ABSTRACT

Let S be a topological space and let E be a Banach space. We denote by $C_b(S;E)$ the Banach space of all bounded and continuous functions $f: S \to E$ equipped with the sup-norm $\|f\| = \sup\{\|f(s)\|; s \in S\}$. Suppose that a closed and convex non-empty subset $V \subset C_b(S;E)$, is given. In § 2 we study the following question and generalizations thereof: assume that we know that, for each $s \in S$, $\overline{V(s)}$ has the relative Chebyshev center property in E, then under what additional hypothesis, does V have the relative Chebyshev center property in $C_b(S;E)$?

To answer this question we introduce property (A) (see Def. 1 below) for a pair$(\mathcal{V},\mathcal{B})$, where $\mathcal{V} = \{\overline{V(s)}; s \in S\}$ and $\mathcal{B}$ is some class of bounded subsets of E.

In § 3 we study the question of continuity (with respect to the Hausdorff metric) of the Chebyshev center map $B \to \text{cent}(B;V)$.

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1. INTRODUCTION

Throughout this paper, S denotes a topological space, E denotes a Banach space, E^* denotes the topological dual of E, and $C_b(S;E)$ denotes the Banach space of all continuous and bounded functions from S into E, equipped with the sup-norm $\|f\| = \sup\{\|f(s)\|;\ s \in S\}$. If $a \in E$ and $r > 0$, then $B(a;r) = \{x \in E;\ \|x - a\| < r\}$ and $\bar{B}(a;r) = \{x \in E;\ \|x - a\| \leq r\}$. If V is some closed and non-empty subset of E, then for any $a \in E$, $\operatorname{dist}(a;V) = \inf\{\|v - a\|;\ v \in V\}$ and $P_V(a) = \{v \in V; \|v - a\| = \operatorname{dist}(a;V)\}$. If $P_V(a) \neq \phi$ for all $a \in E$, we say that V is *proximinal* in E. The elements of $P_V(a)$ are called *best approximants of* a *from* V. If $B \subset E$ is a bounded subset, then

$$\operatorname{rad}(B;V) = \inf\{\sup\{\|v - a\|;\ a \in B\};\ v \in V\}$$

and $\operatorname{cent}(B;V) = \{v \in V;\ \sup\{\|v - a\|;\ a \in B\} = \operatorname{rad}(B;V)\}$. The number $\operatorname{rad}(B;V)$ is called the *relative Chebyshev radius of* B *with respect to* V, and the elements of $\operatorname{cent}(B;V)$ are called *relative Chebyshev centers* of B with respect to V. When $V = E$ we write $\operatorname{rad}(B) = \operatorname{rad}(B;E)$ and $\operatorname{cent}(B) = \operatorname{cent}(B;E)$ and speak of the *Chebyshev radius of* B and *Chebyshev centers of* B. When $\operatorname{cent}(B;V) \neq \phi$ for all bounded non-empty subsets $B \subset E$, then we say that V has the *relative Chebyshev center property in* E. When E has the relative Chebyshev center property in E, we say that E *admits Chebyshev centers*.

If X is any non-empty subset of $C_b(S;E)$ then $X(s) = \{f(s);\ f \in X\} \subset E$, for any $s \in S$, and $\overline{X(s)}$ denotes the closure of $X(s)$ in E. Clearly, if X is convex (resp. a vector subspace), then $\overline{X(s)}$ is closed and convex (resp. a closed vector subspace).

Let V be a closed and convex non-empty subset of $C_b(S;E)$. Let A be a class of bounded non-empty subsets of $C_b(S;E)$ such that, for each $s \in S$ and $B \in A$, $\operatorname{cent}(B(s);\overline{V(s)}) \neq \phi$. In this paper we examine the following

QUESTION. *Under what assumptions is* cent$(B;V) \neq \phi$ *for any* $B \in A$? *In particular, when is* $P_V(f) \neq \phi$ *for any* $f \in C_b(S;E)$?

Similar questions, and particular cases of the question above, have been studied in the literature. See the references at the end of the paper.

DEFINITION 1. Let $\mathcal{V}$ be a class of closed and convex non-empty subsets of a Banach space E, and let $\mathcal{B}$ be a class of bounded non-empty subsets of E. We say that the pair $(\mathcal{V},\mathcal{B})$ *has property* (A) if, for every $\varepsilon > 0$ and $R > 0$, there exists $\delta > 0$ such that, given $V \in \mathcal{V}$, $B \in \mathcal{B}$ and $w \in V$ with $\mathrm{rad}(B;V) \leq R$ and $\|f - w\| < R + \delta$ for all $f \in B$, there exists $v \in V$ such that $\|v - w\| \leq \varepsilon$ and $\|f - v\| \leq R$ for all $f \in B$.

When $\mathcal{V}$ is a singleton, say $\{V\}$, and $(\mathcal{V},\mathcal{B})$ has property (A), then we say that $(V,\mathcal{B})$ has property (A). (See Definition 1, [19]).

Clearly, if $(\mathcal{V},\mathcal{B})$ has property (A) and $\mathcal{B}' \subset \mathcal{B}$, then $(\mathcal{V},\mathcal{B}')$ has property (A). In particular, if $(\mathcal{V},\mathcal{B}(E))$ has property (A), where $\mathcal{B}(E)$ is the class of *all* bounded and non-empty subsets of E, then $(\mathcal{V},\mathcal{B})$ has property (A), for any class $\mathcal{B}$ of bounded non-empty subsets of E.

DEFINITION 2. When $(\mathcal{V},\mathcal{B})$ has property (A) and $\mathcal{B}$ is the class of all singletons of E, i.e., $\mathcal{B} = \{\{f\}; f \in E\}$, then we say that $\mathcal{V}$ *has property* (A). If $\mathcal{V} = \{V\}$ and $\mathcal{V}$ has property (A), then we say that V *has property* (A).

Notice that $\mathcal{V}$ has property (A) if, given $\varepsilon > 0$ and $R > 0$, there exists $\delta > 0$ such that given $V \in \mathcal{V}$, $f \in E$ and $w \in V$ with $\mathrm{dist}(f;V) \leq R$ and $\|f - w\| < R + \delta$ there exists $v \in V$ such that $\|v - w\| \leq \varepsilon$ and $\|f - v\| \leq R$.

In our paper [17], where we announced some of the results presented here, we say that $\mathcal{V}$ has the *weak property* (w P) if, and only if, $\mathcal{V}$ has property (A) above.

When the elements of $\mathcal{V}$ are closed vector subspaces of E and the class $\mathcal{B}$ is invariant under translation, then in order to prove that the pair $(\mathcal{V},\mathcal{B})$ has property (A) it suffices to take $w = 0$ and $R = 1$.

It is easy to see that, when $(\mathcal{V},\mathcal{B})$ has property (A), then $\text{cent}(B;V) \neq \phi$ for any $V \in \mathcal{V}$ and $B \in \mathcal{B}$. When $\mathcal{V}$ has property (A), then each element $V \in \mathcal{V}$ is a proximinal subspace of E, i.e., $P_V(f) \neq \phi$ for any $f \in E$.

Notice also that if $\mathcal{V}$ is such that, for each $V \in \mathcal{V}$, the pair $(V,\mathcal{B})$ has property (A) and the corresponding $\delta = \delta(V,\varepsilon,R)$ does not depend on V, then the pair $(\mathcal{V},\mathcal{B})$ has property (A).

2. SPACES OF CONTINUOUS FUNCTIONS

THEOREM 1. *Let* S *be a paracompact and completely regular Hausdorff space. Let* A *be a class of bounded and equicontinuous non-empty subsets of* $C_b(S;E)$. *Let* $V \subset C_b(S;E)$ *be a closed and convex non-empty subset such that:*

(1) *for any* $B \in A$, *the pair* $(\mathcal{V},\mathcal{B})$ *has property* (A), *where* $\mathcal{V} = \{\overline{V(s)};\ s \in S\}$ *and* $\mathcal{B} = \{B(s);\ s \in S\}$;

(2) *for any* $g \in C_b(S;E)$, *if* $g(s) \in \overline{V(s)}$ *for all* $s \in S$, *then* $g \in V$.

Then $\text{cent}(B;V) \neq \phi$, *for any* $B \in A$.

PROOF. Let $B \in A$ be given, let $R = \text{rad}(B;V)$. We may assume $R > 0$, because $R = 0$ implies that B reduces to an element of V. Define

$$\varphi(s) = \{v \in \overline{V(s)};\ \|f(s) - v\| \leq R\ ,\ \text{ for all } f \in B\}$$

for each $s \in S$. Clearly, $\varphi(s)$ is closed and convex, and since $\text{rad}(B(s);\overline{V(s)}) \leq \text{rad}(B;V)$, it follows that $\varphi(s) \neq \phi$, because the pair $(\mathcal{V},\mathcal{B})$ has property (A) and therefore $\text{cent}(B(s);\overline{V(s)}) \neq \phi$.

We claim that the set-valued mapping φ is lower-semicontinuous. Let $s_o \in S$, $a \in E$ and $r > 0$ be given with $\varphi(s_o) \cap B(a;r) \neq \phi$. Choose w in $\varphi(s_o) \cap B(a;r)$ and then choose $t > 0$ such that $\|w - a\| < t < r$. Let $\varepsilon = r - t > 0$, and let $\delta > 0$ be given by property (A) applied to $\varepsilon > 0$ and $R > 0$. Notice that $\|f(s_o) - w\| \leq R$ for all $f \in B$. By equicontinuity of B at s_o there exists a neighborhood N of s_o in S such that $\|f(s) - f(s_o)\| < \delta$ for all $f \in B$, and $s \in N$. Hence $\|f(s) - w\| < R + \delta$ for all $f \in B$ and $s \in N$. By property (A), given $s \in N$, there is some $v_s \in \overline{V(s)}$ such that $\|f(s) - v(s)\| \leq R$, for all $f \in B$, and $\|v_s - w\| \leq \varepsilon$. Then $\|v_s - a\| \leq \|v_s - w\| + \|w - a\| < \varepsilon + t = r$. Hence v_s belongs to $\varphi(s) \cap B(a;r)$, for each $s \in N$. Consequently, the set-valued mapping φ is lower-semicontinuous.

By Michael's selection theorem [16], there is some $g \in C_b(S;E)$ such that $g(s) \in \varphi(s)$ for each $s \in S$. Hence $\|f(s) - g(s)\| \leq R$ for all $s \in S$, and therefore $\|f - g\| \leq R = \mathrm{rad}(B;V)$. On the other hand, $g(s) \in \overline{V(s)}$ for all $s \in S$, and then by (2), g belongs to V. Hence $g \in \mathrm{cent}(B;V)$.

THEOREM 2. *Let* S *be as in Theorem* 1. *Let* $V \in C_b(S;E)$ *be a closed and convex non-empty subset such that:*

(1) $\mathcal{V} = \{\overline{V(s)};\ s \in S\}$ *has property* (A);

(2) *for any* $g \in C_b(S;E)$, *if* $g(s) \in \overline{V(s)}$ *for all* $s \in S$, *then* $g \in V$.

Then V *is proximinal in* $C_b(S;E)$.

PROOF. Let $\mathcal{A} = \{\{f\};\ f \in C_b(S;E)\}$. Given any $f \in C_b(S;E)$, then $B = \{f\}$ belongs to $\mathcal{A}$ and $B(s) = \{f(s)\}$. It follows from (1), that the pair $(\mathcal{V},\mathcal{B})$ has property (A), where $\mathcal{B} = \{f(s);\ s \in S\}$. By Theorem 1, $P_V(f) \neq \phi$.

LEMMA 1. *Let* $\mathcal{V}$ *be a class of closed and convex non-empty subsets of a Banach space* E.

(1) *If* $\mathcal{V}$ *has property* (A), *then* $\mathcal{V} \cup \{E\}$ *has property* (A).

(2) $\mathcal{V} = \{E,\{0\}\}$ *has property* (A).

PROOF (1) Let $\varepsilon > 0$ and $R > 0$. Choose $\delta > 0$ by property (A) valid for $\mathcal{V}$. Without loss of generality we may assume $\delta \leq \varepsilon$. Let $f \in E$, $w \in V$ and $V \in \mathcal{V} \cup \{E\}$ be given with $\mathrm{dist}(f;V) \leq R$ and $\|f-w\| < R+\delta$. If $V \in \mathcal{V}$ there is nothing to prove. If $V = E$ and $\|f-w\| \leq R$, then $v = w$ belongs to E and $\|f-v\| \leq R$, while $\|v-w\| = 0 < \varepsilon$. Suppose now that $V = E$ and $\|f-w\| > R$. Let $v = f - \dfrac{R}{\|f-w\|}(f-w)$. Then $v \in E = V$, and $\|v-w\| = \|f-w\| - R < \delta \leq \varepsilon$. On the other hand $\|f-v\| = R$.

(2) follows from (1) and the obvious fact that $\{0\}$ has property (A).

COROLLARY 1. *Let* S *be as in Theorem 1, and let* $T \subset S$ *be a closed non-empty subset. Then a closed and convex non-empty subset* V *of* $C_b(S;E)$ *is proximinal in each of the following cases:*

(1) V *satisfies condition* (2) *of Theorem 1 and* $\mathcal{V} = \{\overline{V(t)};\, t \in T\}$ *has property* (A), *while* $\overline{V(s)} = E$ *for each* $s \in S$, $s \notin T$;

(2) $V = \{f \in C_b(S;E);\, f(T) \subset F\}$, *where* $F \subset E$ *is a closed and convex non-empty subset which has property* (A);

(3) $V = C_b(S;F)$ *with* F *as in part* (2);

(4) $V = \{f \in C_b(S;E);\, f(t) = 0$ *for all* $t \in T\}$;

(5) $V = \{f \in C_b(S;E);\, \varphi(f(t)) = 0$ *for all* $t \in T$ *and* $\varphi \in F\}$ *where* $F \subset E^*$ *is such that* $N = \cap\{\varphi^{-1}(0);\, \varphi \in F\}$ *has property* (A) *in* E.

(6) $V = \{f \in C_b(S;E);\ \varphi(f(s)) = 0\}$ *for some* $s \in S$, *and* $\varphi \in E^*$ *is such that* $\varphi^{-1}(0)$ *has property* (A) *in* E.

PROOF. (1) By Lemma 1, part (1), $V \cup \{E\}$ has property (A), and the result follows from Theorem 2.

(2) The subset V clearly satisfies condition 2 of Theorem 1, and the result follows from part (1), since $V(s) = E$ for any $s \in S$, $s \notin T$, because S is completely regular.

(3) Take $T = S$ and apply part (2).

(4) By Lemma 1, the family $\{E, \{0\}\}$ has property (A).

(5)-(6) In both cases V satisfies condition (2) of Theorem 2.

REMARK. Part (4) of Corollary 1 was proved in Prolla [20] by a different method. (See Theorem 2.6, [20]). Yost ([26], Proposition 2.4) proved that $V = \{f \in C(S;E);\ f(t) = 0$ for all $t \in T\}$, for compact S, has the n-ball property in C(S;E) and hence is an M-ideal. His proof applies to $C_b(S;E)$ for paracompact S. Later on (see Theorem 8 below) we shall generalize this result.

THEOREM 3. *Let* S *be a compact Hausdorff space and let* $V \subset C(S;E)$ *be a* $C(S;\mathbb{K})$*-module such that the pair* (V,B) *has property* (A), *where* $V = \{\overline{V(s)};\ s \in S\}$ *and* B *is a class of bounded non-empty subsets of* E. *Then* $\mathrm{cent}(B;V) \neq \phi$ *for any bounded and equicontinuous subset* $B \subset C(S;E)$ *such that* $B(s) \in B$ *for all* $s \in S$.

PROOF. By Theorem 1.11 of Prolla [22], (taking $A = C(S;\mathbb{K})$) any $C(S;\mathbb{K})$-module V verifies condition (2) of Theorem 1.

COROLLARY 2. *Let* S *be a compact Hausdorff space and let* $V \subset C(S;E)$ *be a* $C(S;\mathbb{K})$*-module such that* $(V,K(E))$ *has property* (A) *where* $V = \{\overline{V(s)};\ s \in S\}$ *and* $K(E)$ *is the class of all relatively compact* *non-empty subsets of* E. *Then* $\mathrm{cent}(B;V) \neq \phi$ *for any relatively compact non-empty subset* $B \subset C(S;E)$.

COROLLARY 3. *Let* S *be a compact Hausdorff space and let* $V \subset C(S;E)$ *be a* $C(S;\mathbb{K})$*-module such that* $\mathcal{V}$ *has property* (A), *where* $\mathcal{V} = \{\overline{V(s)};\ s \in S\}$. *Then* V *is proximinal in* C(S;E).

THEOREM 4. *Let* S *be as in Theorem 1, and let* E *be a real Lindenstrauss space. Let* V *be a closed vector subspace of* $C_b(S;E)$ *such that*

(1) *for each* $s \in S$, $\overline{V(s)}$ *is an* M*-ideal in* E;

(2) *for any* $g \in C_b(S;E)$, *if* $g(s) \in \overline{V(s)}$ *for all* $s \in S$, *then* $g \in V$.

Then $\mathrm{cent}(B;V) \neq \phi$, *for any relatively compact non-empty subset* B *of* $C_b(S;E)$. *In particular*, V *is proximinal in* $C_b(S;E)$.

PROOF. Let $\mathcal{A} = \{B \subset C_b(S;E)$; B is relatively compact and non-empty$\}$. Then each element of $\mathcal{A}$ is bounded and equicontinuous. For any $B \in \mathcal{A}$ and $s \in S$, the set $B(s)$ is a relatively compact and non-empty subset of E, i.e., $B(s) \in \mathcal{K}(E)$. From Example 1, [19], the pair $(\overline{V(s)}, \mathcal{K}(E))$ has property (A) with $\delta = \varepsilon$. Hence $(\mathcal{V}, \mathcal{K}(E))$ has property (A) and, a fortiori, the same is true of $(\mathcal{V}, \mathcal{B})$, where $\mathcal{B} = \{B(s);\ s \in S\}$. By Theorem 1, $\mathrm{cent}(B;V) \neq \phi$.

COROLLARY 4. *Let* S *and* E *be as in Theorem 4. Let* B *be a relatively compact and non-empty subset of* $C_b(S;E)$. *Then* $\mathrm{cent}(B;V) \neq \phi$ *in the following cases:*

(1) $V = C_b(S;E)$.

(2) $V = \{f \in C_b(S;E);\ f(T) \subset M\}$, *where* T *is a closed non-empty subset of* S *and* $M \subset E$ *is an* M*-ideal*.

(3) $V = C_b(S;M)$, *where* $M \subset E$ *is an M-ideal*.

(4) $V = \{f \in C_b(S;E);\ f(t) = 0$ *for all* $t \in T\}$, *where* T *is a closed non-empty subset of* S.

(5) S *is compact, and* $V \subset C(S;E)$ *is a* $C(S;\mathbb{K})$*-module such that for each* $s \in S$, $\overline{V(s)}$ *is an M-ideal*.

PROOF. (1)-(4). In each case , $\overline{V(s)}$ is an M-ideal in E, for any $s \in S$, and condition (2) of Theorem 4 is easily verified.

(5) follows from Corollary 2.

THEOREM 5. *Let* S *be as in Theorem 1, and let* E *be a uniformly convex Banach space. Let* V *be a closed and convex non-empty subset of* $C_b(S;E)$ *such that condition (2) of Theorem 1 is verified.*

Then $\mathrm{cent}(B;V) \neq \phi$, *for any equicontinuous and bounded non-empty subset* B *of* $C_b(S;E)$. *In particular,* V *is proximinal in* $C_b(S;E)$.

PROOF. Let $\mathcal{V} = \{\overline{V(s)};\ s \in S\}$, and let $\mathcal{B}(E)$ be the class of all bounded and non-empty subsets of E. We claim that $(\mathcal{V},\mathcal{B}(E))$ has property (A). Let $\varepsilon > 0$ and $R > 0$ be given. Choose $\delta(\varepsilon)$ and $h: E \times E \to E$ by Proposition 1 , Mach [13]. Let $B \in \mathcal{B}(E)$, $W \in \mathcal{V}$ and $w \in W$ be given with $\mathrm{rad}(B;W) \leq R$, and $\|f - w\| < R + \delta$ for all $f \in B$. Choose an element $u \in \mathrm{cent}(B;W)$. Therefore $B \subset \bar{B}(w;R + \delta(\varepsilon)) \cap \bar{B}(u;R) \subset \bar{B}(h(w,u);R)$ by taking $\theta = 0$ in Property (P_2) of Mach [13]. It remains to notice that h maps $W \times W$ into W and $\|h(w,u) - w\| \leq \varepsilon$.

THEOREM 6. *Let* S *be as in Theorem 1, and let* E *be a uniformly convex Banach lattice, and let* $V = [a,b] = \{f \in C_b(S;E); a \leq f \leq b\}$. *If* $V \neq \phi$, *then* $\mathrm{cent}(B;V) \neq \phi$ *for any equicontinuous and bounded non-empty subset* $B \subset C_b(S;E)$. *In particular* , V *is proximinal in* $C_b(S;E)$. (*Here* $a,b \in \ell_\infty(S;E)$.)

PROOF. For each $s \in S$, $\overline{V(s)} \subset \{v \in E \,;\, a(s) \leq v \leq b(s)\}$. On the other hand, if $f \in C_b(S;E)$ is such that $f(s) \in \overline{V(s)}$ for all $s \in S$, then $a(s) \leq f(s) \leq b(s)$ for all $s \in S$ and $f \in [a,b] = V$. Hence V satisfies condition (2) of Theorem 1.

REMARK. Theorem 6 applies to $E = L_p(X,\Sigma,\mu)$, $1 < p < \infty$, for any measure space (X,Σ,μ). It also applies to $E = \mathbb{R}$. Therefore, for normal S, it generalizes the result of Franchetti and Cheney [6, Theorem 3.3] which says that any order interval $[u,v] \subset C_b(S;\mathbb{R})$ is proximinal, if S is normal, u is upper semicontinuous and v is lower semicontinuous. For other cases in which $V \neq \phi$, see Theorem 3.13, Roversi [23]. In case $E = \mathbb{R}$, Theorem 6 above fails to recapture the full generality of Proposition 3.12 of Roversi [23] because we have to assume here that S is paracompact, and B is equicontinuous, whereas Proposition 3.12 of Roversi [23] says that $\mathrm{cent}(B;V) \neq \phi$ for *any* bounded non-empty subset B of $\ell_\infty(S;\mathbb{R})$, if $V = \{f \in C_b(S;\mathbb{R})\,;\, a \leq f \leq b\}$ is non-empty, where a and b are two arbitrary elements of $\ell_\infty(S;\mathbb{R})$. On the other hand, the methods of [23] apply to any uniformly convex Banach lattice. Hence the following generalization of Proposition 3.12 of Roversi [23] is true.

THEOREM 7. *Let X be a non-empty set and let E be a uniformly convex Banach lattice, and let $a,b \in \ell_\infty(X;E)$. Let $W = [a,b] = \{f \in \ell_\infty(X;E)\,;\, a \leq f \leq b\}$. Then W has the relative Chebyshev center property in $\ell_\infty(X;E)$. If X is a topological space and $V = \{f \in C_b(X;E)\,;\, a \leq f \leq b\}$ is non-empty, then V has the relative Chebyshev center property in $\ell_\infty(X;E)$ and, a fortiori, in $C_b(X;E)$.*

COROLLARY 5. *Let S and E be as in Theorem 5. Let B be an equicontinuous and bounded non-empty subset of $C_b(S;E)$. Then $\mathrm{cent}(B;V) \neq \phi$ in the following cases:*

(1) $V = C_b(S;E)$.

(2) $V = \{f \in C_b(S;E);\ f(T) \subset F\}$, *where* T *is a closed non-empty subset of* S *and* F *is a closed and convex non-empty subset of* E.

(3) $V = C_b(S;F)$, *where* F *is any closed and convex non-empty subset of* E.

(4) $V = \{f \in C_b(S;E);\ f(t) = 0$, *for all* $t \in T\}$, *where* T *is a closed non-empty subset of* S.

(5) $V = \{f \in C_b(S;E);\ \varphi(f(t)) = 0$, *for all* $t \in T$, $\varphi \in F\}$, *where* T *is a closed non-empty subset of* S *and* F *is a non-empty subset of* E^*.

(6) $V = \{f \in C_b(S;E);\ \varphi(f(s)) = 0\}$ *for some* $s \in S$ *and* $\varphi \in E^*$.

(7) S *is compact and* $V \subset C(S;E)$ *is a closed* $C(S;\mathbb{K})$-*module*.

COROLLARY 6. *Let* S *be as in Theorem 1. Let* B *be an equicontinuous and bounded non-empty subset of* $C_b(S;E)$. *Then* $\operatorname{cent}(B) \neq \phi$ *in the following cases:*

(1) $(E;\mathcal{B}(E))$ *has property* (A), *where* $\mathcal{B}(E)$ *is the class of all bounded and non-empty subsets of* E.

(2) E *is quasi-uniformly convex.*

PROOF. (1) Apply Theorem 1 with $V = C_b(S;E)$.

(2) If E is quasi-uniformly convex (with respect to itself), then by Example 2 of [19], the pair $(E;\mathcal{B}(E))$ has property (A).

COROLLARY 7. *Let* S *and* B *be as in Corollary 6, and let* V *be a closed vector subspace of* E. *Then* $\operatorname{cent}(B;C_b(S;V)) \neq \phi$ *is the following cases:*

(1) $(V;\mathcal{B}(E))$ *has property* (A), *where* $\mathcal{B}(E)$ *is the class of all bounded and non-empty subsets of* E.

(2) E *is quasi-uniformly convex with respect to* V.

PROOF. (1) Apply Theorem 1 to $C_b(S;V) \subset C_b(S;E)$.

(2) If E is quasi-uniformly convex with respect to V, then by Example 2 of [19], the pair $(V,\mathcal{B}(E))$ has property (A).

REMARK. Corollary 7 was proved in [19].

COROLLARY 8. *Let* S *be as in Theorem 1 and let* $V \subset C_b(S;E)$ *be a closed vector subspace satisfying condition* (2) *of Theorem 1. Then* V *is proximinal in* $C_b(S;E)$ *in each of the following cases:*

(1) *for each* $s \in S$, $\overline{V(s)}$ *has the* $1\frac{1}{2}$*-ball property in* E;

(2) *for each* $s \in S$, $\overline{V(s)}$ *is an M-ideal in* E;

(3) *for each* $s \in S$, $\overline{V(s)}$ *is an M-summand*;

(4) *for each* $s \in S$, $\overline{V(s)}$ *is a semi-L-summand*;

(5) *for each* $s \in S$, $\overline{V(s)}$ *is an L-summand*.

PROOF. (1) Let $\mathcal{V} = \{\overline{V(s)};\, s \in S\}$. Let $\varepsilon > 0$ and $R > 0$ be given. Choose $\delta = \varepsilon$. Let $s \in S$ and $f \in E$ be given with $\mathrm{dist}(f;\overline{V(s)}) \leq R$ and $\|f\| < R + \delta$. Since the $1\frac{1}{2}$-ball property implies proximinality (Lemma 1.1, Yost [27]), $\overline{V(s)} \cap \bar{B}(f;R) \neq \phi$. By the $1\frac{1}{2}$-ball property, $\overline{V(s)} \cap \bar{B}(f;R) \cap \bar{B}(0,\varepsilon) \neq \phi$, and therefore $\mathcal{V}$ has property (A), and the result follows from Theorem 2.

(2) Every M-ideal in E has the $1\frac{1}{2}$-ball property, since it has a much stronger intersection property (see, e.g. Behrends [4], p. 52).

(3) Every M-summand is an M-ideal.

(4) We show that every semi-L-summand has the $1\frac{1}{2}$-ball

property. A semi-L-summand W is a Chebyshev subspace of E such that the metric projection P_W satisfies

$$\|f\| = \|P_W(f)\| + \|f - P_W(f)\|$$

for every $f \in E$ (Lima [12], section 5). Let $a_1 \in W$, $W \cap \bar{B}(a_2,r_2) \neq \phi$ and $\|a_1 - a_2\| < r_1 + r_2$. Let $d = \mathrm{dist}(a_2,W)$. Then $d \leq r_2$ and

$$\|P_W(a_2 - a_1) - (a_2 - a_1)\| = \|P_W(a_2) - a_2\| = d .$$

Hence

$$\|P_W(a_2) - a_1\| = \|P_W(a_2 - a_1)\| = \|a_2 - a_1\| - \|P_W(a_2 - a_1) - (a_2 - a_1)\| <$$
$$< r_1 + r_2 - d .$$

If $d = r_2$, then $\|P_W(a_2) - a_1\| < r_1$ and $P_W(a_2) \in W \cap \bar{B}(a_1;r_1) \cap \bar{B}(a_2;r_2)$.

If $d < r_2$, let $\varepsilon = r_2 - d > 0$. Hence $\|P_W(a_2) - a_1\| < r_1 + \varepsilon$, and for some $v \in W$ we have $\|P_W(a_2) - v\| < \varepsilon$ and $\|a_1 - v\| < r_1$. Now $\|v - a_2\| \leq \|v - P_W(a_2)\| + \|P_W(a_2) - a_2\| < \varepsilon + d = r_2$, and $v \in W \cap \bar{B}(a_1;r_1) \cap \bar{B}(a_2;r_2)$.

(5) Every L-summand is a semi-L-summand.

REMARK. Part (1) of Corollary 8 was proved directly in [21, Theorem 2]. When each $\overline{V(s)}$ is an M-ideal, then more can be said about V. See Theorem 8 below.

DEFINITION 3. Let E be Banach space and let $\mathcal{V}$ be a collection of closed vector subspaces of E. We say that $\mathcal{V}$ has property

(U) if there exists a function $\delta : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$ and

$$((1+\varepsilon)\bar{B}) \cap (V + \bar{B}) \subset B + \delta(\varepsilon)\cdot(\bar{B} \cap V)$$

for every $V \in \mathcal{V}$, where $\bar{B} = \{v \in E; \|v\| \leq 1\}$.

This definition, for $\mathcal{V}$ a singleton $\{V\}$ is due to K. S. Lau [11], who then says that V is a U-proximinal subspace of E. It follows from Proposition 2.3, Lau [11] that when $\mathcal{V}$ has property (U), each element of $\mathcal{V}$ is proximinal in E. In fact, when V is U-proximinal then V admits a continuous proximity map (Lau [11], Corollary 3.5).

REMARK. A closed vector subspace V of a Banach space E is said to be a *semi-M-ideal* if its annihilator $V^0 = \{\varphi \in E^*; \varphi(v) = 0, v \in V\}$ is a semi-L-summand in E^*. This notion is due to Lima [12]. Any M-ideal is a semi-M-ideal, the converse being false in general.

It follows from Theorem 5.4 of Lau [11] that, if each element of $\mathcal{V}$ is a semi-M-ideal, then $\mathcal{V}$ has property (U), with $\delta(\varepsilon) = 2\varepsilon$.

COROLLARY 9. *Let S be as in Theorem 1 and let $V \subset C_b(S;E)$ be a closed vector subspace satisfying condition (2) of Theorem 1. Then V is proximinal in each of the following cases:*

(1) $\mathcal{V} = \{\overline{V(s)}; s \in S\}$ *has property* (U);

(2) *for each* $s \in S$, $\overline{V(s)}$ *is a semi-M-ideal.*

PROOF. The result follows from the following lemma.

LEMMA 2. *If $\mathcal{V}$ has property* (U), *then it has property* (A).

PROOF. Let $\varepsilon > 0$ and $R > 0$ be given. Choose $\eta > 0$ such that $R\delta(\eta) < \varepsilon$ and then choose $\delta > 0$ such that $\delta < \eta R$.

Let $f \in E$ and $V \in \mathcal{V}$ be given with $\text{dist}(f;V) \leq R$ and $\|f\| < R + \delta$. Hence f belongs to $((R+\delta)\bar{B}) \cap (V + R\bar{B})$, since each $V \in \mathcal{V}$ is proximinal (Proposition 2.3, Lau [11]). By property (U), f/R belongs to $\bar{B} + \delta(\eta)\bar{B} \cap V$. Hence f belongs to $R \cdot \bar{B} + \varepsilon \cdot \bar{B} \cap V$ and $\mathcal{V}$ has property (A).

COROLLARY 10. *Let* S *be as in Theorem 1. Let* T *be a closed non-empty subset of* S *and let* F *be a closed vector subspace of* E. *Then* $V = \{f \in C_b(S;E);\ f(T) \subset F\}$ *and* $V = C_b(S;F)$ *are proximinal in* $C_b(S;E)$ *in each of the following cases:*

(0) F *has property* (A).

(1) F *has the* $1\frac{1}{2}$*ball property in* E.

(2) F *is* U*-proximinal in* E.

(3) F *is an* M*-ideal in* E.

(4) F *is an* M*-summand in* E.

(5) F *is a semi-*L*-summand in* E.

(6) F *is an* L*-summand in* E.

(7) F *is a semi-*M*-ideal in* E.

(8) E *is quasi-uniformly convex with respect to* F.

(9) E *is uniformly convex with respect to* F.

(10) E *is uniformly convex.*

(11) E *is uniformly convex in every direction and* F *is finite dimensional.*

COROLLARY 11. *Let* S *be as in Theorem 1. Let* T *be a closed and non-empty subset of* S *and let* F *be a closed vector subspace of* E. *Then* $\{f \in C_b(S;E^*);\ f(T) \subset F^{\circ}\}$ *and* $C_b(S;F^{\circ})$ *are proximinal in* $C_b(S;E^*)$ *in each of the following cases:*

(1) F *has the* 1½-*ball property in* E ;

(2) F *is a semi-M-ideal of* E ;

(3) F *is an M-ideal of* E ;

(4) F *is an M-summand of* E ;

(5) F *is an L-summand of* E .

PROOF: (1) The annihilator F° has the 1½-ball property in E*. See Yost [27], p. 289.

(2) If F is a semi-M-ideal , then its annihilator F° is a semi-L-summand of E*, and the result follows from part (5) of Corollary 10.

(3) If F is an M-ideal, then F° is an L-summand of E*, and the result follows from part (6) of Corollary 10 . Since every L-summand is also a semi-L-summand the result follows also from part (2) above.

(4) Every M-summand is an M-ideal .

(5) If F is an L-summand of E , then its annihilator F° is an M-summand of E* and the result follows from part (4) of Corollary 10.

REMARK. If $V \subset C_b(S;E)$ satisfies condition (2) of Theorem 1, and each $\overline{V(s)}$ is an M-ideal , then it follows from Corollary 8, part (2), that V is proximinal in $C_b(S;E)$. Let us show that in fact V is an M-ideal of $C_b(S;E)$.

THEOREM 8. *Let* S *be as in Theorem 1 and let* V *be a closed vector subspace of* $C_b(S;E)$ *satisfying condition (2) of Theorem 1 and such that, for each* $s \in S$, $\overline{V(s)}$ *is an M-ideal of* $C_b(S;E)$. *Then* V *is an M-ideal of* $C_b(S;E)$.

PROOF. We show that V has the n-ball property for closed balls, for all $n \in \mathbb{N}$. (See Behrends [4], pg. 52). Let $\bar{B}_i = \bar{B}(f_i;r_i)$ $(i = 1,2,\dots,n)$ be n closed balls in $C_b(S;E)$ such that $\bar{B}_i \cap V \neq \phi$ (for all $i = 1,2,\dots,n$) and $\bigcap_{i=1}^{n} B_i \neq \phi$, where $B_i = B(f_i;r_i)$ is the corresponding open ball. We claim that $\bigcap_{i=1}^{n} \bar{B}_i \cap V \neq \phi$. For each $s \in S$, let

$$\varphi(s) = \{a \in E;\ \|f_i(s) - a\| \leq r_i\ ,\ i = 1,2,\dots,n\} \cap \overline{V(s)}$$

Clearly, each $\varphi(s)$ is closed and convex. On the other hand, if we take $g_i \in \bar{B}_i \cap V$, then $\|g_i(s) - f_i(s)\| \leq r_i$ and $g_i(s) \in V(s)$ $(i = 1,\dots,n)$. Hence $\bar{B}(f_i(s);r_i) \cap \overline{V(s)} = \phi$ for all $i = 1,\dots,n$. Finally, take $g \in \bigcap_{i=1}^{n} B_i$. Then $g(s) \in \bigcap_{i=1}^{n} B(f_i(s);r_i)$. Since $\overline{V(s)}$ is an M-ideal, the intersection $\varphi(s) = \bigcap_{i=1}^{n} \bar{B}(f_i(s);r_i) \cap \overline{V(s)} \neq \phi$.

We claim that φ is lower-semicontinuous. Let $s_o \in S$, $a \in E$ and $r > 0$ be given such that $\varphi(s_o) \cap B(a;r) \neq \phi$. Choose $w_o \in \overline{V(s)}$ such that $\|f_i(s_o) - w_o\| \leq r_i$, $\|a - w_o\| < r$. Choose $t > 0$ such that $\|a - w_o\| < t < r$.

Then $\varphi(s_o) \cap B(a;t) \neq \phi$. Hence $K \cap B(a;t) \neq \phi$ is also true, where K is the closed and convex set $\bigcap_{i=1}^{n} \bar{B}(f_i(s_o);r_i)$. On the other hand, its interior K^o is non-empty, since $g(s_o) \in \bigcap_{i=1}^{n} B(f_i(s_o);r_i) = K^o$. Hence $K = \overline{K^o}$, and therefore $\overline{K^o} \cap B(a;t) \neq \phi$ implies that for some $v \in E$,

$$v \in \bigcap_{i=1}^{n} B(f_i(s_o);r_i) \cap B(a;t)\ .$$

By continuity, there is some neighborhood N of s_o in S such that

$$v \in \bigcap_{i=1}^{n} B(f_i(s);r_i) \cap B(a;t)$$

for all $s \in N$. Since $\overline{V(s)} \cap \bar{B}(a;t) \neq \phi$, and $g_i(s) \in \bar{B}(f_i(s);r_i) \cap \overline{V(s)}$ $(i = 1,2,\ldots,n)$, and $\overline{V(s)}$ is an M-ideal and therefore has the $(n+1)$-ball property for closed balls,

$$V(s) \cap [\bigcap_{i=1}^{n} \bar{B}(f_i(s);r_i) \cap \bar{B}(a;t)] \neq \phi$$

for all $s \in N$. But $t < r$ implies that $\varphi(s) \cap B(a;r) \neq \phi$ for all $s \in N$. Hence φ is lower-semicontinuous. By Michael's selection theorem [16], there exists $h \in C_b(S;E)$ such that $h(s) \in \varphi(s)$ for all $s \in S$. Hence $h(s) \in \overline{V(s)}$, for all $s \in S$, and by condition (2) of Theorem 1, $h \in V$. On the other hand, $\|f_i(s) - h(s)\| \leq r_i$ $(i = 1,2,\ldots,n)$, for all $s \in S$. Hence $\|f_i - h\| \leq r_i$ $(i = 1,2,\ldots,n)$ and $h \in \bigcap_{i=1}^{n} \bar{B}_i \cap V$.

COROLLARY 12. *Let* S *be as in Theorem 1 and let* V *be an M-ideal of* E. *Let* T *be a closed non-empty subset of* S. *Then* $\{f \in C_b(S;E);\ f(T) \subset V\}$, *and in particular* $C_b(S;V)$, *is an M-ideal of* $C_b(S;E)$.

COROLLARY 13. *For any Banach space* E, $c_o(E)$ *is an M-ideal in* $\ell_\infty(E)$.

PROOF. Let $\mathbb{N}$ be the natural numbers equipped with the discrete topology, and let S be its one-point compactification. Then $\ell_\infty(E)$ is isometric to $C(S;E)$, and by taking $T = \{\infty\}$ and $V = \{0\}$, then $c_o(E)$ is isometric to $\{f \in C(S;E);\ f(T) \subset V\}$. Corollary 13 is due to D. T. Yost. See Lemma 2.6, Yost [27].

REMARKS. (a) If $V = \{0\}$, then V is an M-ideal of E, and by Corollary 12 $\{f \in C_b(S;E);\ f(t) = 0$, for all $t \in T\}$ is an M-ideal. This was proved for compact S by Yost [27, Proposition 2.4].

(b) The preceding results suggest the following question: if V satisfies condition (2) of Theorem 1, then under what assumption on $\mathcal{V} = \{\overline{V(s)};\ s \in S\}$ will V have property (A) or will V have the 1½-ball property in $C_b(S;E)$? The later question was answered by Theorem 1 of [21]: if each $\overline{V(s)}$ has the 2½-ball property in E then V has the 1½-ball property in $C_b(S;E)$. Let us recall the definition of the 2½-ball property: a closed subspace V of a Banach space E has the 2½-ball property in E if given $f,g \in E$, $r > 0$, $s > 0$ and $\varepsilon > 0$ such that $\bar{B}(f;r) \cap V \neq \phi$, $\bar{B}(g;s) \cap V \neq \phi$ and $B(f;r) \cap B(g;s) \cap B(0;\varepsilon) \neq \phi$ then $\bar{B}(f;r) \cap \bar{B}(g;s) \cap \bar{B}(0;\varepsilon) \cap V \neq \phi$.

The space $V = K(\ell_1)$ of compact operators in ℓ_1 has the 1½-ball property in the space $E = L(\ell_1)$ of all bounded operators in ℓ_1. (See Proposition 2.8 of Yost [27].) However, it does not have the 2½-ball property. Indeed, by a result of Smith and Ward ([25], Theorem 6.2), it fails even the 2-ball property in $L(\ell_1)$.

It was shown in Yost [27], Theorem 2.1, that every Stone--Weierstrass subspace of $C(S;E)$ has the 1½-ball property, for compact S and real Lindenstrauss E. A slight modification of his proof shows that it has the 2½-ball property.

3. CONTINUITY OF THE CHEBYSHEV CENTER MAP

In this section we study the continuity of the Chebyshev center map $B \to \text{cent}(B;V)$. To do this we recall the definition of the Hausdorff metric d_H defined on pairs of bounded non--empty subsets K and L of E

$$d_H(K,L) = \inf\{r > 0 \; ; \; K \subset L + rU \; , \; L \subset K + rU\}$$

where $U = \{v \in E \, ; \, \|v\| \leq 1\}$.

DEFINITION 4. Let V and B be as in Definition 1. We say that *the pair* (V,B) has property (B) if, for every $\varepsilon > 0$ and $R > 0$, there exists $\delta > 0$ such that, given $0 < r \leq R$, $V \in V$, $B \in B$ and $w \in V$ with $\mathrm{rad}(B;V) \leq r$ and $\|f - w\| < r + \delta$ for all $f \in B$, there exists $v \in V$ such that $\|v - w\| \leq \varepsilon$ and $\|f - v\| \leq r$ for all $f \in B$.

When V is a singleton, say $\{V\}$, and (V,B) has property (B), then we say that (V,B) has property (B).

Clearly, property (B) $\Rightarrow$ property (A).

DEFINITION 5. When (V,B) has property (B) and B is the class of all singletons of E, then we say that V *has property* (B). If $V = \{V\}$ has property (B), we say that V *has property* (B).

Notice that V has property (B) if, given $\varepsilon > 0$ and $R > 0$, there exists $\delta > 0$ such that given $0 < r \leq R$, $V \in V$, $f \in E$, and $w \in V$ with $\mathrm{dist}(f;V) \leq r$ and $\|f - w\| < r + \delta$, there exists $v \in V$ such that $\|f - v\| \leq r$ and $\|v - w\| \leq \varepsilon$.

In [17] we say that V has property (P) if, and only if, V has property (B).

EXAMPLE 1. Let V be a class of closed vector subspaces of a Banach space E. In each of the following examples, V has property (B):

(1) every element of V has the $1\frac{1}{2}$-ball property in E.

(2) V has property (U) (see Def. 3) and the function $\delta : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the following condition: given $\varepsilon > 0$ and $R > 0$, there exists $\eta > 0$ such that $r\delta(\eta/r) < \varepsilon$ for all $0 < r \leq R$.

PROOF: (1) Similar to part (1) of Corollary 8.

(2) Let $\varepsilon > 0$ and $R > 0$ be given. Choose $\eta > 0$ such that $r\delta(\eta/r) < \varepsilon$ for all $0 < r \leq R$.

Let $f \in E$, $W \in \mathcal{V}$ and $0 < r \leq R$ be given with $\mathrm{dist}(f;W) \leq r$ and $\|f\| < r + \eta$. By Proposition 2.3 of Lau [11], W is a proximinal subspace of E. Hence f belongs to $((r+\eta)U) \cap (W + rU)$, where U is the closed unit ball of E, and $r^{-1}f$ belongs to $U + \delta(r^{-1}\eta)(U \cap W)$. Therefore f belongs to $rU + \varepsilon(U \cap W)$.

REMARK. Condition (2) of Example 1 is satisfied when $\mathcal{V}$ has property (U) and $\delta(\varepsilon) = k\varepsilon$, for some $k > 0$. If each element of $\mathcal{V}$ is a semi-L-summand, then $\mathcal{V}$ in has property (U) with $\delta(\varepsilon) = \varepsilon$, by Theorem 5.1 of Lau [11]; if each element of $\mathcal{V}$ is a semi-M-ideal, then $\mathcal{V}$ has property (U) with $\delta(\varepsilon) = 2\varepsilon$, by Theorem 5.4 of Lau [11].

LEMMA 3. *If the pair* $(\mathcal{V},\mathcal{B})$ *has property* (B), *then given* $\varepsilon > 0$ *and* $R > 0$, *there exists* $0 < \delta = \delta(\varepsilon,R) \leq \varepsilon$ *such that, given* $B \in \mathcal{B}$, $V \in \mathcal{V}$ *and* $w \in V$ *such that* $\mathrm{rad}(B;V) \leq R$, *and* $\|f - w\| < \mathrm{rad}(B;V) + \delta$, *for all* $f \in B$, *there exists* $v \in \mathrm{cent}(B;V)$ *such that* $\|v - w\| \leq \varepsilon$.

PROOF. Let $\varepsilon > 0$ and $R > 0$ be given. Choose $\delta > 0$ by property (B). We may assume that $\delta \leq \varepsilon$. Let $B \in \mathcal{B}$, $V \in \mathcal{V}$ and $w \in V$ be given with $\mathrm{rad}(B;V) \leq R$ and $\|f - w\| < \mathrm{rad}(B;V) + \delta$. If $\mathrm{rad}(B;V) = 0$, then B is a singleton, say $\{f\}$, and $f \in V$. Then $\mathrm{cent}(B;V) = \{f\}$, and $\|f - w\| < \delta \leq \varepsilon$. If $r = \mathrm{rad}(B;V) > 0$, then by property (B), there exists $v \in V$ such that $\|f - v\| \leq r$ for all $f \in B$, and $\|v - w\| \leq \varepsilon$. But then $v \in \mathrm{cent}(B;V)$.

COROLLARY 14. *If* $\mathcal{V}$ *has property* (B), *then given* $\varepsilon > 0$ *and* $R > 0$, *there exists* $0 < \delta = \delta(\varepsilon,R) \leq \varepsilon$ *such that given* $f \in E$

$v \in V$ and $w \in V$ with $\operatorname{dist}(f;V) \le R$ *and* $\|f - w\| < \operatorname{dist}(f;V) + \delta$, *there exists* $v \in P_V(f)$ *such that* $\|v - w\| \le \varepsilon$.

COROLLARY 15. *If V has property* (B) *and each element of V is a convex Chebyshev subset of* E, *then given $\varepsilon > 0$ and $R > 0$ there exists* $0 < \delta = \delta(\varepsilon,R) \le \varepsilon$ *such that given* $f \in E$, $V \in V$ *and* $w \in V$ *with* $\operatorname{dist}(f;V) \le R$ *and* $\|f - w\| < \operatorname{dist}(f;V) + \delta$ *then* $\|P_V(f) - w\| \le \varepsilon$.

THEOREM 9. *If the pair* (V,B) *has property* (B), *then the collection of set-valued mappings* $B \to \operatorname{cent}(B;V)$, $V \in V$, *is uniformly d_H-equicontinuous on* $\{B \in B;\ \operatorname{rad}(B;V) \le R$ *for all* $V \in V\}$, *for each* $R > 0$.

PROOF. Let $\varepsilon > 0$ be given. Determine $\delta(\varepsilon,R) > 0$ by property (B). Take $\delta < \delta(\varepsilon,R)/2$. Let K and L be in B, with $d_H(K,L) < \delta$, and $\operatorname{rad}(K,V) \le R$, $\operatorname{rad}(L,V) \le R$ for all $V \in V$. Notice that $d_H(K,L) < \delta$ implies $K \subset L + \rho U$ and $L \subset K + \rho U$ for some $\rho \le \delta$, where $U = \{x \in E;\ \|x\| \le 1\}$.

Let $V \in V$ be given. Choose $v \in \operatorname{cent}(K;V)$ and $w \in \operatorname{cent}(L;V)$. Then, for any $g \in L$ we have $g = f + x$, where $f \in K$ and $\|x\| \le \rho$. Hence

$$(1)\qquad \|v - g\| \le \|v - f\| + \|f - g\| \le$$

$$\le \sup_{k\in K} \|v - k\| + \rho = \operatorname{rad}(K;V) + \rho$$

$$\le \sup_{k\in K} \|k - w\| + \rho.$$

On the other hand, for any $k \in K$ we have $k = m + y$, where $m \in L$ and $\|y\| \le \rho$. Hence

$$(2) \qquad \|w - k\| \leq \|w - m\| + \|m - k\| \leq$$

$$\leq \sup_{f \in L} \|w - f\| + \rho = \operatorname{rad}(L;V) + \rho .$$

From (1) and (2) we get

$$\|v - g\| \leq \operatorname{rad}(L;V) + 2\rho \leq \operatorname{rad}(L;V) + \delta(\varepsilon;R)$$

for all $g \in L$.

Since $\operatorname{rad}(L,V) \leq R$ we can apply Lemma 3 and there exists $s \in \operatorname{cent}(L;V)$ such that $\|s - v\| \leq \varepsilon$. Hence v belongs to $\operatorname{cent}(L;V) + \varepsilon U$, where $U = \{x \in E ; \|x\| \leq 1\}$. Therefore $\operatorname{cent}(L;V) \subset \operatorname{cent}(L;V) + \varepsilon U$. Similarly, $\operatorname{cent}(L;V) \subset \operatorname{cent}(K,V) + \varepsilon U$, and then $d_H(\operatorname{cent}(K;V), \operatorname{cent}(L;V)) \leq \varepsilon$ for all $V \in \mathcal{V}$.

COROLLARY 16. *If $\mathcal{V}$ has property* (B), *then the family* $\{P_V ; V \in \mathcal{V}\}$ *is uniformly* d_H*-equicontinuous on bounded subsets of* E.

REMARK. Since d_H-continuity implies lower semicontinuity and each Banach space is a metric space, hence a paracompact space, each P_V, for $V \in \mathcal{V}$, is lower semicontinuous and admits a continuous proximity map, whenever $\mathcal{V}$ has property (B).

COROLLARY 17. *Let $\mathcal{V}$ be a family of convex Chebyshev subsets with property* (B). *Then* $\{P_V ; V \in \mathcal{V}\}$ *is uniformly equicontinuous on bounded sets.*

REMARK. It follows from Corollary 15 that if property (B) is valid for the family of all Chebyshev subsets of E, then the space E has property (*) of [10, Theorem 1]. In particular, when E is uniformly convex, then the family $\mathcal{V}$ of all closed and convex non-empty subsets has property (B). In this case Corollary 17 reduces to Theorem 4 of Holmes [10].

COROLLARY 18. *If the pair* $(V, \mathcal{B})$ *has property* (B), *then for each* $V \in V$ *and* $R > 0$, *the set-valued mapping* $B \to \mathrm{cent}(B;V)$ *is uniformly* d_H*-continuous on the set* $\{B \in \mathcal{B}; \mathrm{rad}(B;V) \leq R\}$.

COROLLARY 19. *If* V *has property* (B), *then for each* $V \in V$ *and* $R > 0$, *the metric projection* P_V *is uniformly* d_H*-continuous on the set* $\{f \in E \ \mathrm{dist}(f;V) \leq R\}$.

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