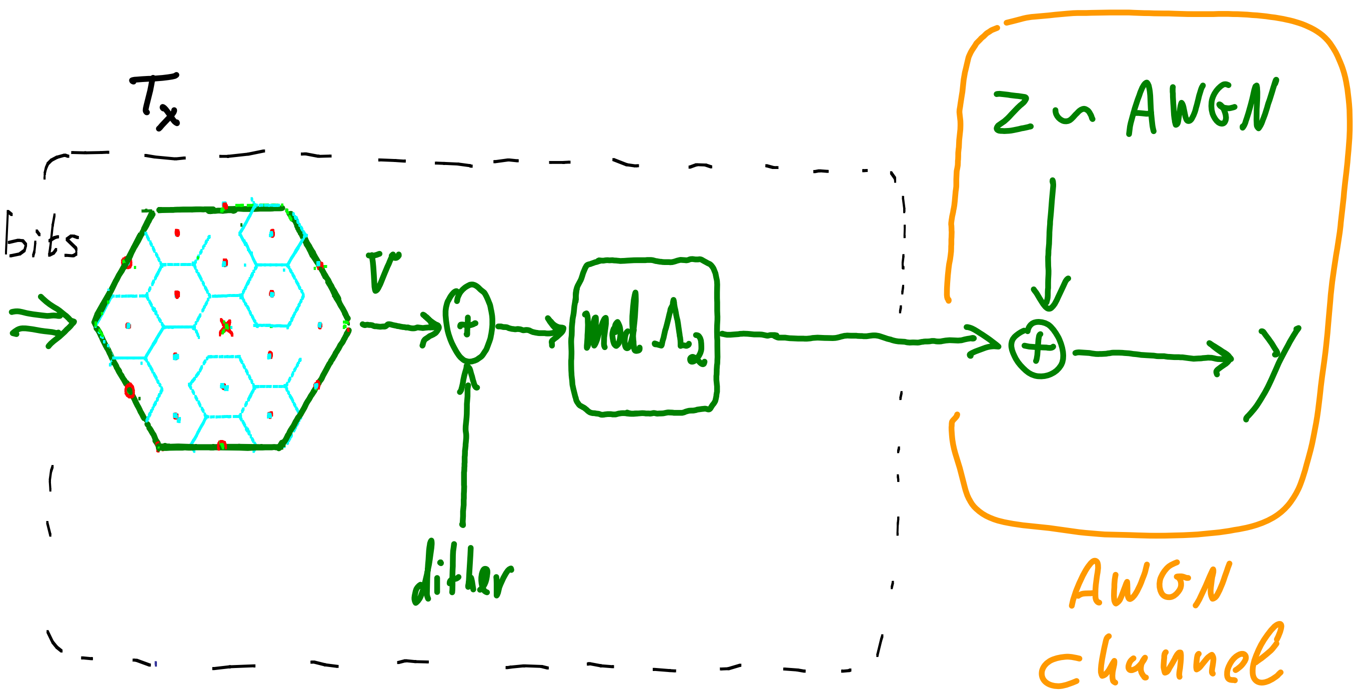
Crypto Lemma $\Rightarrow$ for any $v \in \mathcal{C}_{\Lambda}$:

$$X = [V + \text{dither}] \bmod \Lambda_2 \sim \text{Unif}(\mathcal{V}_0)$$

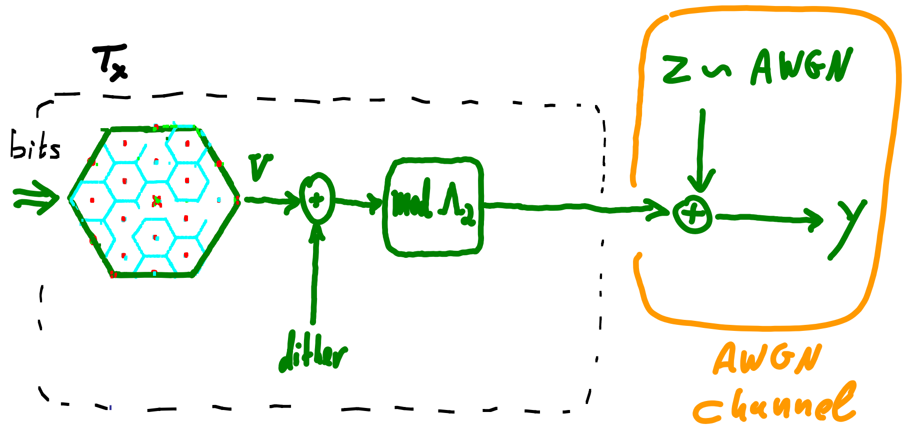
$\Rightarrow$ T_x Power $= \frac{1}{n} E\|X\|^2 = \sigma^2_{\Lambda_2}$

Dithered Voronoi Modulation:

Decoding



Dithered Voronoi Modulation: Decoding



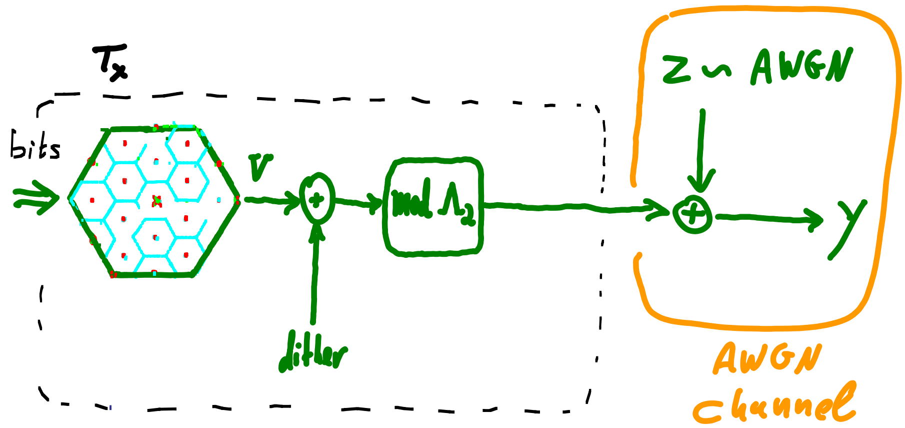
Maximum-Likelihood Decoding:

$$\hat{v}_{ML} = \underset{v \in \mathcal{C}_\Lambda}{\arg\max} \; \| y - [v + \text{dither}] \bmod \Lambda_2 \|$$

$\Rightarrow$ constrained search $\Rightarrow$ complex!

reduction to $\mathcal{V}_0$

Dithered Voronoi Modulation: Decoding



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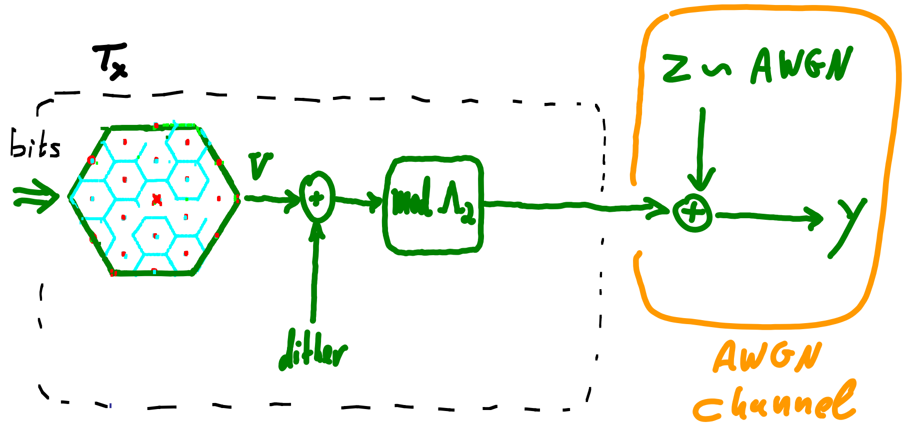


Maximum-Likelihood Decoding:

$$\hat{v}_{ML} = \underset{v \in \mathcal{C}_\Lambda}{\arg\min} \; \| y - [v + \text{dither}] \bmod \Lambda_2 \|$$

⇒ constrained search ⇒ complex!

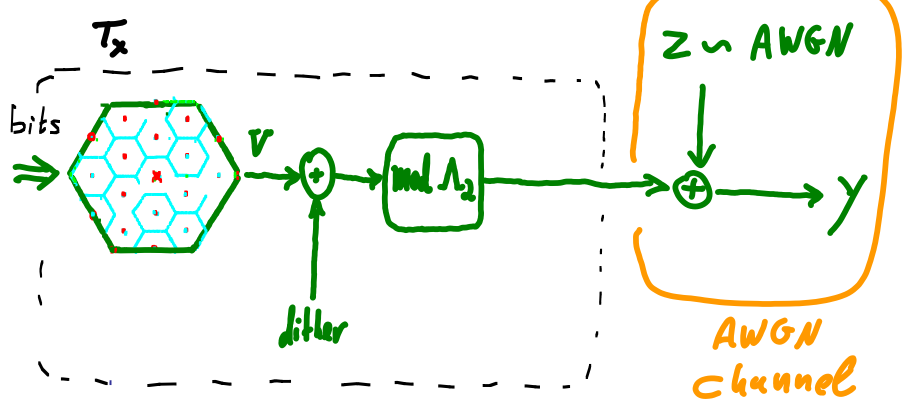
Lattice Decoding:

$$\hat{v}_{LD} = \left(\underset{\lambda \in \Lambda_1}{\arg\min} \; \| y - [\lambda + \text{dither}] \| \right) \bmod \Lambda_2$$

reduction to set of representation

⇒ un-constrained search ⇒ easier?

Dithered Voronoi Modulation: Decoding



Coset Decoding:

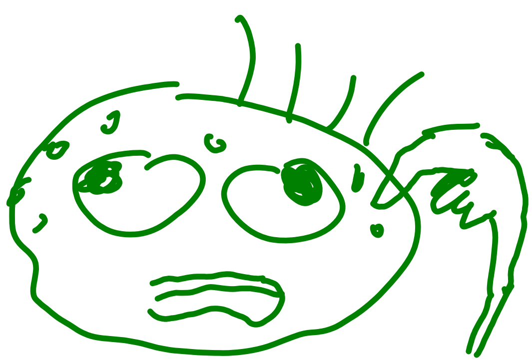
$$\hat{v}_{CD} = \underset{\text{coset} \in \Lambda_1/\Lambda_2}{\arg\max} \sum_{\lambda \in \text{coset}} e^{-\frac{\| \quad \|^2}{2\sigma^2}}$$

Open Question 5:

Simple Decoding of

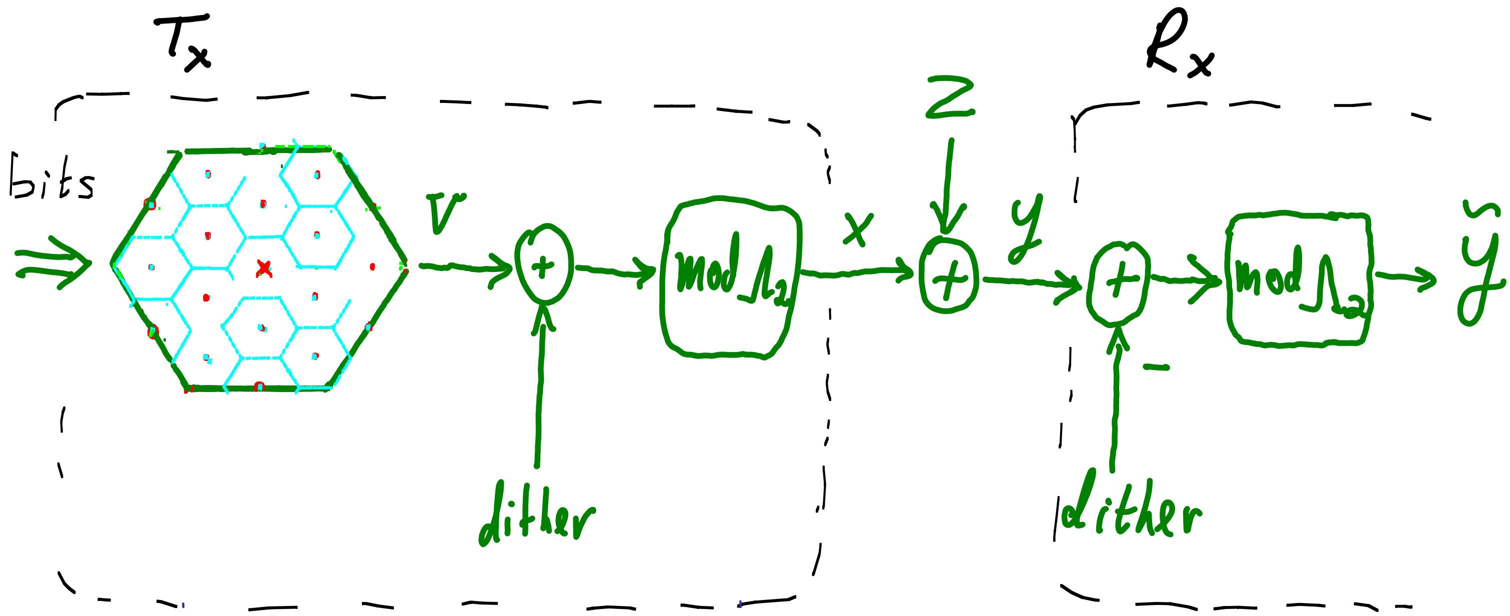
<u>Shaped</u> Lattices ?

<u>Remark</u>: keep symmetry of $P_{e,i}$, power$_i$!



?

Dithered Voronoi Modulation: Equivalent Channel



Tx: $x = [v + \text{dither}] \bmod \Lambda_2$

reduction to V_0

Channel: $y = x + z$

Rx: $\tilde{y} = [y - \text{dither}] \bmod \Lambda_2$

reduction to (any) set of coset rep.

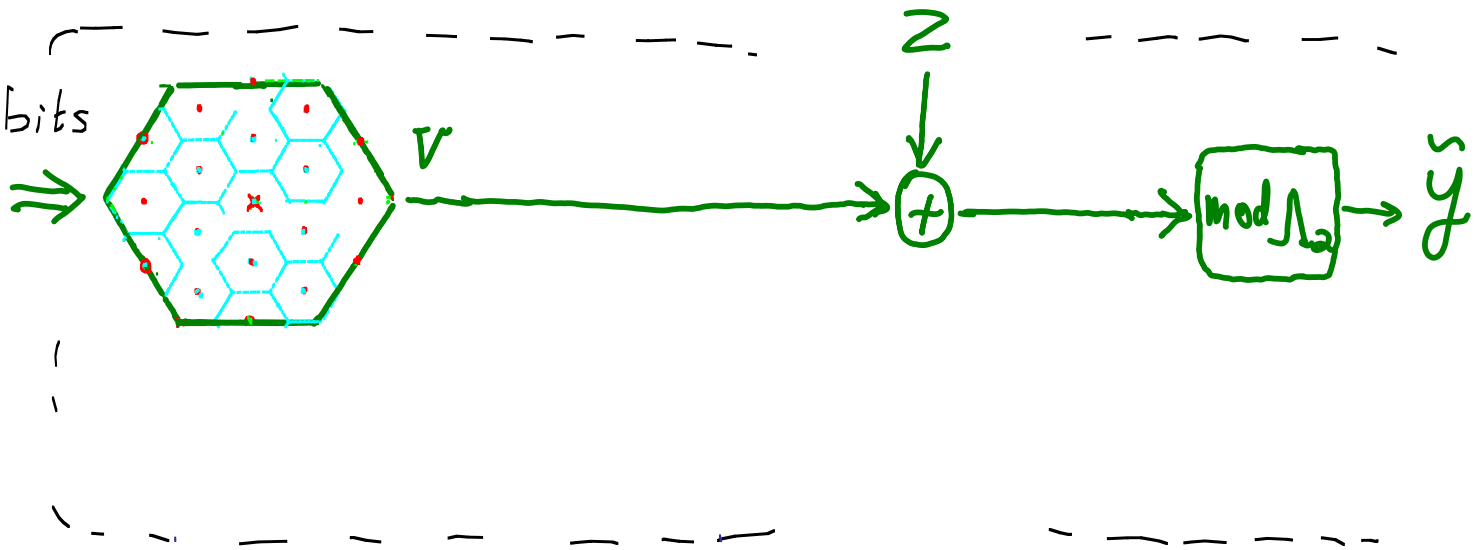
$\hat{v}_{LD} = Q_{\Lambda_1}(\tilde{y})$

Dithered Voronoi Modulation: Equivalent Channel

Distributive Property:

$$\big(a + b \bmod \Lambda\big) \bmod \Lambda = (a+b) \bmod \Lambda$$

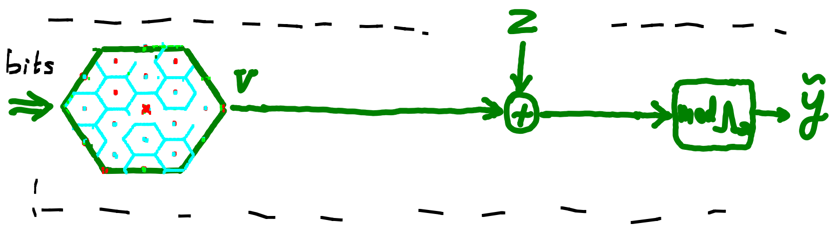
$\Rightarrow$



Dithered Voronoi Modulation : Performance

Λ_1 = good for $\mathcal{N}(0, \sigma_z^2)$ $\Rightarrow$ $P_e < \varepsilon \ \forall v$

Λ_2 = good for quantization $\Rightarrow$ $\sigma_{\Lambda_2}^2$ small w.r.t V_2



$$\text{Rate} = \frac{1}{n}\log\left(\frac{V_2}{V_1}\right) \quad \text{bit/channel use}$$

NSM(Λ_2)
VNR(Λ_1)

$$= \underbrace{\frac{1}{2}\log\left(\frac{P}{\sigma_z^2}\right)}_{\text{AWGN capacity @ High SNR}} - \underbrace{\frac{1}{2}\log\left(G(\Lambda_2)\cdot\mu(\Lambda_1,\varepsilon)\right)}_{\text{LOSS}}$$

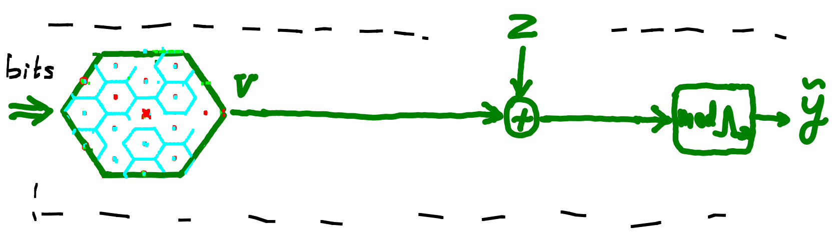
LOSS $\xrightarrow[n \to \infty]{} 0$

for good lattices ....

Dithered Voronoi Modulation : Performance

Λ_1 = good for $\mathcal{N}(0, \sigma_z^2)$ $\Rightarrow$ $P_e < \varepsilon \ \forall v$

Λ_2 = good for quantization $\Rightarrow$ $\sigma_{\Lambda_2}^2$ small w.r.t V_2



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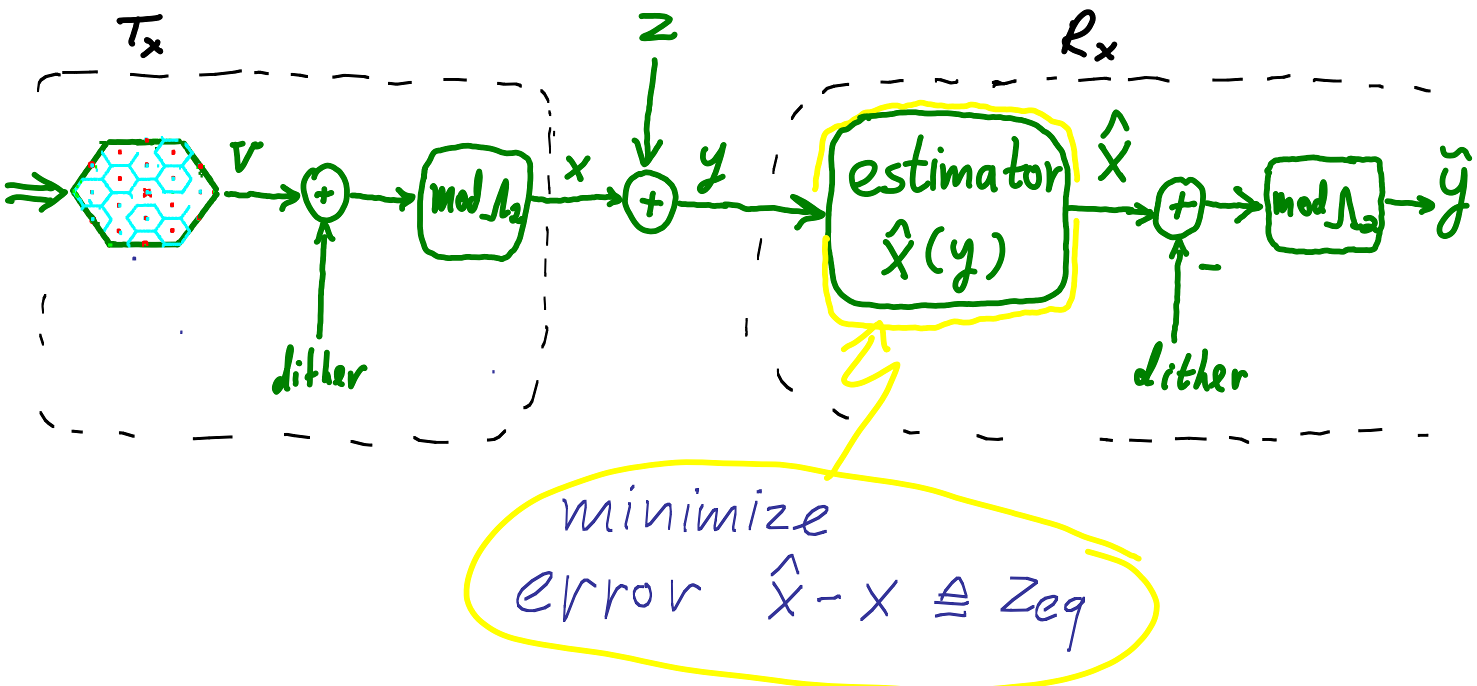
LOSS $\longrightarrow 0$ ($n \to \infty$) for good lattices ....

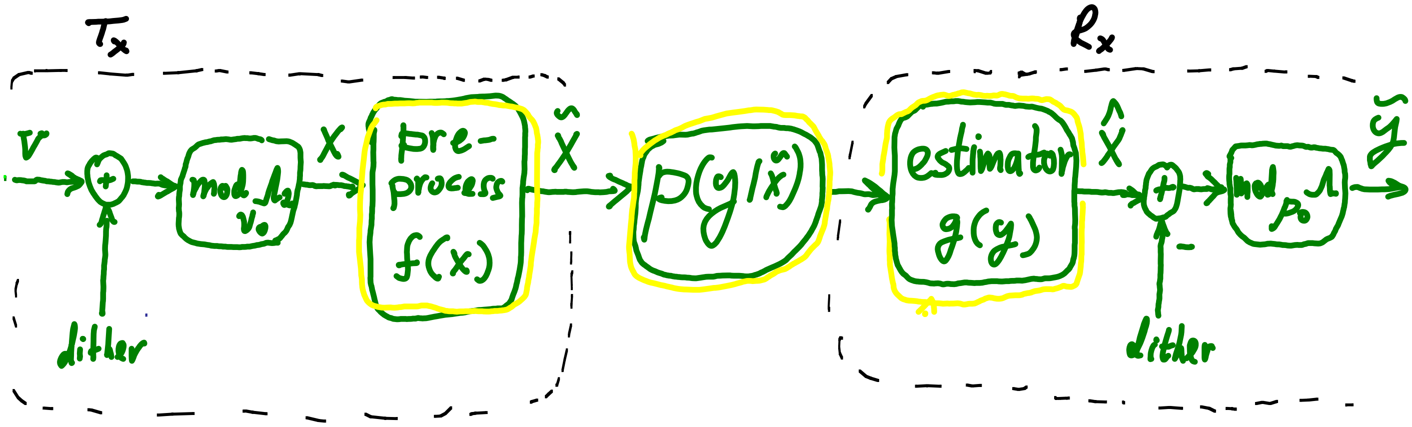
Shaping loss $\triangleq \frac{1}{2}\log(2\pi e\, G(\Lambda_2))$

coding loss $\triangleq \frac{1}{2}\log(\mu(\Lambda_1, P_e)/2\pi e)$

Achieving $\frac{1}{2}\log(1+SNR)$ @ general SNR

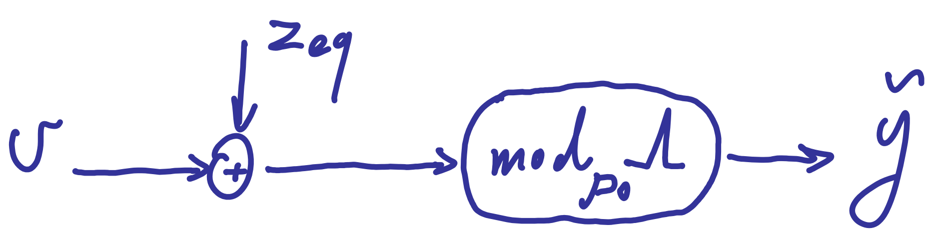


The Equivalent Mod-Λ Channel



Thm. [Mod-Λ channel]

The channel $v \to \tilde{y}$ is equivalent to a mod-Λ channel



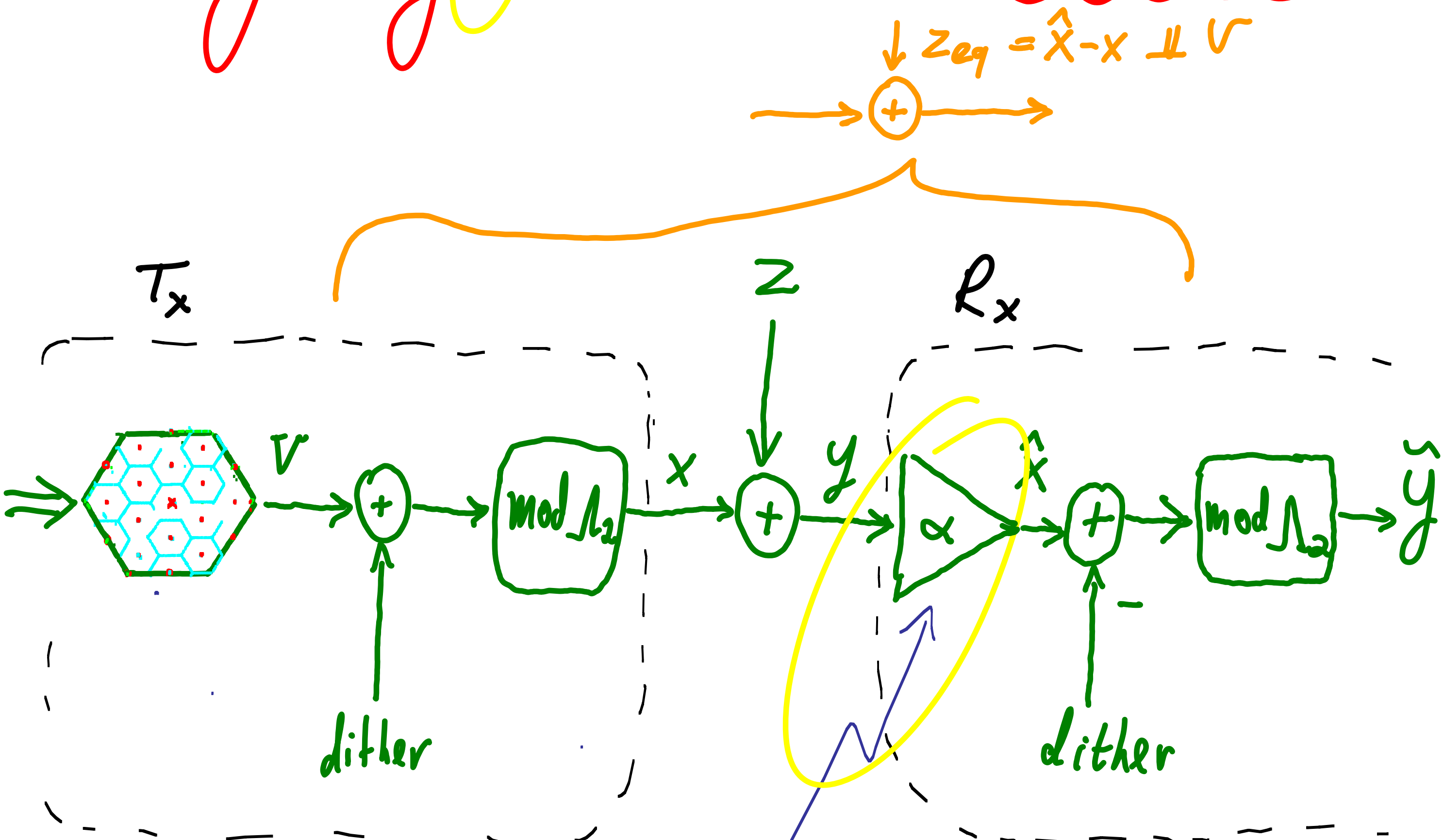
$$Z_{eq} \overset{dist.}{=} \hat{X} - X \perp\!\!\!\perp v$$

$$X \sim \mathrm{unif}(V_0) \quad , \qquad Y|X \sim p(y|f(x))$$

$$\hat{X} = g(p(y|f(x)))$$

On-Board Calculation...

Achieving $\frac{1}{2}\log(1+SNR)$ with Linear Estimation



$$\mathrm{Var}\Big\{\underbrace{\alpha Z}_{\text{AWGN}} + \underbrace{(1-\alpha)\cdot \text{dither}}_{\text{"self noise"}}\Big\} \geq \sigma_Z^2 \cdot \frac{SNR}{1+SNR}$$

equality @ $\alpha = \alpha_{\text{wienr}}$

$$\Rightarrow \text{Rate} = \underbrace{\tfrac{1}{2}\log(1+SNR)}_{\text{AWGN capacity}} - \underbrace{\text{Shaping \& Coding loss}}_{\xrightarrow{n\to\infty} 0}$$

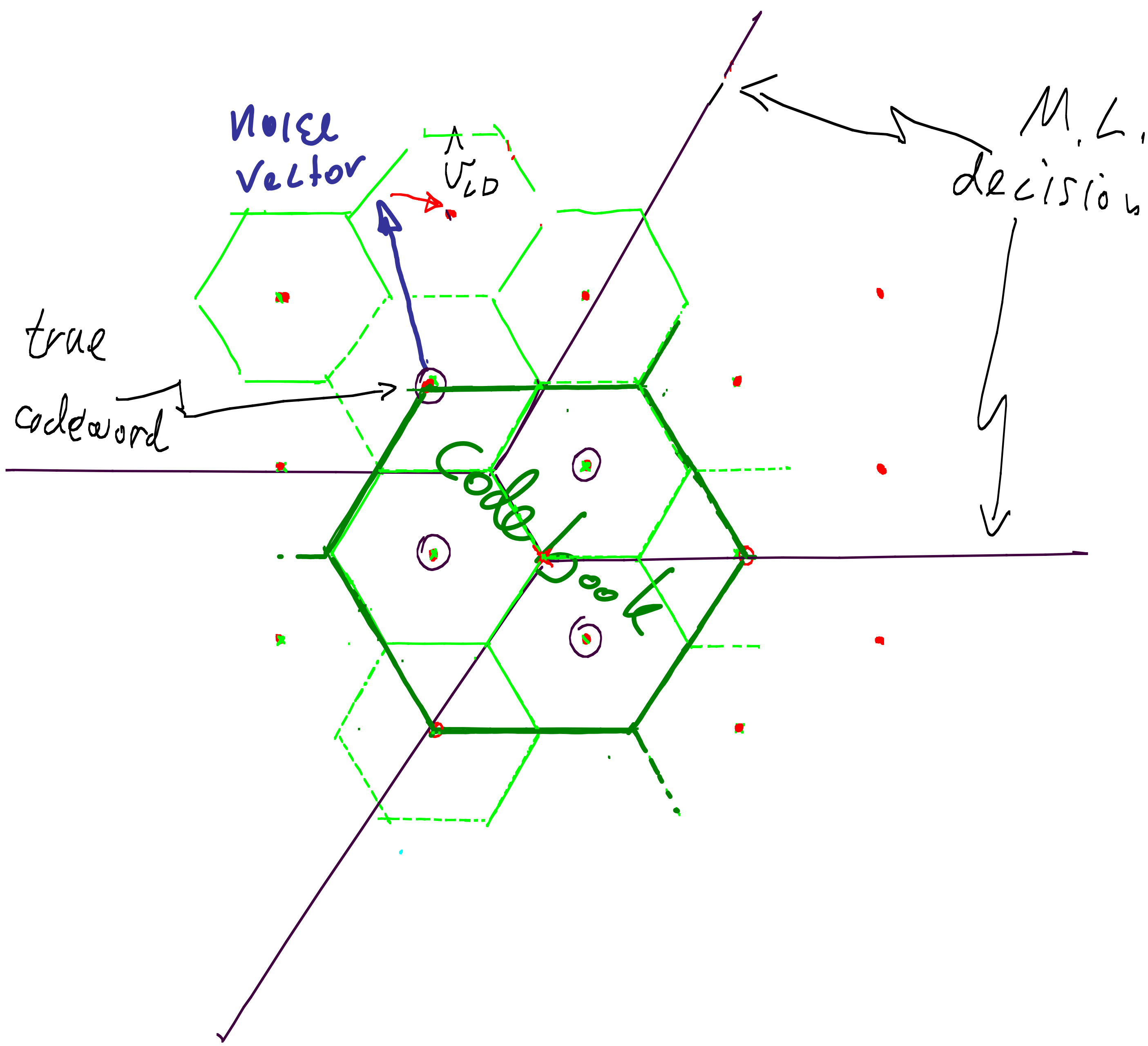
Open Questions:

* Performance for finite q, k, n ?

* Bounds on Volum to Noise Ratio relative to mixture noise $\mu(\Lambda, p_e, \alpha)$?

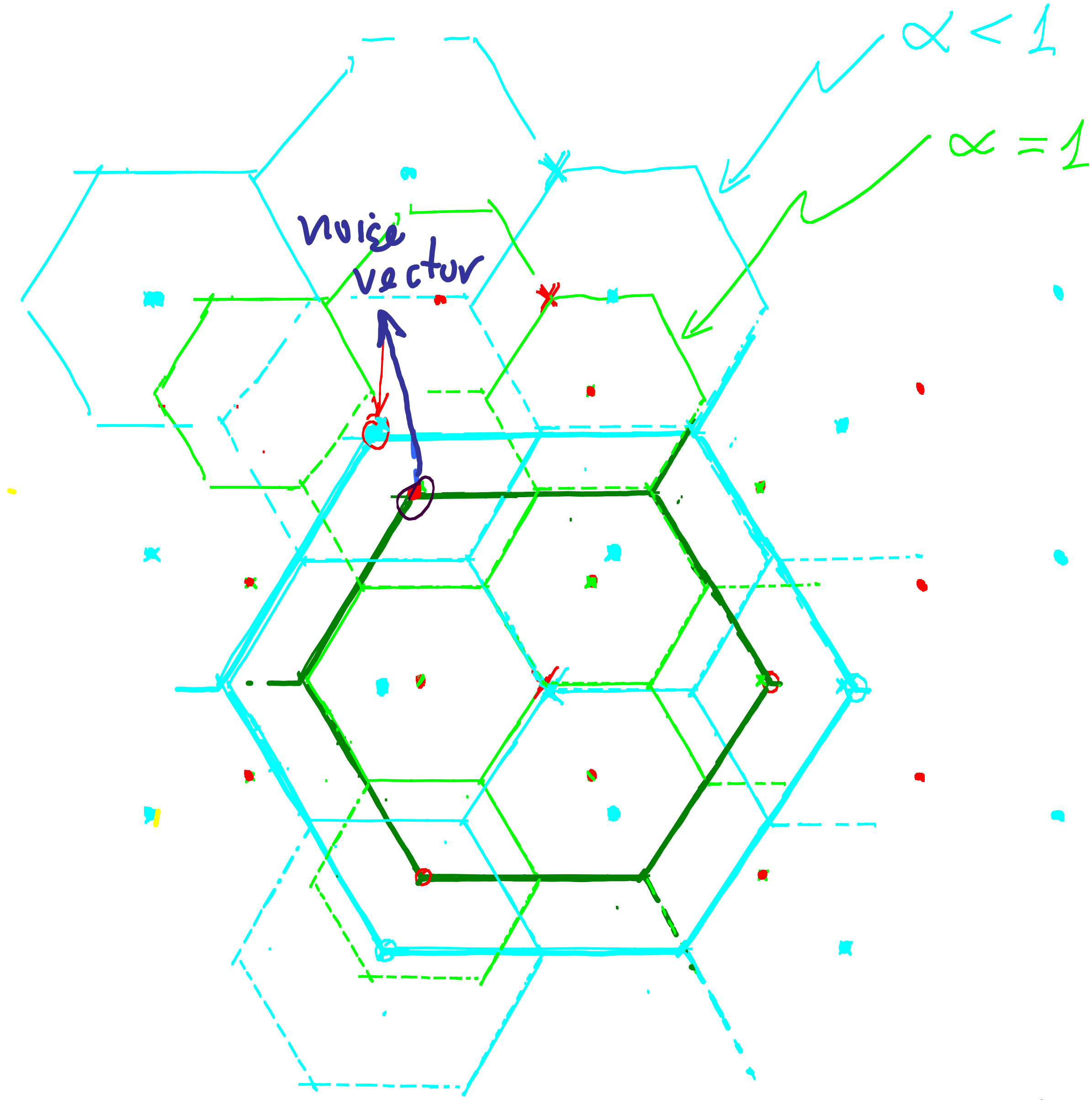
* Good nested pairs @ general SNR ?

Geometric picture:
ML versus Lattice Decoding ($\alpha = 1$)



Geometric Picture:

Linear Scaling $(\alpha=1) \Rightarrow$ Lattice Inflation



Lattice @ Decoder Inflated by $\frac{1}{\alpha}$

But Noise is Not Quite Gaussian ...

$\alpha = 1$

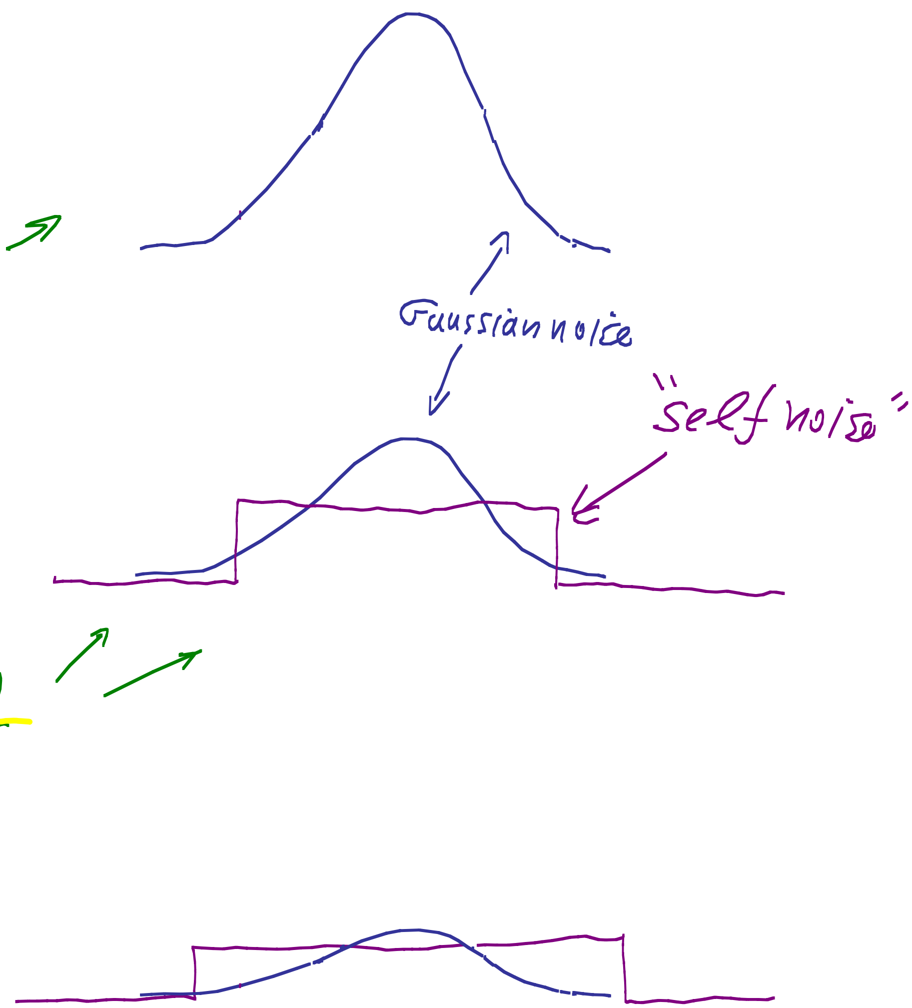
pure Gaussian

$\alpha = \alpha_{wiener}$

minimum-energy mixture

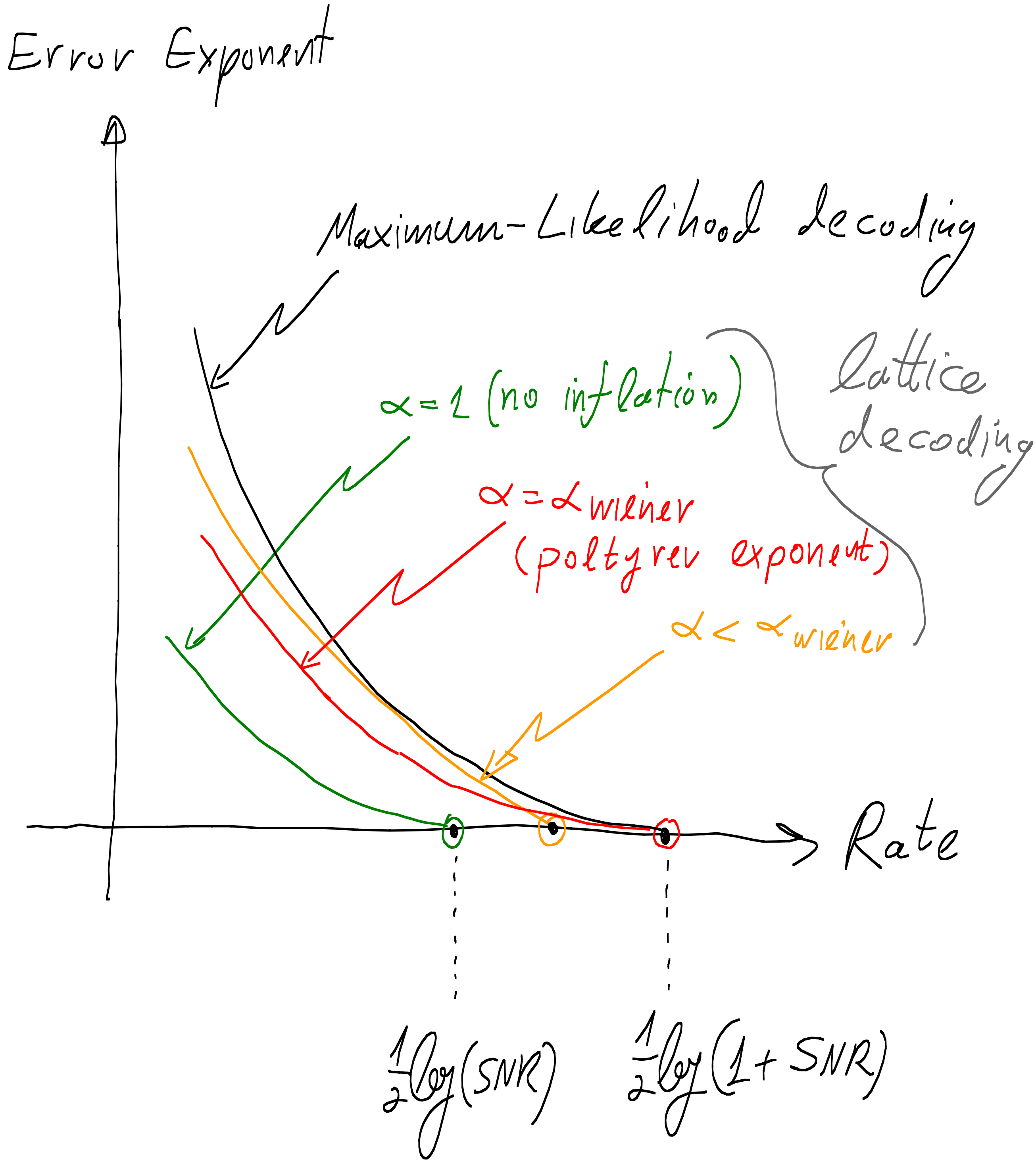
$\alpha < \alpha_{wiener}$

reduced-tail mixture

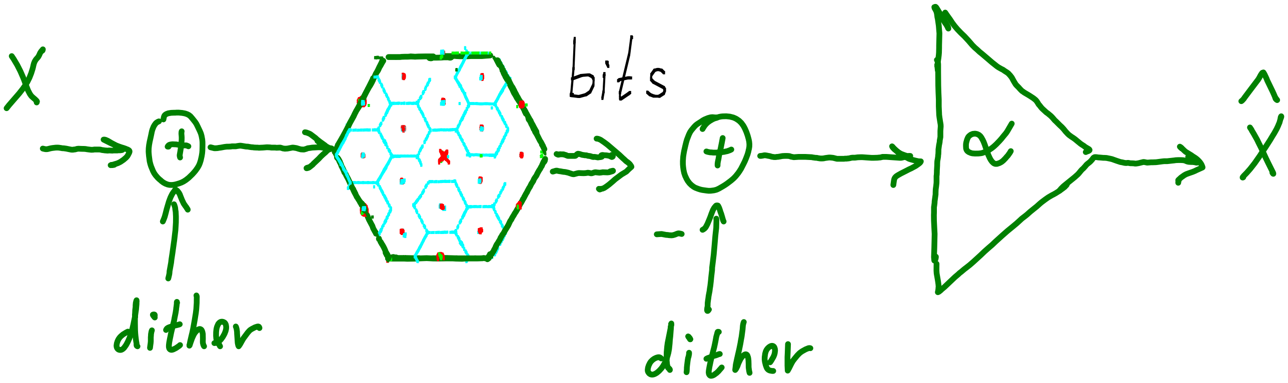


$\Rightarrow$ inflation-coefficient α affects equivalent noise distribution

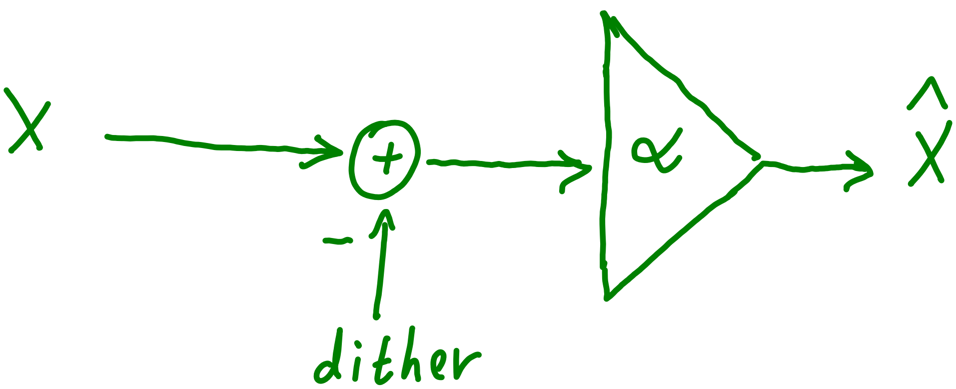
Effect of α on Lattice-Decoding



Voronoi Quantization



Crypto Lemma ⇒ Equivalent channel



$$\text{Distortion} = \mathrm{Var}\{\alpha \cdot \text{dither} + (1-\alpha)X\} = \frac{\sigma_x^2 \sigma_\Lambda^2}{\sigma_x^2 + \sigma_\Lambda^2}$$

@ $\alpha = \alpha_{\text{wiener}}$

$$\text{Rate} = \underbrace{\frac{1}{2}\log\left(\frac{\sigma_x^2}{D}\right)}_{\text{Q-Gaussian RDF}} - \underbrace{\frac{1}{2}\log\left(G(\Lambda_1)\cdot\mu(\Lambda_2, Pe, \alpha)\right)}_{\text{loss} \to 0 \text{ as } n \to \infty}$$

Next file ... →