Estimates for n-widths of multiplier operators of multiple Walsh series

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Abstract

Estimates for Kolmogorov and Gelfand n-widths of multiplier operators of multiple Walsh series are obtained. Upper and lower bounds are established for n-widths of general multiplier operators. These results are applied to get upper and lower bounds for n-widths of specific multiplier operators, which generate sets of finitely and infinitely differentiable functions in the dyadic sense. It is shown that these estimates have order sharp in various important cases.

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1 Introduction

In [2, 3, 16, 17, 18, 19, 20, 21, 22] techniques to study asymptotic estimates for n-widths of multiplier operators defined for functions on the torus and on two-points homogeneous spaces $\big(\mathbb{S}^d, \mathbb{P}^d(\mathbb{R}), \mathbb{P}^d(\mathbb{C}), \mathbb{P}^d(\mathbb{H}), P^{16}(\mathrm{Cay})\big)$ were obtained. In the present paper we continue these studies considering now multiplier operators of multiple Walsh series.

The studies on asymptotic estimates for Kolmogorov n-widths of Sobolev classes on the circle were performed by several important mathematicians such as Rudin, Stechkin, Gluskin, Ismagilov, Maiorov, Makovoz and Scholz in [9, 12, 24, 25, 34, 36, 37], they were initiated by Kolmogorov [15] in 1936 and completed by Kashin [13, 14] in 1977. Several techniques were applied in these different cases, among them we highlight a technique of discretization due to Maiorov and the Borsuk theorem. Observing the historical evolution of the study of n-widths, it is possible to note that it has been an usual practice to use different techniques in proofs of lower and upper bounds and in estimates for classes of finitely and infinitely differentiable functions (see, e.g. [31]). One of the objectives of this work is to give an unified treatment in the study of n-widths of sets of functions determined by multiplier operators of multiple Walsh series.

The Walsh functions are defined on the unit interval $[0,1)$ and they form a complete orthonormal system in $L^2[0,1)$ (see [29, 35]). This system can be applied in different situations, such as: data transmission, filtering, image enhancement, signal analysis and pattern recognition. The Walsh functions are easy to deploy on high-speed computers and can be used with little storage space. This is due in part to the fact that the Walsh functions assume only the values +1 and -1 (see [1]). However, the one-dimensional theory has not been sufficient, so multidimensional Walsh analysis, at least for the two-dimensional case, has been developed. It is usual a problem in Fourier analysis to be studied initially in the trigonometric case and then in the Walsh case. In several papers we find studies involving multiple Walsh series which were first studied for multiple trigonometric series, for example, Goginava [10, 11] extended for the d-dimensional case of the Walsh series, the result obtained for the two-dimensional trigonometric series, where for $f \in L\log L([0,2\pi]^2)$ the means $\sigma_n f = (1/n)\sum_{j=1}^{n} S_{j,j}(f)$, of the partial sums $S_{j,j}(f)$, converge a.e. to f as $n \to \infty$, studied by Marcinkiewicz [26]. On the order hand, the Cantor-Lebesgue theorem states that, if a trigonometric series converges to a finite sum on a subset of positive measure, its coefficients tend to zero. The possibility of extending this theorem to the case of multiple series has

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been investigated by several authors, Plotnikov [32] has extended this result for the coefficients of multiple Walsh series. Moreover, Ghodadra [7] defined the notion of bounded p-variation, $p \geq 1$ for a function from a rectangle $[a_1, b_1] \times \cdots \times [a_m, b_m]$ to $\mathbb{C}$ and studied the order of magnitude of trigonometric Fourier coefficients of such functions from $[0, 2\pi]^m$ to $\mathbb{C}$, in [8], he has studied the order of magnitude of Walsh-Fourier coefficients for a function of bounded p-variation form $[0,1]^m$ to $\mathbb{C}$. Other articles in which are obtained results for Walsh series that have already been made for trigonometric series are [6, 33, 38].

Let $\mathbb{N}$ denote the set of positive integers and $\mathbb{N}_0$ the set of non-negative integers. For $d \in \mathbb{N}$ and $\mathbf{k} \in \mathbb{N}_0^d$ let $|\mathbf{k}| = \left(k_1^2 + \cdots + k_d^2\right)^{1/2}$ and $|\mathbf{k}|_* = \max_{1 \leq j \leq d} k_j$. Given $l, N \in \mathbb{N}_0$ we define $A_l = \{\mathbf{k} \in \mathbb{N}_0^d : |\mathbf{k}| \leq l\}$, $A_l^* = \{\mathbf{k} \in \mathbb{N}_0^d : |\mathbf{k}|_* \leq l\}$, $A_{-1} = A_{-1}^* = \emptyset$. In Section 2 we give the definition of the multiple Walsh functions $\psi_{\mathbf{k}} : I^d \to \mathbb{R}$, $I = [0, 1)$, $\mathbf{k} \in \mathbb{N}_0^d$. Let $\mathcal{H}_l = [\psi_{\mathbf{k}} : \mathbf{k} \in A_l \setminus A_{l-1}]$ be the linear space generated by the functions $\psi_{\mathbf{k}}$ with $\mathbf{k} \in A_l \setminus A_{l-1}$ and let $d_l = \dim \mathcal{H}_l$, $\mathcal{T}_N = \bigoplus_{l=0}^{N} \mathcal{H}_l$, $\mathcal{H} = \bigcup_{N=1}^{\infty} \mathcal{T}_N$. Analogous we define $\mathcal{H}_l^*$, d_l^*, $\mathcal{T}_N^*$ and $\mathcal{H}^* = \mathcal{H}$. Given a real function λ defined on the interval $[0, \infty)$ we consider the sequences $\Lambda = \{\lambda_{\mathbf{k}}\}_{\mathbf{k} \in \mathbb{N}_0^d}$ and $\Lambda^* = \{\lambda_{\mathbf{k}}^*\}_{\mathbf{k} \in \mathbb{N}_0^d}$ where $\lambda_{\mathbf{k}} = \lambda(|\mathbf{k}|)$, $\lambda_{\mathbf{k}}^* = \lambda(|\mathbf{k}|_*)$ and the linear operators associated with these sequences $\Lambda, \Lambda^* : \mathcal{H} \to \mathcal{H}$ defined by $\Lambda(\sum a_{\mathbf{k}} \psi_{\mathbf{k}}) = \sum \lambda_{\mathbf{k}} a_{\mathbf{k}} \psi_{\mathbf{k}}$, $\Lambda^*(\sum a_{\mathbf{k}} \psi_{\mathbf{k}}) = \sum \lambda_{\mathbf{k}}^* a_{\mathbf{k}} \psi_{\mathbf{k}}$. If Λ and Λ^* are bounded from $L^p(I^d)$ to $L^q(I^d)$ we also denote the extensions on $L^p(I^d)$ by Λ and Λ^*. In the present paper, we study estimates for the n-widths of Kolmogorov and Gelfand for the multiplier operators Λ and Λ^*.

In Section 2 we give some definitions and basic results and we prove a theorem which provides estimates for Levy means of special norms on $\mathbb{R}^n$ which we introduce using the multiple Walsh functions.

In Section 3 we prove two theorems where upper and lower bounds are established for n-widths of general multiplier operators. The main tool to prove these results are the estimates for Levy means proved in Section 2. These results provide an unified treatment in the study of upper and lower bounds for n-widths of multiplier operators of multiple Walsh series.

In the last section we apply the results in Section 3 to estimate the n-width of Kolmogorov of sets of differentiable functions in the dyadic sense on I^d. Let $\Lambda^{(1)} = \{\lambda_{\mathbf{k}}\}_{\mathbf{k} \in \mathbb{N}_0^d}$, $\Lambda_*^{(1)} = \{\lambda_{\mathbf{k}}^*\}_{\mathbf{k} \in \mathbb{N}_0^d}$, where $\lambda(t) = t^{-\gamma} (\log_2 t)^{-\xi}$ for $t > 1$ and $\lambda(t) = 0$ for $1 \leq t \leq 1$, with $\gamma > 0$, $\xi \geq 0$, and let $\Lambda^{(2)} = \{\lambda_{\mathbf{k}}\}_{\mathbf{k} \in \mathbb{N}_0^d}$, $\Lambda_*^{(2)} = \{\lambda_{\mathbf{k}}^*\}_{\mathbf{k} \in \mathbb{N}_0^d}$, where $\lambda(t) = e^{-\gamma t^r}$, with $\gamma, r > 0$. Let U_p denote the closed unit ball in $L^p(I^d)$ and let $d_n(A, X)$ denote the Kolmogorov n-width of the subset A of a Banach space X. We have that $\Lambda^{(1)} U_p$ and $\Lambda_*^{(1)} U_p$ are sets of finitely differentiable functions in the dyadic sense, in particular, are Sobolev-type classes if $\xi = 0$, and $\Lambda^{(2)} U_p$ and $\Lambda_*^{(2)} U_p$ are sets of infinitely differentiable functions in the dyadic sense. The following two theorems are the result of our applications.

Theorem 1.1. *If* $\gamma > \max\{d/2, d/p\}$, $1 \leq p \leq \infty$, $2 \leq q \leq \infty$, *then for every* $n \in \mathbb{N}$,

$$d_n(\Lambda^{(1)} U_p, L^q) \ll n^{-\gamma/d + (1/p - 1/2)_+} (\log_2 n)^{-\xi} \begin{cases} q^{1/2}, & 2 \leq q < \infty, \\ (\log_2 n)^{1/2}, & q = \infty. \end{cases} \tag{1.1}$$

If $\gamma > d/2$, *then for all* $n \in \mathbb{N}$, *we have that*

$$d_n(\Lambda^{(1)} U_p, L^q) \gg n^{-\gamma/d} (\log_2 n)^{-\xi} K_n, \tag{1.2}$$

where

$$K_n = \begin{cases} 1, & 1 \leq p \leq 2, \, 1 < q \leq 2, \\ 1, & 2 \leq p < \infty, \, 2 \leq q \leq \infty, \\ 1, & 1 \leq p \leq 2 \leq q \leq \infty, \\ (\log_2 n)^{-1/2}, & 1 \leq p \leq 2, \, q = 1, \\ (\log_2 n)^{-1/2}, & p = \infty, \, 2 \leq q \leq \infty. \end{cases}$$

Theorem 1.2. *Let* $\phi_n = \dim \mathcal{T}_n$, $\omega_n = \phi_n - \phi_n^{1-r/d} - 1$ *and let* K_n *be as in Theorem 1.1. Then*

$$d_{[\omega_n]}(\Lambda^{(2)} U_p, L^q) \gg e^{-\mathcal{R} \phi_n^{r/d}} K_n, \quad r > 0, n \in \mathbb{N}, \tag{1.3}$$

$$d_{[\omega_n]}(\Lambda^{(2)}U_p, L^q) \gg e^{-\mathcal{R}\omega_n^{r/d}} K_n, \quad 0 < r \leq d, n \in \mathbb{N}, \tag{1.4}$$

$$d_{[\phi_n - 1]}(\Lambda^{(2)}U_p, L^q) \gg e^{-\mathcal{R}\phi_n^{r/d}} K_n, \quad r \geq d, n \in \mathbb{N}, \tag{1.5}$$

$$d_n(\Lambda^{(2)}U_p, L^q) \gg e^{-\mathcal{R}n^{r/d}} K_n, \quad 0 < r \leq 1, n \in \mathbb{N}. \tag{1.6}$$

For $0 < r \leq 1$ *and all* $n \in \mathbb{N}$ *we have that*

$$d_n(\Lambda^{(2)}U_p, L^q) \ll e^{-\mathcal{R}n^{r/d}} n^{(1-r/d)(1/p-1/2)_+} \begin{cases} q^{1/2}, & 1 \leq p \leq \infty, 2 \leq q < \infty, \\ (\log_2 n)^{1/2}, & 1 \leq p \leq \infty, q = \infty, \end{cases} \tag{1.7}$$

and for $r > 1$ *and all* $n \in \mathbb{N}$,

$$d_{\phi_n}(\Lambda^{(2)}U_p, L^q) \ll e^{-\gamma n^r} \begin{cases} n^{(d-1)(1/p-1/q)}, & 1 \leq p \leq 2 \leq q \leq \infty, \\ n^{(d-1)(1/2-1/q)}, & 2 \leq p, q \leq \infty, \end{cases} \tag{1.8}$$

where

$$\mathcal{R} = \gamma \left(\frac{d\Gamma(d/2)}{2\pi^{d/2}} \right)^{r/d}.$$

The results of the Theorem 1.1 also hold for the operator $\Lambda_*^{(1)}$ and the results of the Theorem 1.2 hold if we change the constant $\mathcal{R}$ by the constant $\mathcal{R}_* = \gamma 2^{-r}$.

The proofs in this paper are given only for the operators of type Λ. In the Remark 4.5 we explain as the results can be proved for the operators of type Λ_*.

We proved that, the estimates for the Kolmogorov n-widths are sharp in various important situations. We get for $2 \leq p, q < \infty$ and $0 < r \leq 1$ that $d_n(\Lambda^{(1)}U_p, L^q) \asymp d_n(\Lambda_*^{(1)}U_p, L^q) \asymp n^{-\gamma/d}(\log_2 n)^{-\xi}$, $d_n(\Lambda^{(2)}U_p, L^q) \asymp e^{-\mathcal{R}k^{r/d}}$, $d_n(\Lambda_*^{(2)}U_p, L^q) \asymp e^{-\mathcal{R}_* k^{r/d}}$.

In this paper there are several universal constants which enter into the estimates. These positive constants are mostly denoted by the letters $C, C_1, C_2, \ldots$. We did not carefully distinguish between the different constants, neither did we try to get good estimates for them. The same letter will be used to denote different universal constants in different parts of the paper. For ease of notation we will write $a_n \gg b_n$ for two sequences if $a_n \geq Cb_n$ for $n \in \mathbb{N}$, $a_n \ll b_n$ if $a_n \leq Cb_n$ for $n \in \mathbb{N}$, and $a_n \asymp b_n$ if $a_n \ll b_n$ and $a_n \gg b_n$. Also, we shall put $(a)_+ = a$ if $a > 0$ and $(a)_+ = 0$ if $a \leq 0$.

2 Main definitions and some results

Given $n \in \mathbb{N}_0$, there exists only one sequence $\{n_k\}_{k \in \mathbb{N}_0}$, such that $n = \sum_{k=0}^{\infty} 2^k n_k$, where $n_k \in \{0, 1\}$ for all $k \in \mathbb{N}_0$. The elements of this sequence are called the binary coefficients of n. For $n, m \in \mathbb{N}_0$ and $\{n_k\}_{k \in \mathbb{N}_0}$, $\{m_k\}_{k \in \mathbb{N}_0}$, the binary coefficients of n and m, respectively, we define the dyadic addition of n and m by

$$n \oplus m = \sum_{k=0}^{\infty} 2^k \left| n_k - m_k \right|.$$

Consider the dyadic intervals

$$I_{n,i} = \left[\frac{i}{2^n}, \frac{i+1}{2^n} \right), \quad n \in \mathbb{N}_0, \ 0 \leq i < 2^n.$$

The Rademacher system $\{r_n\}_{n \in \mathbb{N}_0}$, $r_n : [0, 1) \to \mathbb{R}$, is defined by

$$r_n(x) = (-1)^i, \quad x \in I_{n,i}, \ 0 \leq i < 2^{n+1}, \tag{2.1}$$

and the Walsh-Paley system $\{\psi_n\}_{n\in\mathbb{N}_0}$ on $[0,1)$ is defined by $\psi_0 := 1$ and

$$\psi_n := \prod_{k=0}^{\infty} r_k^{n_k}, \quad n > 0,$$

where $\{n_k\}_{k\in\mathbb{N}_0}$ are the binary coefficients of n. Now, given $x \in [0,1)$, we can write $x = \sum_{k=1}^{\infty} x_k 2^{-k}$, where $x_k \in \{0,1\}$. The sequence $\{x_k\}_{k\in\mathbb{N}}$ is called the dyadic expansion of x. The dyadic addition of x and y in $[0,1)$ is defined by

$$x \oplus y = \sum_{k=1}^{\infty} 2^{-k} |x_k - y_k| .$$

For $x, y \in [0,1)$ and $n, m \in \mathbb{N}$, we have that

$$\psi_n(x \oplus y) = \psi_n(x)\psi_n(y), \quad \psi_{n\oplus m}(x) = \psi_n(x)\psi_m(x).$$

We consider $[0,1)$ endowed with the Lebesgue measure $d\lambda(t)$. The Walsh-Paley functions $\{\psi_n\}_{n\in\mathbb{N}_0}$ form a complete orthonormal set in $L^2[0,1)$.

Let's consider I^d, $I = [0,1)$, endowed with the Lebesgue measure $d\nu(\mathbf{x}) = d\lambda(x_1)\dots d\lambda(x_d)$. Given $\mathbf{n} = (n_1,\dots,n_d), \mathbf{m} = (m_1,\dots,m_d) \in \mathbb{N}_0^d$, we define the dyadic addition of $\mathbf{n}$ and $\mathbf{m}$ by

$$\mathbf{n} \oplus \mathbf{m} = (n_1 \oplus m_1, n_2 \oplus m_2, \dots, n_d \oplus m_d),$$

and for all $\mathbf{x} = (x_1,\dots,x_d), \mathbf{y} = (y_1,\dots,y_d) \in I^d$ we define the dyadic addition of $\mathbf{x}$ and $\mathbf{y}$ by

$$\mathbf{x} \oplus \mathbf{y} = (x_1 \oplus y_1, x_2 \oplus y_2, \dots, x_d \oplus y_d).$$

The Walsh-Paley functions on I^d are defined as follows

$$\psi_{\mathbf{n}}(\mathbf{x}) = \psi_{n_1}(x_1)\cdots\psi_{n_d}(x_d), \quad \mathbf{n} \in \mathbb{N}_0^d,\ \mathbf{x} \in \mathbb{R}^d.$$

We denoted by $L^p = L^p(I^d)$, $1 \le p \le \infty$, the vector space consisting of all measurable functions f defined on I^d and with values in $\mathbb{C}$, satisfying

$$\|f\|_p = \|f\|_{L^p(I^d)} = \left(\int_{I^d} |f(\mathbf{x})|^p \, d\nu(\mathbf{x})\right)^{1/p} < \infty, \qquad 1 \le p < \infty,$$

$$\|f\|_\infty = \|f\|_{L^\infty(I^d)} = \operatorname*{ess\,sup}_{\mathbf{x}\in I^d} |f(\mathbf{x})| < \infty.$$

We write $U_p = \{\varphi \in L^p : \|\varphi\|_p \le 1\}$. Given $f \in L^1(I^d)$, we define the d-dimensional Walsh-Fourier series of f by

$$\sum_{\mathbf{m}\in\mathbb{N}_0^d} \widehat{f}(\mathbf{m})\psi_{\mathbf{m}}, \quad \widehat{f}(\mathbf{m}) = \int_{I^d} f(\mathbf{x})\psi_{\mathbf{m}}(\mathbf{x}) d\nu(\mathbf{x}).$$

We have that

$$\psi_{\mathbf{n}\oplus\mathbf{m}}(\mathbf{x}) = \psi_{\mathbf{n}}(\mathbf{x})\psi_{\mathbf{m}}(\mathbf{x}),\ \ \psi_{\mathbf{n}}(\mathbf{x}\oplus\mathbf{y}) = \psi_{\mathbf{n}}(\mathbf{x})\psi_{\mathbf{n}}(\mathbf{y}), \quad \mathbf{n},\mathbf{m} \in \mathbb{N}_0^d\ \ \mathbf{x},\mathbf{y} \in I^d,$$

$$\int_{I^d} \psi_{\mathbf{k}}(\mathbf{x})\psi_{\mathbf{m}}(\mathbf{x}) d\nu(\mathbf{x}) = \delta_{\mathbf{k},\mathbf{m}}, \quad \mathbf{k},\mathbf{m} \in \mathbb{N}_0^d.$$

Moreover, the Walsh-Paley system $\{\psi_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{N}_0^d}$ form a complete orthonormal set in $L^2(I^d)$. The convolution product of two functions $f, g \in L^1(I^d)$, denoted by $f * g$, is defined by

$$(f * g)(\mathbf{x}) = \int_{I^d} f(\mathbf{y})g(\mathbf{x}\oplus\mathbf{y}) d\nu(\mathbf{y}).$$

For $f, g \in L^1(I^d)$, the Young's inequality says that,

$$\|f * g\|_r \leq \|f\|_q \|g\|_p ,$$

where $1/q = 1/p + 1/r - 1$ e $1 \leq p, q, r \leq \infty$. If R is a non-negative real number, the spherical Dirichlet kernel D_R on I^d is defined by

$$D_R := \sum_{\substack{\mathbf{m}\in\mathbb{N}_0^d \\ |\mathbf{m}|\leq R}} \psi_{\mathbf{m}}.$$

If $f \in L^1(I^d)$, we define the spherical partial sum of the Walsh-Fourier series of the function f by

$$S_R(f) = f * D_R = \sum_{\substack{\mathbf{m}\in\mathbb{N}_0^d \\ |\mathbf{m}|\leq R}} \widehat{f}(\mathbf{m})\psi_{\mathbf{m}}.$$

In [35] is defined the dyadic derivative as follows. Given a function f defined on $[0, 1)$, let

$$d_n f(x) := \sum_{j=0}^{n-1} 2^{j-1}(f(x) - f(x \oplus 2^{-j-1})), \quad x \in [0, 1), n \in \mathbb{N}_0.$$

A function f is said to be differentiable in the dyadic sense at $x \in [0, 1)$ if $f^{[1]}(x) = \lim_{n\to\infty} d_n f(x)$ exists and is finite. $f^{[1]}(x)$ is called the dyadic derivative of f at x (see [35]). Derivatives of higher order are defined inductively for $m = 2, 3, \ldots$, by $f^{[m]} := (f^{[m-1]})^{[1]}$. This definition was extended to the d-dimensional case by Butzer and Engels (see [4]). Let $\mathbf{e}_q = (0, 0, \ldots, 1, \ldots, 0)$ be the vector of $\mathbb{R}^n$ with 1 on the qth coordinate and zero on the other coordinates. For a function f defined on I^d and $n \in \mathbb{N}$ and $1 \leq q \leq d$ let

$$d_{n,p} f(\mathbf{x}) := \sum_{j=0}^{n-1} 2^{j-1}[f(\mathbf{x}) - f(\mathbf{x} \oplus \mathbf{e}_q 2^{-j-1})], \quad \mathbf{x} \in I^d.$$

If the limit

$$\frac{\partial}{\partial x_q} f(\mathbf{x}) = \lim_{n\to\infty} d_{n,p} f(\mathbf{x}), \quad 1 \leq q \leq d,$$

exists and is finite at $x \in I^d$, we say that this limit is the first dyadic partial derivative, with respect to the qth coordinate, of f in x. Partial derivatives of higher order are defined successively for $r \in \mathbb{N}$ by

$$\frac{\partial^r}{\partial x_q^r} f(\mathbf{x}) = \frac{\partial}{\partial x_q}\left(\frac{\partial^{r-1}}{\partial x_q^{r-1}} f(\mathbf{x})\right), \quad r \in \mathbb{N}.$$

Every Walsh function is dyadically differentiable and for $m \in \mathbb{N}$, $k = 0, 1, 2, \ldots$ and $x \in [0, 1)$, we get $\psi_k^{[m]} = k^m \psi_k$ (see [35]). The d-dimensional version for $r \in \mathbb{N}$, $\mathbf{k} \in \mathbb{N}_0^d$ and $\mathbf{x} \in I^d$, is given by

$$\frac{\partial^r}{\partial x_q^r} \psi_{\mathbf{k}}(\mathbf{x}) = (k_q)^r \psi_{\mathbf{k}}(\mathbf{x}),$$

where $\mathbf{k} = (k_1, \ldots, k_q, \ldots, k_d)$, $1 \leq q \leq d$.

Theorem 2.1. *Let $f \in L^1(I^d)$ and $r \in \mathbb{N}$ such that*

$$\sum_{\mathbf{k}\in\mathbb{N}_0^d} (k_q)^r \left|\widehat{f}(\mathbf{k})\right| < \infty.$$

Then, for almost everything $\mathbf{x} \in I^d$, the dyadic partial derivative of order r of f in $\mathbf{x}$ there exists and is given by

$$\frac{\partial^r}{\partial x_q^r} f(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{N}_0^d} (k_q)^r \widehat{f}(\mathbf{k}) \psi_{\mathbf{k}}(\mathbf{x}).$$

Moreover, $f, \partial f/\partial x_q, \ldots, \partial^r f/\partial x_q^r \in L^p(I^d)$, for $1 \le p \le \infty$ and $1 \le q \le d$, and the above convergence occurs in the norm of $L^p(I^d)$, $1 \le p \le \infty$.

Consider two Banach spaces X and Y. The norm of X will be denoted by $\|\cdot\|$ or $\|\cdot\|_X$ and the closed unit ball $\{x \in X : \|x\| \le 1\}$ by B_X. We begin by recalling well-known definitions. Let A be a compact, convex and centrally symmetric subset of X. The Kolmogorov n-width of A in X is defined by

$$d_n(A, X) = \inf_{X_n} \sup_{x \in A} \inf_{y \in X_n} \|x - y\|_X \, .$$

where X_n runs over all subspaces of X of dimension n. The Gelfand n-width of A in X is defined by

$$d^n(A, X) = \inf_{L^n} \sup_{x \in A \cap L^n} \|x\|_X \, ,$$

where L_n runs over all subspaces of X of co-dimension n. The Bernstein n-width of A in X is defined as

$$b_n(A, X) = \sup_{X_{n+1}} \sup\{\lambda : \lambda B \cap X_{n+1} \subset A\},$$

where X_{n+1} is any $(n+1)$-dimensional subspace of X. The following inequality is always valid:

$$b_n(A, X) \le \min\{d_n(A, X), d^n(A, X)\}. \tag{2.2}$$

If $T \in \mathcal{L}(X, Y)$, we define the n-widths of Kolmogorov, Gelfand and Bernstein of T, respectively, by

$$d_n(T) = d_n(T(B_X), Y), \quad d^n(T) = d^n(T(B_X), Y), \quad b_n(T) = b_n(T(B_X), Y).$$

Given $T \in \mathcal{L}(X, Y)$, let $T' \in \mathcal{L}(Y', X')$ be its adjoint operator, where X' and Y' denote the duals of the spaces X and Y, respectively. If T is compact or Y is reflexive, then (see [31], p. 34)

$$d_n(T) = d^n(T'). \tag{2.3}$$

The number of points with integer coordinates in the closed ball of radius R and centered at the origin of $\mathbb{R}^d$, is given by (see [5], [9], [27])

$$N_d(R) = \left(\frac{2\pi^{d/2}}{d\Gamma(d/2)}\right) R^d + E_d(R), \quad \text{where} \quad E_d(R) \le \left(\frac{4\sqrt{d}\pi^{d/2}}{\Gamma(d/2)}\right) R^{d-1} + C R^{d-3}.$$

As a consequence, we have that $d_l \asymp l^{d-1}$ and

$$\frac{2\pi^{d/2}}{d\Gamma(d/2)} N^d \le \dim \mathcal{T}_N \le \frac{2\pi^{d/2}}{d\Gamma(d/2)} N^d + C_1 N^{d-1}. \tag{2.4}$$

Let $\Lambda = \{\lambda_{\mathbf{k}}\}_{\mathbf{k} \in \mathbb{N}_0^d}$, $\lambda_{\mathbf{k}} \in \mathbb{R}$, and $1 \le p, q \le \infty$. If for all $\varphi \in L^p(I^d)$ there is a function $f = \Lambda\varphi \in L^q(I^d)$ with formal Walsh-Fourier expansion given by

$$f \sim \sum_{\mathbf{k} \in \mathbb{N}_0^d} \lambda_{\mathbf{k}} \widehat{\varphi}(\mathbf{k}) \psi_{\mathbf{k}},$$

such that $\|\Lambda\|_{p,q} := \sup\{\|\Lambda\varphi\|_q : \varphi \in U_p\} < \infty$, we say that the multiplier operator Λ is bounded from $L^p(I^d)$ into $L^q(I^d)$, with norm $\|\Lambda\|_{p,q}$. Let $\lambda : [0, \infty) \to \mathbb{R}$ be a function and for every $\mathbf{k} \in \mathbb{N}_0^d$ let

$$\lambda_{\mathbf{k}} = \lambda(|\mathbf{k}|), \quad \lambda^*_{\mathbf{k}} = \lambda(|\mathbf{k}|_*).$$

We will consider multiplier operators associated with sequences of the type $\Lambda = \{\lambda_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{N}_0^d}$ e $\Lambda_* = \{\lambda^*_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{N}_0^d}$.

Let us write $\|x\|_2 = \left(\sum_{i=1}^n |x_i|^2\right)^{1/2}$, for the euclidean norm of the element $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and by S^{n-1} the unit euclidean sphere $\{x \in \mathbb{R}^n : \|x\|_{(2)} = 1\}$ in $\mathbb{R}^n$. The Levy Mean for a norm $\|\cdot\|$ on $\mathbb{R}^n$ is defined by

$$M(\|\cdot\|) := \left(\int_{S^{n-1}} \|x\|^2 \, d\mu(x)\right)^{1/2},$$

where μ denotes the normalized Lebesgue measure on S^{n-1}.

Given $M_1, M_2 \in \mathbb{N}$, with $M_1 < M_2$, we will use the following notations:

$$\mathcal{T}_{M_1,M_2} = \bigoplus_{l=M_1+1}^{M_2} \mathcal{H}_l, \quad D_{M_1,M_2}(\mathbf{x}) = D_{M_2}(\mathbf{x}) - D_{M_1}(\mathbf{x}) \quad \text{and} \quad n = \dim \mathcal{T}_{M_1,M_2}.$$

Remark 2.2. *Let $A_l \setminus A_{l-1} = \{\mathbf{m}^l_j : 1 \le j \le d_l\}$ where the elements $\mathbf{m}^l_j$ are chosen satisfying $|\mathbf{m}^l_j| \le |\mathbf{m}^l_{j+1}|$ for $1 \le j \le d_l - 1$. Then $\{\psi_{\mathbf{m}^l_j} : 1 \le j \le d_l\}$ is an orthonormal basis of $\mathcal{H}_l$. We consider the orthonormal basis*

$$\Upsilon = \Upsilon_{M_1,M_2} = \{\psi^l_j = \psi_{\mathbf{m}^l_j} : M_1 + 1 \le l \le M_2, 1 \le j \le d_l\}$$

of $\mathcal{T}_{M_1,M_2}$ endowed with the order $\psi^{M_1+1}_1, \ldots, \psi^{M_1+1}_{d_{M_1+1}}, \psi^{M_1+2}_1, \ldots, \psi^{M_1+2}_{d_{M_1+2}}, \ldots, \psi^{M_2}_1, \ldots, \psi^{M_2}_{d_{M_2}}$. We denote

$$\xi_k = \psi^{l+1}_t, \quad k = t + \sum_{j=M_1+1}^{l} d_j, \quad 1 \le t \le d_{l+1}, \quad M_1 \le l < M_2,$$

and hence $\Upsilon = \{\xi_k\}_{k=1}^n$. Let $J : \mathbb{R}^n \to \mathcal{T}_{M_1,M_2}$ be the coordinate isomorphism that assigns to $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$ the function

$$J(\alpha) = J(\alpha_1, \ldots, \alpha_n) = \sum_{k=1}^n \alpha_k \xi_k \in \mathcal{T}_{M_1,M_2}.$$

Consider a function $\lambda : [0, \infty) \to \mathbb{R}$ such that, $\lambda(t) \neq 0$, for $t \ge 0$, and let $\Lambda = \{\lambda_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{N}_0^d}$ be the sequence of multipliers defined by $\lambda_{\mathbf{k}} = \lambda(|\mathbf{k}|)$. Consider $\{\lambda^l_j = \lambda_{\mathbf{m}^l_j} : M_1 + 1 \le l \le M_2, 1 \le j \le d_l\}$ endowed with the order $\lambda^{M_1+1}_1, \ldots, \lambda^{M_1+1}_{d_{M_1+1}}, \lambda^{M_1+2}_1, \ldots, \lambda^{M_1+2}_{d_{M_1+2}}, \ldots, \lambda^{M_2}_1, \ldots, \lambda^{M_2}_{d_{M_2}}$. We denote

$$\lambda_k = \lambda^{l+1}_t = \lambda(|\mathbf{m}^{l+1}_t|), \;\; k = t + \sum_{j=M_1+1}^{l} d_j, \;\; 1 \le t \le d_{l+1}, \;\; M_1 \le l < M_2$$

and we get $\Lambda_n = \{\lambda_k\}_{k=1}^n$. Now consider the multiplier operator Λ_n defined on $\mathcal{T}_{M_1,M_2}$ by

$$\Lambda_n\left(\sum_{j=1}^n \alpha_j \xi_j\right) = \sum_{j=1}^n \lambda_j \alpha_j \xi_j. \tag{2.5}$$

Also, we define the multiplier operator $\widetilde{\Lambda}_n$ on $\mathbb{R}^n$, by

$$\widetilde{\Lambda}_n(\alpha) = \widetilde{\Lambda}_n(\alpha_1, \ldots, \alpha_n) = (\lambda_1\alpha_1, \ldots, \lambda_n\alpha_n). \tag{2.6}$$

Given $\varphi \in \mathcal{T}_{M_1,M_2}$ and $1 \leq p \leq \infty$, we define

$$\|\varphi\|_{\Lambda_n,p} = \|\Lambda_n \varphi\|_p \,.$$

The application $\mathcal{T}_{M_1,M_2} \ni \varphi \to \|\varphi\|_{\Lambda_n,p}$ is a norm on $\mathcal{T}_{M_1,M_2}$. For $\alpha \in \mathbb{R}^n$, we define

$$\|\alpha\|_{(\Lambda_n,p)} = \|J(\alpha)\|_{\Lambda_n,p} = \|\Lambda_n J(\alpha)\|_p \,,$$

and we have that the application $\mathbb{R}^n \ni \alpha \to \|\alpha\|_{(\Lambda_n,p)}$ is a norm on $\mathbb{R}^n$. We will denote

$$B^n_{\Lambda_n,p} = B^n_{\Lambda,p} = \{\varphi \in \mathcal{T}_{M_1,M_2} : \|\varphi\|_{\Lambda_n,p} \leq 1\}, \quad B^n_{(\Lambda_n,p)} = B^n_{(\Lambda,p)} = \{\alpha \in \mathbb{R}^n : \|\alpha\|_{(\Lambda_n,p)} \leq 1\}.$$

If Λ_n is the identity operator I, we will write $\|\cdot\|_{I,p} = \|\cdot\|_p$, $\|\cdot\|_{(I,p)} = \|\cdot\|_{(p)}$, $B^n_{I,p} = B^n_p$ and $B^n_{(I,p)} = B^n_{(p)}$.

For $t_n \in \mathcal{T}_{M_1,M_2}$, we have that $t_n = t_n * D_{M_1,M_2}$ and from Young's inequality we obtain

$$\|t_n\|_\infty = \|t_n * D_{M_1,M_2}\|_\infty \leq \|t_n\|_1 \|D_{M_1,M_2}\|_\infty \,.$$

But, $D_{M_1,M_2} = D_{M_1,M_2} * D_{M_1,M_2}$, and using again Young's inequality, we get

$$\|D_{M_1,M_2}\|_\infty \leq \|D_{M_1,M_2}\|_2^2 = \sum_{s=M_1+1}^{M_2} \dim \mathcal{H}_s = n,$$

and therefore $\|t_n\|_\infty \leq n \|t_n\|_1$. Thus, if I denotes the identity operator, it follows that

$$\|I(t_n)\|_\infty \leq n \|t_n\|_1 \quad \text{and} \quad \|I(t_n)\|_\infty \leq \|t_n\|_\infty \,.$$

Applying the Riesz-Thorin Interpolation Theorem to the pair of inequalities above, we get

$$\|t_n\|_\infty \leq n^{1/p} \|t_n\|_p \,, \quad 1 \leq p \leq \infty; \tag{2.7}$$

and

$$\|t_n\|_q \leq n^{1/2-1/q} \|t_n\|_2 \,, \quad 2 \leq q \leq \infty. \tag{2.8}$$

Lemma 2.3. *([23], p. 585) Let $\{r_k\}_{k=1}^\infty$ be the sequence of Rademacher functions defined in (2.1). Then for $m = 1, 2, \ldots$; $i = 1, 2, \ldots, n$, let*

$$\delta_i^m(\theta) = m^{-1/2}(r_{(i-1)m}(\theta) + \cdots + r_{im-1}(\theta)), \quad \theta \in [0, 1).$$

Given a continuous function $h : \mathbb{R}^n \to \mathbb{R}$ satisfying

$$h(x_1, \ldots, x_n) e^{-\sum_{k=1}^n |x_k|} \to 0,$$

uniformly when $\sum_{k=1}^n |x_k| \to 0$, we have that

$$\int_{\mathbb{R}^n} h(x) d\gamma(x) = \lim_{m\to\infty} \int_0^1 h((2\pi)^{-1/2}(\delta_1^m(\theta), \ldots, \delta_n^m(\theta))) d\lambda(\theta),$$

where $d\gamma(x) = e^{-\pi\|x\|^2} dx$ is the Gaussian measure on $\mathbb{R}^n$.

Theorem 2.4. *Let $n = \dim \mathcal{T}_{M_1,M_2}$, $\Upsilon_{M_1,M_2} = \{\xi_k\}_{k=1}^n$ be the orthonormal system of $\mathcal{T}_{M_1,M_2}$ and $\lambda : [0, \infty) \to \mathbb{R}$ such that $t \to |\lambda(t)|$ is a monotonic function and consider and the multiplier operator Λ_n on $\mathcal{T}_{M_1,M_2}$ defined in (2.5). If $t \to |\lambda(t)|$ is non-increasing, then there is an absolute constant $C > 0$, such that:*

(i) *if $2 \leq p < \infty$, then*

$$n^{-1/2}\left(\sum_{j=M_1+1}^{M_2} |\lambda(l)|^2 d_l\right)^{1/2} \leq M(\|\cdot\|_{(\Lambda_n,p)}) \leq Cp^{1/2}n^{-1/2}\left(\sum_{l=M_1+1}^{M_2} |\lambda(l-1)|^2 d_l\right)^{1/2};$$

(ii) *if $1 \leq p \leq 2$, then*

$$\frac{1}{2}n^{-1/2}\left(\sum_{l=M_1+1}^{M_2} |\lambda(l)|^2 d_l\right)^{1/2} \leq M(\|\cdot\|_{(\Lambda_n,p)}) \leq n^{-1/2}\left(\sum_{l=M_1+1}^{M_2} |\lambda(l-1)|^2 d_l\right)^{1/2};$$

(iii) *if $p = \infty$, then*

$$n^{-1/2}\left(\sum_{l=M_1+1}^{M_2} |\lambda(l)|^2 d_l\right)^{1/2} \leq M(\|\cdot\|_{(\Lambda_n,\infty)}) \leq C(\log_2 n)^{1/2}n^{-1/2}\left(\sum_{M_1+1}^{M_2} |\lambda(l-1)|^2 d_l\right)^{1/2};$$

(iv) *if $p = 2$, then*

$$n^{-1/2}\left(\sum_{l=M_1+1}^{M_2} |\lambda(l)|^2 d_l\right)^{1/2} \leq M(\|\cdot\|_{(\Lambda_n,2)}) \leq n^{-1/2}\left(\sum_{l=M_1+1}^{M_2} |\lambda(l-1)|^2 d_l\right)^{1/2}.$$

If $t \to |\lambda(t)|$ is non-decreasing, then we obtain the estimates in (i), (ii), (iii) and (iv), exchanging $\lambda(l)$ for $\lambda(l-1)$.

Proof. Consider $t \to |\lambda(t)|$ non-increasing. Firstly we will prove the inequality in (iv). For $x = (x_1, \dots, x_n) = (x_1^{M_1+1}, \dots, x_{d_{M_1+1}}^{M_1+1}, \dots, x_1^{M_2}, \dots, x_{d_{M_2}}^{M2}) \in \mathbb{R}^n$, we get

$$\begin{aligned}
\|x\|_{(\Lambda_n,2)}^2 &= \left\|\sum_{j=1}^{n} \lambda_j x_j \xi_j\right\|_2^2 = \sum_{j=1}^{n} |\lambda_j|^2 |x_j|^2 \|\xi_j\|_2^2 \\
&\leq \sum_{l=M_1+1}^{M_2} \left(\sup_{1\leq j\leq d_l} \lambda(|\mathbf{m}_j^l|)\right)^2 \sum_{j=1}^{d_l}(x_j^l)^2 \\
&\leq \sum_{l=M_1+1}^{M_2} |\lambda(l-1)|^2 \sum_{j=1}^{d_l}(x_j^l)^2,
\end{aligned}$$

and hence

$$\int_{S^{n-1}} \|x\|_{(\Lambda_n,2)}^2 \, d\mu(x) \leq \sum_{l=M_1+1}^{M_2} |\lambda(l-1)|^2 \sum_{j=1}^{d_l} \int_{S^{n-1}} (x_j^l)^2 d\mu(x). \tag{2.9}$$

But

$$1 = \int_{S^{n-1}} \|x\|_{(2)}^2 \, d\mu(x) = \sum_{j=1}^{n} \int_{S^{n-1}} x_j^2 d\mu(x) = n \int_{S^{n-1}} x_j^2 d\mu(x),$$

for $j = 1, \dots, n$ and therefore

$$\int_{S^{n-1}} x_j^2 d\mu(x) = \frac{1}{n} \quad , j = 1, 2, \dots, n. \tag{2.10}$$

By (2.9) and (2.10), we get

$$\int_{S^{n-1}} \|x\|_{(\Lambda_n,2)} \, d\mu(x) \leq \frac{1}{n} \sum_{l=M_1+1}^{M_2} |\lambda(l-1)|^2 d_l.$$

Thus,

$$M(\|\cdot\|_{(\Lambda_n,2)}) = \left(\int_{S^{n-1}} \|x\|^2_{(\Lambda_n,2)}\, d\mu(x)\right)^{1/2} \leq n^{-1/2}\left(\sum_{l=M_1+1}^{M_2} |\lambda(l-1)|^2\, d_l\right)^{1/2},$$

and in the same way

$$M(\|\cdot\|_{(\Lambda_n,2)}) \geq n^{-1/2}\left(\sum_{l=M_1+1}^{M_2} |\lambda(l)|^2\, d_l\right)^{1/2}.$$

Therefore we get (iv).

Given $f \in C(S^{n-1})$, we define its extension $\widetilde{f}$ on $\mathbb{R}^n \backslash \{0\}$, by $\widetilde{f}(x) = \|x\|^2_{(2)}\, f(x/\,\|x\|_{(2)})$. It is known that

$$\int_{S^{n-1}} f(x) d\mu(x) = \frac{2\pi}{n} \int_{\mathbb{R}^n} \widetilde{f}(x) d\gamma(x),$$

where $d\gamma$ is the Gaussian measure on $\mathbb{R}^n$. If $f(x) = \|x\|^2_{(\Lambda_n,p)}$, $x = (x_1, \ldots, x_n) \in S^{n-1}$, we obtain

$$\int_{S^{n-1}} \|x\|^2_{(\Lambda_n,p)}\, d\mu(x) = \frac{2\pi}{n} \int_{\mathbb{R}^n} \|x\|^2_{(\Lambda_n,p)}\, d\gamma(x). \tag{2.11}$$

Note that if $x \in \mathbb{R}^n$, then $\|x\|_{(\Lambda_n,p)} \leq \left(\sup_{1\leq j\leq n} |\lambda_j|\right) \sum_{j=1}^n |x_j|$, and hence

$$\widetilde{f}(x) e^{-\sum_{k=1}^n |x_k|} = \|x\|^2_{(\Lambda_n,p)}\, e^{-\sum_{k=1}^n |x_k|} \to 0,$$

uniformly when $\sum_{k=1}^n |x_k| \to 0$. Thus, applying Lemma 2.3 we obtain

$$\int_{\mathbb{R}^n} \widetilde{f}(x) d\gamma(x) = \lim_{m\to\infty} \int_0^1 \left\| (2\pi)^{-1/2} (\delta_1^m(\theta), \ldots, \delta_n^m(\theta) \right\|^2_{(\Lambda_n,p)} d\lambda(\theta). \tag{2.12}$$

From (2.11) and (2.12), it follows that

$$\int_{S^{n-1}} \|x\|^2_{(\Lambda_n,p)}\, d\mu(x) = n^{-1} \lim_{m\to\infty} \int_0^1 \|(\delta_1^m(\theta), \ldots, \delta_n^m(\theta))\|^2_{(\Lambda_n,p)}\, d\lambda(\theta). \tag{2.13}$$

We have that

$$\begin{aligned} \|(\delta_1^m(\theta), \ldots, \delta_n^m(\theta))\|^2_{(\Lambda_n,p)} &= \|\Lambda_n J(\delta_1^m(\theta), \ldots, \delta_n^m(\theta))\|_p^2 = \left\| \sum_{j=1}^n \lambda_j \delta_j^m(\theta) \xi_j \right\|_p^2 \\ &= \left(\int_{I^d} \left| \sum_{j=1}^n \lambda_j \delta_j^m(\theta) \xi_j(\mathbf{x}) \right|^p d\nu(\mathbf{x}) \right)^{2/p}. \end{aligned} \tag{2.14}$$

Now, for $\mathbf{x} \in I^d$, let $\varphi_{(j-1)m+i}(\mathbf{x}) = m^{-1/2} \xi_j(\mathbf{x})$ and $\widetilde{\lambda}_{(j-1)m+i} = \lambda_j$, for $j = 1, 2, \ldots, n$, $i = 1, 2, \ldots, m$ and $m = 1, 2, \ldots$. Hence, we get

$$\sum_{j=1}^n \lambda_j \delta_j^m(\theta) \xi_j(\mathbf{x}) = \sum_{j=1}^{nm} \varphi_j(\mathbf{x}) \widetilde{\lambda}_j r_{j-1}(\theta). \tag{2.15}$$

From (2.13)-(2.15), for $1 \leq p \leq \infty$ it follows that

$$M^2(\|\cdot\|_{(\Lambda_n,p)}) = \lim_{m\to\infty} n^{-1} \int_0^1 \left(\int_{I^d} \left| \sum_{j=1}^{nm} \varphi_j(\mathbf{x}) \widetilde{\lambda}_j r_{j-1}(\theta) \right|^p d\nu(\mathbf{x}) \right)^{2/p} d\lambda(\theta). \tag{2.16}$$

Therefore, from (2.16), Jensen's inequality and Khintchine's inequality ([30], p. 41), we obtain for $2 \le p \le \infty$

$$\begin{aligned} M(\|\cdot\|_{(\Lambda_n,p)}) &= n^{-1/2} \lim_{m\to\infty} \left(\int_0^1 \left(\int_{I^d} \left| \sum_{j=1}^{nm} \varphi_j(\mathbf{x}) \widetilde{\lambda}_j r_{j-1}(\theta) \right|^p d\nu(\mathbf{x}) \right)^{2/p} d\lambda(\theta) \right)^{1/2} \\ &\le c(p) n^{-1/2} \lim_{m\to\infty} \left(\int_{I^d} \left(\sum_{j=1}^{nm} \left| \varphi_j(\mathbf{x}) \widetilde{\lambda}_j \right|^2 \right)^{p/2} d\nu(\mathbf{x}) \right)^{1/p}. \end{aligned} \tag{2.17}$$

But,

$$\begin{aligned} \sum_{j=1}^{nm} \left| \varphi_j(\mathbf{x}) \widetilde{\lambda}_j \right|^2 &= \sum_{j=1}^{n} \sum_{i=1}^{m} \left| \varphi_{(j-1)m+i}(\mathbf{x}) \widetilde{\lambda}_{(j-1)m+i} \right|^2 = \sum_{j=1}^{n} |\lambda_j|^2 |\xi_j(\mathbf{x})|^2 \\ &\le \sum_{l=M_1+1}^{M_2} |\lambda(l-1)|^2 \sum_{j=1}^{d_l} |\xi_j(\mathbf{x})|^2 = \sum_{l=M_1+1}^{M_2} |\lambda(l-1)|^2 d_l \end{aligned}$$

and analogously

$$\sum_{j=1}^{nm} \left| \varphi_j(\mathbf{x}) \widetilde{\lambda}_j \right|^2 \ge \sum_{l=M_1+1}^{M_2} |\lambda(l)|^2 d_l.$$

Therefore,

$$\sum_{l=M_1+1}^{M_2} |\lambda(l)|^2 d_l \le \sum_{j=1}^{nm} \left| \varphi_j(\mathbf{x}) \widetilde{\lambda}_j \right|^2 \le \sum_{l=M_1+1}^{M_2} |\lambda(l-1)|^2 d_l. \tag{2.18}$$

Thus, by (2.17) and (2.18) we have that

$$M(\|\cdot\|_{(\Lambda_n,p)}) \le C p^{1/2} n^{-1/2} \left(\sum_{l=M_1+1}^{M_2} |\lambda(l-1)|^2 d_l \right)^{1/2}, \tag{2.19}$$

where the universal constant C of the last inequality is obtained from the fact that $c(p) \asymp p^{1/2}$. Hence, we obtain the upper estimate in (i). For $p = 1$, we get from (2.16), from Jensen's and Khintchine's inequality and by (2.18)

$$\begin{aligned} M^2(\|\cdot\|_{(\Lambda_n,1)}) &= n^{-1} \lim_{m\to\infty} \int_0^1 \left(\int_{I^d} \left| \sum_{j=1}^{nm} \varphi_j(\mathbf{x}) \widetilde{\lambda}_j r_{j-1}(\theta) \right| d\nu(\mathbf{x}) \right)^2 d\lambda(\theta) \\ &\ge n^{-1} \lim_{m\to\infty} \left(\int_{I^d} \int_0^1 \left| \sum_{j=1}^{nm} \varphi_j(\mathbf{x}) \widetilde{\lambda}_j r_{j-1}(\theta) \right| d\lambda(\theta) d\nu(\mathbf{x}) \right)^2 \\ &\ge n^{-1} (b(1))^2 \left(\int_{I^d} \left(\sum_{l=M_1+1}^{M_2} |\lambda(l)|^2 d_l \right)^{1/2} d\nu(\mathbf{x}) \right)^2 \\ &\ge n^{-1} \left(\frac{1}{2} \right)^2 \left(\sum_{l=M_1+1}^{M_1} |\lambda(l)|^2 d_l \right). \end{aligned}$$

Since the levy Means is an increasing function of p, it follows that for $1 \le p \le 2$,

$$M(\|\cdot\|_{(\Lambda_n,p)}) \ge M(\|\cdot\|_{(\Lambda_n,1)}) \ge n^{-1/2} \left(\frac{1}{2} \right) \left(\sum_{l=M_1+1}^{M_2} |\lambda(l)|^2 d_l \right)^{1/2}.$$

Therefore, the lower estimate in (ii) is proved. Finally, we consider $p = \infty$ and $q = \log_2 n$, applying (2.7) and (2.19) we get

$$
\begin{aligned}
M(\|\cdot\|_{(\Lambda_n,\infty)}) &\leq \left(\int_{S^{n-1}} n^{2/q} \|\Lambda_n J(x)\|_q^2 \, d\mu(x)\right)^{1/2} \\
&\leq C_1 q^{1/2} n^{1/q} n^{-1/2} \left(\sum_{l=M_1+1}^{M_2} |\lambda(l-1)|^2 \, d_l\right)^{1/2} \\
&= C(\log_2 n)^{1/2} n^{-1/2} \left(\sum_{l=M_1+1}^{M_2} |\lambda(l-1)|^2\right)^{1/2},
\end{aligned}
$$

and thus the upper estimate in (iii) is proved. If the function $t \to |\lambda(t)|$ is non-decreasing, the proof is analogous. $\square$

3 Estimates for n-widths of general multiplier operators

Consider a norm $\|\cdot\|$ on $\mathbb{R}^n$ and denote the Banach space $(\mathbb{R}^n, \|\cdot\|)$ by E and its unity ball by B_E. The dual norm of $\|\cdot\|$ is defined by

$$\|x\|^\circ = \sup\{|\langle x, y\rangle| : y \in B_E\},$$

where $\langle x, y\rangle$ is the usual inner product of the elements $x, y \in \mathbb{R}^n$. The dual space $(\mathbb{R}^n, \|\cdot\|^\circ)$ of E will be denoted by E°.

For the operators Λ_n, J, $\widetilde{\Lambda}_n$ defined at the beginning of the previous section we have that $\Lambda_n \circ J = J \circ \widetilde{\Lambda}_n$ and $J^{-1} \circ \Lambda_n = \widetilde{\Lambda}_n \circ J^{-1}$.

Theorem 3.1. *([28]) There exists an absolute constant $C > 0$ such that, for every $0 < \rho < 1$, there exists a subspace $F_k \subset \mathbb{R}^n$, with $\dim F_k = k > \rho n$, such that*

$$\|x\|_{(2)} \leq CM(\|\cdot\|^\circ)(1-\rho)^{-1/2} \|x\|, \quad \forall x \in F_k.$$

Theorem 3.2. *Let $1 \leq q \leq p \leq 2$, $0 < \rho < 1$, $n = \dim \mathcal{T}_N$, $\mathcal{T}_N = \bigoplus_{l=0}^N \mathcal{H}_l$, $d_k = \dim \mathcal{H}_k$, $\lambda : [0, \infty) \to \mathbb{R}$ such that $t \to |\lambda(t)|$ is a non-increasing function, and let $\Lambda = \{\lambda_{\mathbf{k}}\}_{\mathbf{k} \in \mathbb{N}_0^d}$, $\lambda_{\mathbf{k}} = \lambda(|\mathbf{k}|)$, $\lambda_{\mathbf{k}} \neq 0$ for all $\mathbf{k} \in \mathbb{N}_0^d$. Then there is an absolute constant $C > 0$ such that*

$$\min\{d_{[\rho n-1]}(\Lambda U_p, L^q), d^{[\rho n-1]}(\Lambda U_p, L^q)\} \geq C(1-\rho)^{1/2} n^{1/2} \left(\sum_{l=1}^N |\lambda(l)|^{-2} \, d_l\right)^{-1/2} k_{q,n},$$

with

$$k_{q,n} = \begin{cases} (1 - 1/q)^{1/2}, & q > 1, \\ (\log_2 n)^{-1/2}, & q = 1, \end{cases}$$

where $[\rho n - 1]$ denotes the integer part of the number $\rho n - 1$ and $U_p = \{f \in L^p(I^d) : \|f\|_p \leq 1\}$.

Proof. Consider $x, y \in \mathbb{R}^n = J^{-1}(\mathcal{T}_N)$. Using the fact that J is a isomorphism and from Hölder's inequality, it follows that

$$
\begin{aligned}
\|x\|^\circ_{(\Lambda_n,q)} &= \sup\{|\langle x, y\rangle| : y \in B^n_{(\Lambda_n,q)}\} = \sup\left\{\left|\int_{I^d} (\Lambda_n^{-1} J(x)) J(\bar{y}) d\nu\right| : J(\bar{y}) \in B_q^n\right\} \\
&\leq \sup\left\{\left\|\Lambda_n^{-1} J(x)\right\|_{q'} \|J(\bar{y})\|_q : J(\bar{y}) \in B_q^n\right\} \leq \left\|\Lambda_n^{-1} J(x)\right\|_{q'} = \|x\|_{(\Lambda_n^{-1},q')}, \qquad (3.1)
\end{aligned}
$$

where $1/q + 1/q' = 1$ and $\Lambda^{-1} = \{\lambda_{\mathbf{k}}^{-1}\}_{\mathbf{k}\in\mathbb{N}_0^d}$. Now, given $0 < \rho < 1$, we can apply Theorem 3.1 and get a subspace F_k of $\mathbb{R}^n$ with $\dim F_k = k > \rho n$, such that

$$\|x\|_{(2)} \le CM(\|\cdot\|^{\circ}_{(\Lambda_n,q)})(1-\rho)^{-1/2}\|x\|_{(\Lambda_n,q)}, \quad x \in F_k,$$

and using (3.1), we have that

$$\|x\|_{(2)} \le CM(\|\cdot\|_{(\Lambda_n^{-1},q')})(1-\rho)^{-1/2}\|x\|_{(\Lambda_n,q)}, \quad x \in F_k.$$

Hence, taking $\epsilon = \big(C\big)^{-1}(1-\rho)^{1/2}\big(M(\|\cdot\|_{(\Lambda_n^{-1},q')})\big)^{-1}$, we get that $\epsilon\|x\|_{(2)} \le \|x\|_{(\Lambda_n,q)}$, for all $x \in F_k$. Thus, if $x \in B^n_{(\Lambda_n,q)} \cap F_k$, $\epsilon\|x\|_{(2)} \le \|x\|_{(\Lambda_n,q)} \le 1$, and we have $\epsilon x \in B^n_{(2)} = J^{-1}(B^n_2)$, which implies $x \in \epsilon^{-1}B^n_{(2)}$, and therefore

$$\epsilon B^n_{(\Lambda_n,q)} \cap F_k \subset B^n_{(2)}. \tag{3.2}$$

Using Theorem 2.4, we get

$$M(\|\cdot\|_{(\Lambda_n^{-1},q')}) \le C_1 n^{-1/2}\left(\sum_{l=1}^{N}|\lambda(l)|^{-2}d_l\right)^{1/2}\begin{cases}(q')^{1/2}, & q' < \infty,\\ (\log_2 n)^{1/2}, & q' = \infty.\end{cases}$$

But $1/q + 1/q' = 1$ and $1 \le q \le 2$, and therefore

$$\epsilon \ge C_2(1-\rho)^{1/2}n^{1/2}\left(\sum_{l=1}^{N}|\lambda(l)|^{-2}d_l\right)^{-1/2}\begin{cases}(1-1/q)^{1/2}, & 1 < q \le 2,\\ (\log_2 n)^{-1/2}, & q = 1.\end{cases} \tag{3.3}$$

Considering that $\Lambda U_2 \subset \Lambda U_p$, it follows from basic properties of n-widths and (2.2) that

$$\min\{d_{[\rho n-1]}(\Lambda U_p, L^q), d^{[\rho n-1]}(\Lambda U_p, L^q)\} \ge b_{[\rho n-1]}(\Lambda U_p, L^q) \ge b_{[\rho n-1]}(\Lambda U_2, L^q).$$

Since, $\dim F_k = k > \rho n \ge [\rho n - 1] + 1$, then, by the definition of Bernstein n-width, by (3.1) and (3.2), it follows that

$$\begin{aligned} b_{[\rho n-1]}(\Lambda U_2, L^q) \ge b_{k-1}(\Lambda U_2, L^q) &\ge \sup_{\substack{L_k\subset L^q\cap\mathcal{T}_N\\ \dim L_k = k}}\{\beta > 0 : \beta(B^n_q \cap L_k) \subset \Lambda_n B^n_2\}\\ &= \sup_{\substack{J^{-1}(L_k)\subset\mathbb{R}^n\\ \dim L_k = k}}\{\beta > 0 : J^{-1}(\beta(B^n_q \cap L_k)) \subset J^{-1}\Lambda_n B^n_2\}\\ &= \sup_{\substack{\widetilde{L}_k\subset\mathbb{R}^n\\ \dim \widetilde{L}_k = k}}\{\beta > 0 : \widetilde{\Lambda}_n^{-1}\beta(B^n_{(q)} \cap \widetilde{L}_k) \subset B^n_{(2)}\}\\ &= \sup_{\substack{\widetilde{L}_k\subset\mathbb{R}^n\\ \dim \widetilde{L}_k = k}}\{\beta > 0 : \beta(B^n_{(\Lambda_n,q)} \cap \widetilde{L}_k) \subset B^n_{(2)}\}\\ &\ge \sup\{\beta > 0 : \beta(B^n_{(\Lambda_n,q)} \cap F_k) \subset B^n_{(2)}\} \ge \epsilon. \end{aligned}$$

Consequently, using (3.3) we conclude the proof of the theorem. ☐

Corollary 3.3. *In the conditions of Theorem 3.2, we have that*

$$d_{[\rho n-1]}(\Lambda U_p, L^q) \ge C(1-\rho)^{1/2}n^{1/2}\left(\sum_{l=1}^{N}|\lambda(l)|^{-2}d_l\right)^{-1/2}K_n,$$

where K_n is given in Theorem 1.1.

Proof. The proof follows by Theorem 3.2, by (2.3) and using basic properties of n-widths. □

Theorem 3.4. *Let $\lambda : [0,\infty) \to \mathbb{R}$ such that $t \to |\lambda(t)|$ is non-increasing function and let $\Lambda = \{\lambda_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{N}_0^d}$, $\lambda_{\mathbf{k}} = \lambda(|\mathbf{k}|)$. Suppose that $1 \le p \le 2 \le q \le \infty$ and that the multiplier operator Λ is bounded from $L^1(I^d)$ to $L^2(I^d)$ and let. Let $\{N_k\}_{k=0}^{\infty}$ and $\{m_k\}_{k=0}^{M}$ be sequences of natural numbers such that $M \in \mathbb{N}$, $N_k < N_{k+1}$, $N_0 = 0$ and $\sum_{k=0}^{M} m_k \le \beta$, for $\beta \in \mathbb{N}$. Then there exists an absolute constant $C > 0$, such that*

$$d_\beta(\Lambda U_p, L^q) \le C\left(\sum_{k=1}^{M} |\lambda(N_k)|\, \varrho_{m_k} + \sum_{k=M+1}^{\infty} |\lambda(N_k)|\, \theta_{N_k,N_{k+1}}^{1/p-1/q}\right),$$

where

$$\varrho_{m_k} = \frac{\theta_{N_k,N_{k+1}}^{1/p}}{(m_k)^{1/2}} \begin{cases} q^{1/2}, & 2 \le q < \infty, \\ (\log_2 \theta_{N_k,N_{k+1}})^{1/2}, & q = \infty, \end{cases}$$

and

$$\theta_{N_k,N_{k+1}} = \sum_{s=N_k+1}^{N_{k+1}} \dim \mathcal{H}_s, \quad k \ge 1.$$

Proof. Let $M_1, M_2 \in \mathbb{N}$ with $M_1 < M_2$, $\mathcal{T}_{0,M_2} = \mathcal{T}_{M_2} = \bigoplus_{l=0}^{M_2} \mathcal{H}_l$ and for $M_1 \ge 1$, let $\mathcal{T}_{M_1,M_2} = \bigoplus_{l=M_1+1}^{M_2} \mathcal{H}_l$, $n = \dim \mathcal{T}_{M_1,M_2} = \sum_{k=M_1+1}^{M_2} d_k$, $B_p^n := U_p \cap \mathcal{T}_{M_1,M_2}$, $B_{(p)}^n := J^{-1}(B_p^n)$, $\lambda_0 = 0$, and fix $0 < \rho < 1$. By Theorem 3.1, there exists a subspace F_k of $\mathbb{R}^n$, with $\dim F_k = k > \rho n$, such that, for all $x \in F_k$,

$$\|x\|_{(2)} \le C_1 M(\|\cdot\|_{(q)})(1-\rho)^{-1/2}\, \|x\|_{(q)}^{\circ}.$$

For $m = n - k$, we have that $(1-\rho)^{-1/2} < (n/m)^{1/2}$ and hence

$$\|x\|_{(2)} \le C_1 M(\|\cdot\|_{(q)}) \left(\frac{n}{m}\right)^{1/2} \|x\|_{(q)}^{\circ}.$$

Applying Theorem 2.4 for $\Lambda_n = Id$, we get

$$M(\|\cdot\|_{(q)}) \le C_2 \begin{cases} q^{1/2}, & 2 \le q < \infty, \\ (\log_2 n)^{1/2}, & q = \infty, \end{cases}$$

and consequently

$$\|x\|_{(2)} \le C_3 \left(\frac{n}{m}\right)^{1/2} \|x\|_{(q)}^{\circ} \begin{cases} q^{1/2}, & 2 \le q < \infty, \\ (\log_2 n)^{1/2}, & q = \infty, \end{cases} \qquad x \in F_k. \tag{3.4}$$

Therefore by (2.3) and (3.4),

$$\begin{aligned} d_m(B_2^n, L^q \cap \mathcal{T}_{M_1,M_2}) &= d_m(B_{(2)}^n, (\mathbb{R}^n, \|\cdot\|_{(q)})) = d^m((B_{(q)}^n)^{\circ}, (\mathbb{R}^n, \|\cdot\|_{(2)})) \\ &\le \sup_{x\in(B_{(q)}^n)^{\circ}\cap F_k} \|x\|_{(2)} \\ &\le C_3 \left(\frac{n}{m}\right)^{1/2} \begin{cases} q^{1/2}, & 2 \le q < \infty, \\ (\log_2 n)^{1/2}, & q = \infty. \end{cases} \end{aligned} \tag{3.5}$$

Denote $B_p^{N_k,N_{k+1}} = U_p \cap \mathcal{T}_{N_k,N_{k+1}}$ and for $f \in \Lambda U_p \subset L^2(I^d)$ let $S_N(f)$ be the N-th Walsh-Fourier spherical sum of f, that is,

$$S_N(f) = \sum_{\substack{\mathbf{k}\in\mathbb{N}_0^d \\ |\mathbf{k}|\le N}} \widehat{f}(\mathbf{k})\psi_{\mathbf{k}}.$$

For $k \geq 1$, let $\phi_{N_k,N_{k+1}}(f) = S_{N_{k+1}}(f) - S_{N_k}(f)$ and $\phi_{N_0,N_1}(f) = S_{N_1}(f)$. Since $S_{N_s}(f) \to f$ in $L^2(I^d)$ when $s \to \infty$, we obtain

$$\sum_{k=s}^{\infty} \phi_{N_k,N_{k+1}}(f) = f - S_{N_s}(f) \quad \text{and} \quad \lim_{s\to\infty} \left\| \sum_{k=s}^{\infty} \phi_{N_k,N_{k+1}}(f) \right\|_2 = 0.$$

Then, if $f = \Lambda\varphi \in \Lambda U_p \subset L^2(I^d)$, we have $f = \sum_{k=0}^{\infty} \phi_{N_k,N_{k+1}} \circ \Lambda(\varphi)$, and hence

$$\Lambda U_p \subset \bigoplus_{k=0}^{\infty} (\phi_{N_k,N_{k+1}} \circ \Lambda) U_p. \tag{3.6}$$

Since $\phi_{N_k,N_{k+1}} \circ \Lambda(U_p) = \Lambda \circ \phi_{N_k,N_{k+1}}(U_p)$, by (3.6)

$$\Lambda U_p \subset \bigoplus_{k=0}^{\infty} (\Lambda \circ \phi_{N_k,N_{k+1}}) U_p. \tag{3.7}$$

Now, using the fact that $t \to |\lambda(t)|$ is a non-increasing function, for $\varphi \in U_p$ we get

$$\begin{aligned}
\left\|(\Lambda \circ \phi_{N_k,N_{k+1}})\varphi\right\|_2 &\leq \left(\sum_{l=N_k+1}^{N_{k+1}} |\lambda(l-1)|^2 \sum_{\mathbf{j}\in A_l \setminus A_{l-1}} |\widehat{\varphi}(\mathbf{j})|^2 \right)^{1/2} \\
&\leq |\lambda(N_k)| \left(\sum_{l=N_k+1}^{N_{k+1}} \sum_{\mathbf{j}\in A_l \setminus A_{l-1}} |\widehat{\varphi}(\mathbf{j})|^2 \right)^{1/2} \\
&= |\lambda(N_k)| \left\| \phi_{N_k,N_{k+1}} \varphi \right\|_2 .
\end{aligned} \tag{3.8}$$

Applying Young's inequality it follows that

$$\left\|\phi_{N_k,N_{k+1}}\varphi\right\|_2 = \left\|D_{N_k,N_{k+1}} * \varphi\right\|_2 \leq \|\varphi\|_p \left\|D_{N_k,N_{k+1}}\right\|_{1/(3/2-1/p)},$$

and also we get

$$\begin{aligned}
\left\|D_{N_k,N_{k+1}}\right\|_2^2 &= \sum_{\mathbf{k}\in A_{N_{k+1}} \setminus A_{N_k}} \sum_{\mathbf{j}\in A_{N_{k+1}} \setminus A_{N_k}} \int_{I^d} \psi_{\mathbf{k}}(\mathbf{x}) \psi_{\mathbf{j}}(\mathbf{x}) d\nu(\mathbf{x}) \\
&= \sum_{s=N_k+1}^{N_k} \dim \mathcal{H}_s = \theta_{N_k,N_{k+1}},
\end{aligned}$$

therefore for $p = 1$, we obtain

$$\left\|\phi_{N_k,N_{k+1}}\varphi\right\|_2 \leq \theta_{N_k,N_{k+1}}^{1/2} \|\varphi\|_1 .$$

For $p = 2$ we have $\varphi \in U_2 \subset L^2(I^d)$ and hence

$$\left\|\phi_{N_k,N_{k+1}}\varphi\right\|_2 = \left(\sum_{\mathbf{j}\in A_{N_{k+1}} \setminus A_{N_k}} |\widehat{\varphi}(\mathbf{j})|^2 \right)^{1/2} \leq \left(\sum_{\mathbf{j}\in \mathbb{N}_0^d} |\widehat{\varphi}(\mathbf{j})|^2 \right)^{1/2} = \|\varphi\|_2 .$$

Applying the Riesz-Thorin Interpolation Theorem to the last two inequalities, we get

$$\left\|\phi_{N_k,N_{k+1}}\varphi\right\|_2 \leq \theta_{N_k,N_{k+1}}^{1/p-1/2} \|\varphi\|_p \leq \theta_{N_k,N_{k+1}}^{1/p-1/2}, \quad 1 \leq p \leq 2.$$

Thus, from (3.8) it follows that $(\Lambda \circ \phi_{N_k,N_{k+1}})U_p \subset |\lambda(N_k)|\, \theta_{N_k,N_{k+1}}^{1/p-1/2} B_2^{N_k,N_{k+1}}$, and by (3.7)

$$\Lambda U_p \subset \bigoplus_{k=0}^{\infty} |\lambda(N_k)|\, \theta_{N_k,N_{k+1}}^{1/p-1/2} B_2^{N_k,N_{k+1}}. \tag{3.9}$$

Since $2 \le q \le \infty$, using (2.8), we get for $\varphi \in B_2^{N_k,N_{k+1}}$ that

$$\|\varphi\|_q = \left\|\phi_{N_k,N_{k+1}}\varphi\right\|_q \le \phi_{N_k,N_{k+1}}^{1/2-1/q} \left\|\phi_{N_k,N_{k+1}}\varphi\right\|_2 \le \theta_{N_k,N_{k+1}}^{1/2-1/q},$$

and hence $B_2^{N_k,N_{k+1}} \subset \theta_{N_k,N_{k+1}}^{1/2-1/q} B_q^{N_k,N_{k+1}}$. Therefore by (3.9),

$$\Lambda U_p \subset \bigoplus_{k=0}^{M} \theta_{N_k,N_{k+1}}^{1/p-1/2} |\lambda(N_k)|\, B_2^{N_k,N_{k+1}} + \bigoplus_{k=M+1}^{\infty} |\lambda(N_k)|\, \theta_{N_k,N_{k+1}}^{1/p-1/q} B_q^{N_k,N_{k+1}}, \tag{3.10}$$

Now, using properties of n-widths, (3.4) and (3.10) we obtain

$$\begin{aligned}
d_\beta(\Lambda U_p, L^q) &\le \sum_{k=0}^{M} |\lambda(N_k)|\, \theta_{N_k,N_{k+1}}^{1/p-1/2} d_{m_k}(B_2^{N_k,N_{k+1}}, L^q \cap \mathcal{T}_{N_k,N_{k+1}}) \\
&\quad + \sum_{k=M+1}^{\infty} |\lambda(N_k)|\, \theta_{N_k,N_{k+1}}^{1/p-1/q} d_0(B_q^{N_k,N_{k+1}}, L^q \cap \mathcal{T}_{N_k,N_{k+1}}) \\
&\le C\left(\sum_{k=0}^{M} |\lambda(N_k)|\, \varrho_{m_k} + \sum_{k=M+1}^{\infty} |\lambda(N_k)|\, \theta_{N_k,N_{k+1}}^{1/p-1/q}\right).
\end{aligned}$$

□

Remark 3.5. *Let us now improve the estimate obtained in the previous theorem by specifying the sequences N_k and m_k. Given $N \in \mathbb{N}$, we define $N_1 = N$ and*

$$N_{k+1} = \min\{l \in \mathbb{N} : 2\,|\lambda(l)| \le |\lambda(N_k)|\},$$

thus $2\,|\lambda(N_{k+1})| \le |\lambda(N_k)|$ and therefore $|\lambda(N_{k+1})| \le 2^{-k}\,|\lambda(N)|$. For $\epsilon > 0$, we define

$$M = \left[\frac{\log_2(\theta_{N_1,N_2})}{\epsilon}\right], \qquad m_k = [2^{-\epsilon k}\theta_{N_1,N_2}] + 1, \quad k = 1, 2, \ldots, M,$$

and $m_0 = \theta_{N_0,N_1} = \theta_{0,N}$. We have that

$$\sum_{k=1}^{M} m_k \le M + \theta_{N_1,N_2} \sum_{k=1}^{\infty} (2^{-\epsilon})^k \le \frac{\log_2(\theta_{N_1,N_2})}{\epsilon} + \theta_{N_1,N_2}\frac{1}{1-2^{-\epsilon}} \le C_\epsilon \theta_{N_1,N_2}. \tag{3.11}$$

Now let

$$\beta = M_0 + \sum_{k=1}^{M} m_k = \sum_{s=0}^{N} \dim \mathcal{H}_s + \sum_{k=1}^{M} m_k.$$

Then by Theorem 3.4 and denoting $d_\beta := d_\beta(\Lambda U_p, L^q)$, we get

$$\begin{aligned}
d_\beta &\le C\sum_{k=1}^{M} |\lambda(N_k)|\, \varrho_{m_k} + C\sum_{k=M+1}^{\infty} |\lambda(N_k)|\, \theta_{N_k,N_{k+1}}^{1/p-1/q} \\
&\le 2C\,|\lambda(N)| \sum_{k=1}^{M} 2^{-k(1-\epsilon/2)} \frac{\theta_{N_k,N_{k+1}}^{1/p}}{\theta_{N_1,N_2}^{1/2}} \begin{cases} q^{1/2}, & 2 \le q < \infty, \\ (\log_2 \theta_{N_k,N_{k+1}})^{1/2}, & q = \infty, \end{cases} \\
&\quad + 2C\,|\lambda(N)| \sum_{k=M+1}^{\infty} 2^{-k}\theta_{N_k,N_{k+1}}^{1/p-1/q}.
\end{aligned}$$

Definition 3.6. *Let N_k and M be as in Remark 3.5. We say that $\Lambda = \{\lambda_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{N}_0^d} \in K_{\epsilon,p}$, for $\epsilon > 0$ and $1 \le p \le 2$, if $|\lambda(k+1)| \le |\lambda(k)|$, $N_k < N_{k+1}$ for all $k \in \mathbb{N}$ and if for all $N \in \mathbb{N}$ we have*

$$\sum_{k=1}^{M} 2^{-k(1-\epsilon/2)} \frac{\theta_{N_k,N_{k+1}}^{1/p}}{\theta_{N_1,N_2}^{1/2}} \le C_{\epsilon,p}\theta_{N_1,N_2}^{1/p-1/2}.$$

The following result is an immediate consequence of Theorem 3.4 and Remark 3.5.

Corollary 3.7. *Let $\Lambda = \{\lambda_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{N}_0^d}$ and $\theta_{N_k,N_{k+1}}$ be as in Theorem 3.4, and let $\epsilon > 0$, N_k, M, $\{m_k\}_{k=0}^M$ and β be as in Remark 3.5 and $1 \le p \le 2 \le q \le \infty$. Suppose that $\Lambda \in K_{\epsilon,p}$, for some $\epsilon > 0$. Then there exists a constant $C_{\epsilon,p} > 0$, such that*

$$d_\beta(\Lambda U_p, L^q) \le C_{\epsilon,p}\,|\lambda(N)|\,\theta_{N_1,N_2}^{1/p-1/2}\begin{cases} q^{1/2}, & 2 \le q < \infty,\\ \sup_{1\le k\le M}(\log_2\theta_{N_k,N_{k+1}})^{1/2}, & q = \infty,\end{cases}$$
$$+\,C_{\epsilon,p}\,|\lambda(N)|\sum_{k=M+1}^{\infty} 2^{-k}\theta_{N_k,N_{k+1}}^{1/p-1/q}.$$

4 Proofs of the Theorems 1.1 and 1.2

Remark 4.1. *Consider $\Lambda^{(1)} = \{\lambda_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{N}_0^d}$, with $\lambda_{\mathbf{k}} = \lambda(|\mathbf{k}|)$, where $\lambda(t) = t^{-\gamma}(\log_2 t)^{-\xi}$ for $t > 1$ and $\lambda(t) = 0$ for $0 \le t \le 1$, with $\gamma, \xi \in \mathbb{R}$, $\gamma > d/2$, $\xi > 0$. We will prove that $\Lambda^{(1)}$ is a bounded operator from $L^1(I^d)$ to $L^2(I^d)$. Given $\varphi \in U_1$, for each $n \in \mathbb{N}$, we defined*

$$\varphi_n = \widehat{\varphi}(\mathbf{0}) + \sum_{l=1}^{n}\sum_{\mathbf{k}\in A_l\setminus A_{l-1}} \widehat{\varphi}(\mathbf{k})\psi_{\mathbf{k}}.$$

We have that

$$\Lambda^{(1)}\varphi_n = \sum_{l=2}^{n}\sum_{\mathbf{k}\in A_l\setminus A_{l-1}} \lambda_{\mathbf{k}}\widehat{\varphi}(\mathbf{k})\psi_{\mathbf{k}},$$

Since $|\widehat{\varphi}(\mathbf{k})| \le \|\varphi\|_1 \le 1$ and $d_l \asymp l^{d-1}$, we get for $\gamma > d/2$ and $m, n \in \mathbb{N}$ with $n < m$, that

$$\begin{aligned}\left\|\Lambda^{(1)}\varphi_m - \Lambda^{(1)}\varphi_n\right\|_2^2 &= \int_{I^d}\left|\Lambda^{(1)}\varphi_m(\mathbf{x}) - \Lambda^{(1)}\varphi_n(\mathbf{x})\right|^2 d\nu(\mathbf{x}) = \sum_{l=n+1}^{m}\sum_{\mathbf{k}\in A_l\setminus A_{l-1}}\lambda_{\mathbf{k}}^2\,|\widehat{\varphi}(\mathbf{k})|^2\\ &\le \|\varphi\|_1^2\sum_{l=n+1}^{m}\sum_{\mathbf{k}\in A_l\setminus A_{l-1}}\lambda_{\mathbf{k}}^2 \le \sum_{l=n+1}^{m}|\lambda(l-1)|^2\,d_l\\ &\le C_1\sum_{l=n+1}^{m}(l-1)^{-2\gamma}(\log_2(l-1))^{-2\xi}l^{d-1}\\ &\le C_2\sum_{l=n}^{m}l^{-2\gamma+d-1} \le C_3,\end{aligned}$$

and thus

$$\lim_{m,n\to\infty}\left\|\Lambda^{(1)}\varphi_m - \Lambda^{(1)}\varphi_n\right\|_2^2 = 0.$$

Hence, $\{\Lambda^{(1)}\varphi_n\}_{n=1}^\infty$ is a sequence of Cauchy in $L^2(I^d)$, and therefore it converges in $L^2(I^d)$. We write

$$\Lambda^{(1)}\varphi = \lim_{n\to\infty}\Lambda^{(1)}\varphi_n = \sum_{l=2}^{\infty}\sum_{\mathbf{k}\in A_l\setminus A_{l-1}}\lambda_{\mathbf{k}}\widehat{\varphi}(\mathbf{k})\psi_{\mathbf{k}}, \tag{4.1}$$

where the convergence occurs in the norm of $L^2(I^d)$. Therefore

$$\left\|\Lambda^{(1)}\varphi\right\|_2^2 = \left\|\lim_{n\to\infty}\Lambda^{(1)}\varphi_n\right\|_2^2 = \lim_{n\to\infty}\left\|\Lambda^{(1)}\varphi_n\right\|_2^2 \le C.$$

Remark 4.2. *Let $\varphi \in L^1(I^d)$, $m \in \mathbb{N}$ and $\gamma > m+d$. If $\mathbf{k} \in A_l \backslash A_{l-1}$, then $\lambda(|\mathbf{k}|) \le \lambda(l-1) \le (l-1)^{-\gamma} \le Cl^{-\gamma}$ and $k_q \le l$ for all $1 \le q \le d$, hence*

$$|\lambda_{\mathbf{k}}\widehat{\varphi}(\mathbf{k})|\, k_q^m = \lambda(|\mathbf{k}|)\, |\widehat{\varphi}(\mathbf{k})|\, k_q^m \le Cl^{-\gamma}\, \|\varphi\|_1\, k_q^m \le Cl^{-\gamma+m}\, \|\varphi\|_1 .$$

Then, since $d_l \asymp l^{d-1}$, we get

$$\sum_{\mathbf{k}\in A_l\backslash A_{l-1}} |\lambda_{\mathbf{k}}\widehat{\varphi}(\mathbf{k})|\, k_q^m \le C\,\|\varphi\|_1 \sum_{\mathbf{k}\in A_l\backslash A_{l-1}} l^{-\gamma+m} \le C' l^{-\gamma+m+d-1},$$

but $-\gamma+m+d-1 < -1$, therefore it follows that

$$\sum_{l=2}^{\infty}\sum_{\mathbf{k}\in A_l\backslash A_{l-1}} |\lambda_{\mathbf{k}}\widehat{\varphi}(\mathbf{k})|\, k_q^m \le C' \sum_{l=2}^{\infty} l^{-\gamma+m+d-1} < \infty.$$

Thus, applying Theorem 2.1 we obtain that the dyadic partial derivative $\partial^m(\Lambda^{(1)}\varphi)/\partial x_q^m$ there exists and

$$\frac{\partial^m}{\partial x_q^m}(\Lambda^{(1)}\varphi) = \sum_{l=2}^{\infty}\sum_{\mathbf{k}\in A_l\backslash A_{l-1}} \lambda_{\mathbf{k}}\widehat{\varphi}(\mathbf{k}) k_q^m \psi_{\mathbf{k}}.$$

Therefore, the dyadic partial derivative $\partial^m(\Lambda^{(1)}\varphi)/\partial x_q^m(\mathbf{x})$ there exists for almost all $\mathbf{x} \in I^d$ and $1 \le q \le d$, if $\gamma > m+d$, that is, $\Lambda^{(1)}\varphi$ is a function that has partial derivatives until the order m, in the dyadic sense, for all $\varphi \in L^1(I^d)$, and the above series converges in the norm of L^p, $1 \le p \le \infty$.

Proof of Theorem 1.1. We will prove (1.1). Since $\gamma > d/2$, it follows from Remark 4.1 that the multiplier operator $\Lambda^{(1)}$ is bounded from $L^1(I^d)$ to $L^2(I^d)$, which is a necessary condition for the application of Corollary 3.7.

Let us fix $\delta > 0$ and let $\lambda_1, \lambda_2 : (1,+\infty) \to \mathbb{R}$ be defined by $\lambda_1(t) = t^{-\gamma}$ and $\lambda_2(t) = t^{-\gamma-\delta}$. Let $a > 1$ and $b, b_1, b_2 \in \mathbb{R}$ such that $2\lambda(b) = \lambda(a)$, $2\lambda_1(b_1) = \lambda_1(a)$ and $2\lambda_2(b_2) = \lambda_2(a)$. The functions $\lambda, \lambda_1, \lambda_2$ are decreasing, and then $b, b_1, b_2 > a$, $b_1 = 2^{1/\gamma}a$, $b_2 = 2^{1/(\gamma+\delta)}a$ and $b_1 = b(\log_2 b/\log_2 a)^{\xi/\gamma}$. Since $b > a$, $b_1 > b$ and $b > b_2$ we get $2^{1/(\gamma+\delta)}a < b < 2^{1/\gamma}a$. Therefore for $a = N_k$, we have $2^{1/(\gamma+\delta)}N_k < b < 2^{1/\gamma}N_k$. But, since $N_k \le N_{k+1}-1 < b \le N_{k+1} < b+1$, it follows that

$$2^{1/(\gamma+\delta)}N_k < N_{k+1} < 2^{1/\gamma}N_k + 1, \quad k \ge 1. \tag{4.2}$$

Integrating the function x^{d-1} and using that $d_l \asymp l^{d-1}$, we get

$$\theta_{N_k,N_{k+1}} \asymp \sum_{s=N_k+1}^{N_{k+1}} s^{d-1} \le \sum_{s=0}^{N_{k+1}-1} s^{d-1} + N_{k+1}^{d-1} \asymp N_{k+1}^d.$$

Now

$$\begin{aligned}\theta_{N_k,N_{k+1}} &\ge \int_{(N_{k+1}+N_k)/2}^{N_{k+1}} x^{d-1}dx \\ &\asymp N_{k+1}^d - \frac{N_{k+1}^d}{2^d}\left[1 + d\left(\frac{N_k}{N_{k+1}}\right) + \frac{d(d-1)}{2!}\left(\frac{N_k}{N_{k+1}}\right)^2 + \cdots + \left(\frac{N_k}{N_{k+1}}\right)^d\right] \\ &\ge N_{k+1}^d(1 - C_{\gamma,\delta,d}),\end{aligned} \tag{4.3}$$

where $0 < C_{\gamma,\delta,d} < 1$. Therefore

$$\theta_{N_k,N_{k+1}} \asymp N_{k+1}^d, \quad k \geq 1. \tag{4.4}$$

From (4.2) we get

$$2^{k/(\gamma+\delta)} N \leq N_{k+1} \leq C_\gamma 2^{k/\gamma} N, \quad k \geq 1. \tag{4.5}$$

Consider $M = [\epsilon^{-1} \log_2 \theta_{N_1,N_2}]$ as in Remark 3.5. Then $M \asymp \epsilon^{-1} \log_2 N \asymp \epsilon^{-1} \log_2 n$ and hence, by (4.4) and (4.5)

$$\sigma = |\lambda(N)| \sum_{k=M+1}^{\infty} 2^{-k} \theta_{N_k,N_{k+1}}^{1/p-1/q} \ll N^{-\gamma+d(1/p-1/q)} (\log_2 N)^{-\xi} \sum_{k=[C\epsilon^{-1}\log_2 N]}^{\infty} 2^{-k(1-d(1/p-1/q)/\gamma)}. \tag{4.6}$$

We have that $1 \leq p \leq 2 \leq q \leq q \leq \infty$ and $\gamma > d/2$, and then $0 \leq 1/p-1/q \leq 1$ and $1-(d/\gamma)(1/p-1/q) > 0$. Therefore

$$\begin{aligned} \sum_{k=[C\epsilon^{-1}\log_2 N]}^{\infty} 2^{-k(1-(d/\gamma)(1/p-1/q))} &\asymp 2^{-(1-(d/\gamma)(1/p-1/q))C\epsilon^{-1}\log_2 N} \frac{1}{1-2^{-(1-(d/\gamma)(1/p-1/q))}} \\ &\ll N^{-C\epsilon^{-1}(1-(d/\gamma)(1/p-1/q))} \end{aligned}$$

and thus we obtain by (4.6) that $\sigma \ll N^{-\gamma+d(1/p-1/q)-C\epsilon^{-1}(1-(d/\gamma)(1/p-1/q))} (\log_2 N)^{-\xi}$. Hence, for $0 < \epsilon < (C(1-(d/\gamma)(1/p-1/q))/(d(1/p-1/q))$, it follows that

$$\sigma \ll N^{-\gamma} (\log_2 N)^{-\xi}. \tag{4.7}$$

From (4.4), (4.5) and $M \asymp \epsilon^{-1} \log_2 N$, we get

$$\sum_{k=1}^{M} 2^{-k(1-\epsilon/2)} \frac{\theta_{N_k,N_{k+1}}^{1/p}}{\theta_{N_1,N_2}^{1/2}} \ll N^{d(1/p-1/2)} \sum_{k=1}^{C\epsilon^{-1}\log_2 N} 2^{-k(1-\epsilon/2-d/\gamma p)}. \tag{4.8}$$

Now $1 - d/\gamma p > 0$, since $\gamma/d > 1/p$, and then $t = -(1-\epsilon/2-d/\gamma p) < 0$ if we take $0 < \epsilon/2 < 1 - d/\gamma p$. Therefore from (4.8), (4.4) and (4.5),

$$\sum_{k=1}^{M} 2^{-k(1-\epsilon/2)} \frac{\theta_{N_k,N_{k+1}}^{1/p}}{\theta_{N_1,N_2}^{1/2}} \leq C_{\epsilon,p} \theta_{N_1,N_2}^{1/p-1/2}, \tag{4.9}$$

and hence $\Lambda^{(1)} \in K_{\epsilon,p}$. Thus applying Corollary 3.7 and using (4.7) we get

$$d_\beta(\Lambda^{(1)} U_p, L^q) \ll |\lambda(N)| \, \theta_{N_1,N_2}^{1/p-1/2} \begin{cases} q^{1/2}, & 2 \leq q < \infty, \\ \sup_{1\leq k\leq M} (\log_2 \theta_{N_k,N_{k+1}})^{1/2}, & q = \infty, \end{cases} + N^{-\gamma} (\log_2 N)^{-\xi}.$$

By (4.4), (4.5), by the definition of M, and since $n \asymp N^d$, we get, for $1 \leq k \leq M$ that

$$\theta_{N_k,N_{k+1}} \ll (2^{k/\gamma} N)^d \leq (2^{M/\gamma} N)^d \leq N^{d+dC/\gamma\epsilon} = (N^d)^{1+C/\gamma\epsilon} \ll n^{1+C/\gamma\epsilon},$$

and then $\log_2 \theta_{N_k,N_{k+1}} \ll \log_2 n$. Now, by (4.4) and (4.5), it follows that $\theta_{N_1,N_2}^{1/p-1/2} \ll N^{d(1/p-1/2)}$ and hence

$$d_\beta(\Lambda^{(1)} U_p, L^q) \ll N^{-\gamma} N^{d(1/p-1/2)} (\log_2 N)^{-\xi} \begin{cases} q^{1/2}, & 2 \leq q < \infty, \\ (\log_2 n)^{1/2}, & q = \infty, \end{cases} + N^{-\gamma} (\log_2 N)^{-\xi}.$$

But $n \asymp N^d$, and thus

$$d_\beta(\Lambda^{(1)}U_p, L^q) \ll n^{-\gamma/d+(1/p-1/2)}(\log_2 n)^{-\xi} \begin{cases} q^{1/2}, & 2 \le q < \infty, \\ (\log_2 n)^{1/2}, & q = \infty. \end{cases} \tag{4.10}$$

Since $n \asymp N^d$ and $M \asymp \epsilon^{-1}\log_2 N$, from the Remark 3.5, by (4.4) and (4.5) we get

$$\beta \ll N^d + 2^{d/\gamma}N^d \sum_{j=1}^{[C\epsilon^{-1}\log_2 N]} (2^{-\epsilon})^j \ll N^d \asymp n,$$

and then by (4.10) we obtain (1.1) for $1 \le p \le 2$. If $2 \le p \le \infty$, then $\Lambda^{(1)}U_p \subset \Lambda^{(1)}U_2$ and hence (1.1) follows for $2 \le p \le \infty$.

Now we will prove (1.2). Since $d_k \asymp k^{d-1}$ and $n \asymp N^d$ we get

$$\begin{aligned} \left(\sum_{k=1}^{N} |\lambda_k|^{-2} d_k\right)^{-1/2} &\asymp \left(\sum_{k=1}^{N} (k^{-\gamma}(\log_2 k)^{-\xi})^{-2} k^{d-1}\right)^{-1/2} \gg (N^{-\gamma-d/2})(\log_2 N)^{-\xi} \\ &\asymp n^{-\gamma/d-1/2}(\log_2 n)^{-\xi}. \end{aligned}$$

By Corollary 3.3 for $\rho = 1/2$ we get

$$\begin{aligned} d_{[(n-2)/2]}(\Lambda^{(1)}U_p, L^q) &\ge C n^{1/2}\left(1 - \frac{1}{2}\right)^{1/2} \left(\sum_{k=1}^{N} |\lambda_k|^{-2} d_k\right)^{-1/2} K_n \\ &\gg n^{-\gamma/d}(\log_2 n)^{-\xi} K_n. \end{aligned}$$

Then for $m, n \in \mathbb{N}$ such that $n \ge 4$ and $[(n-3)/2] \le m \le [(n-2)/2]$ it follows that

$$d_m(\Lambda^{(1)}U_p, L^q) \ge d_{[(n-2)/2]}(\Lambda^{(1)}U_p, L^q) \gg m^{-\gamma}(\log_2 m)^{-\xi} K_m,$$

concluding the proof of (1.2). $\square$

Remark 4.3. *Let $\Lambda^{(2)} = \{\lambda_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{N}_0^d}$, with $\lambda_{\mathbf{k}} = \lambda(|\mathbf{k}|)$, where $\lambda : [0,\infty) \to \mathbb{R}$ is given by $\lambda(t) = e^{-\gamma t^r}$, $\gamma, r > 0$. We will prove that $\Lambda^{(2)}$ is a bounded operator from $L^p(I^d)$ to $L^q(I^d)$, for $1 \le p, q \le \infty$. It is sufficient to show that $\Lambda^{(2)}$ is bounded when $p = 1$ and $q = \infty$. Let $\varphi \in U_1$ and for all $n \in \mathbb{N}$, let*

$$\varphi_n = \widehat{\varphi}(\mathbf{0}) + \sum_{l=1}^{n} \sum_{\mathbf{k}\in A_l\setminus A_{l-1}} \widehat{\varphi}(\mathbf{k})\psi_{\mathbf{k}}.$$

Therefore

$$\Lambda^{(2)}\varphi_n = \widehat{\varphi}(\mathbf{0}) + \sum_{l=1}^{n} \sum_{\mathbf{k}\in A_l\setminus A_{l-1}} \lambda_{\mathbf{k}}\widehat{\varphi}(\mathbf{k})\psi_{\mathbf{k}}.$$

Since $d_l \asymp l^{d-1}$ and $|\widehat{\varphi}(\mathbf{k})| \le \|\varphi\|_1 \le 1$, then for $n, m \in \mathbb{N}$, with $n < m$, and all $\mathbf{x} \in I^d$, we get

$$\begin{aligned} \left|\Lambda^{(2)}\varphi_m(\mathbf{x}) - \Lambda^{(2)}\varphi_n(\mathbf{x})\right| &\le \sum_{l=n+1}^{m} \|\varphi\|_1 \sum_{\mathbf{k}\in A_l\setminus A_{l-1}} \lambda_{\mathbf{k}} \le \sum_{l=n+1}^{m} e^{-\gamma(l-1)^r} d_l \\ &\le C_1 \sum_{l=n}^{m} e^{-\gamma l^r} l^{d-1}. \end{aligned}$$

Let $n_0 \in \mathbb{N}$ such that $e^{-\gamma l^r} \leq C_2 l^{-d-1}$ for all $l \geq n_0$. Then, if $n \geq n_0$, we get

$$\left|\Lambda^{(2)}\varphi_m(\mathbf{x}) - \Lambda^{(2)}\varphi_n(\mathbf{x})\right| \leq C_3 \sum_{l=n}^{m} l^{-2}, \tag{4.11}$$

and hence

$$\lim_{n,m\to\infty} \left|\Lambda^{(2)}\varphi_m(\mathbf{x}) - \Lambda^{(2)}\varphi_n(\mathbf{x})\right| = 0,$$

that is $\{\Lambda^{(2)}\varphi_n(\mathbf{x})\}_{n=1}^{\infty}$ is a sequence of Cauchy in $\mathbb{R}$, and therefore converges in $\mathbb{R}$. We write

$$\Lambda^{(2)}\varphi(\mathbf{x}) = \sum_{l=0}^{\infty} \sum_{\mathbf{k}\in A_l\setminus A_{l-1}} \lambda_{\mathbf{k}}\widehat{\varphi}(\mathbf{k})\psi_{\mathbf{k}}(\mathbf{x}), \quad \mathbf{x} \in I^d. \tag{4.12}$$

Therefore from (4.11),

$$\left\|\Lambda^{(2)}\varphi\right\|_{\infty} \leq \sup_{\mathbf{x}\in I^d} \left|\Lambda^{(2)}\varphi(\mathbf{x})\right| \leq C_4.$$

Remark 4.4. *Let $\varphi \in L^1(I^d)$ and $m \in \mathbb{N}$. If $\mathbf{k} \in A_l \setminus A_{l-1}$, then $\lambda(|\mathbf{k}|) \leq \lambda(l-1) = e^{-\gamma(l-1)^r}$ and $k_q \leq l$ for all $1 \leq q \leq d$. Hence, we get*

$$|\lambda_{\mathbf{k}}\widehat{\varphi}(\mathbf{k})|\, k_q^m = \lambda(|\mathbf{k}|)\, |\widehat{\varphi}(\mathbf{k})|\, k_q^m \leq e^{-\gamma(l-1)^r} l^m \|\varphi\|_1 .$$

Consider $C_1 > 0$ a constant such that $e^{-\gamma(l-1)^r} l^{m+d+1} \leq C_1$ for all $l \in \mathbb{N}$. Then, we obtain

$$\begin{aligned}\sum_{l=0}^{\infty} \sum_{\mathbf{k}\in A_l\setminus A_{l-1}} |\lambda_{\mathbf{k}}\widehat{\varphi}(\mathbf{k})|\, k_q^m &\leq \|\varphi\|_1 \sum_{l=0}^{\infty} \sum_{\mathbf{k}\in A_l\setminus A_{l-1}} e^{-\gamma(l-1)^r} l^m \\ &\leq C_2 \|\varphi\|_1 \sum_{l=1}^{\infty} e^{-\gamma(l-1)^r} l^m l^{d-1} \\ &\leq C_1 C_2 \|\varphi\|_1 \sum_{l=1}^{\infty} l^{-2} < \infty.\end{aligned}$$

Hence, applying Theorem 2.1 we obtain that the dyadic partial derivative of order m of $\Lambda^{(2)}\varphi$ there exists and

$$\frac{\partial^m}{\partial x_q^m}(\Lambda^{(2)}\varphi) = \sum_{l=0}^{\infty} \sum_{\mathbf{k}\in A_l\setminus A_{l-1}} \lambda_{\mathbf{k}}\widehat{\varphi}(\mathbf{k}) \frac{\partial^m}{\partial x_q^m}\psi_{\mathbf{k}} = \sum_{l=0}^{\infty} \sum_{\mathbf{k}\in A_l\setminus A_{l-1}} \lambda_{\mathbf{k}}\widehat{\varphi}(\mathbf{k}) k_q^m \psi_{\mathbf{k}}.$$

Therefore, the dyadic partial derivative $\partial^m(\Lambda^{(2)}\varphi)/\partial x_q^m(\mathbf{x})$ there exists for almost all $\mathbf{x} \in I^d$ and all $m \in \mathbb{N}$, that is, $\Lambda^{(2)}\varphi$ is a function that has dyadic partial derivative of any order for all $\varphi \in L^1(I^d)$, and the above series converges in the norm of L^p, $1 \leq p \leq \infty$.

Proof of Theorem 1.2. For $\theta, \eta, r > 0$ and $\eta \geq r-1$, we get

$$\begin{aligned}\sum_{k=1}^{N} e^{\theta k^r} k^\eta &= \sum_{k=1}^{N} \left(\frac{1}{\theta r} k^{1-r+\eta}\right) e^{\theta k^r} \theta r k^{r-1} \leq (\theta r)^{-1} N^{1-r+\eta} \sum_{k=1}^{N} e^{\theta k^r} \theta r k^{r-1} \\ &\ll N^{1-r+\eta} \int_1^{N+1} e^{\theta x^r} \theta r x^{r-1} dx \ll N^{1-r+\eta} e^{\theta(N+1)^r}.\end{aligned} \tag{4.13}$$

Note that

$$(N+1)^r = N^r + \frac{r}{N}\left(1 + \frac{(r-1)}{2!}\frac{1}{N} + \frac{(r-1)(r-2)}{3!}\frac{1}{N^2} + \cdots\right) \leq N^r + C_1 N^{-1},$$

and hence $e^{\theta(N+1)^r} \leq C_2 e^{\theta N^r}$. Therefore by (4.13), for $\eta \geq r-1$, we obtain

$$\sum_{k=1}^{N} e^{\theta k^r} \ll N^{1-r+\eta} e^{\theta N^r}. \tag{4.14}$$

Consider $N \in \mathbb{N}$ fixed and let $n = \dim \mathcal{T}_N$. By (2.4) it follows that

$$\left(\frac{2\pi^{d/2}}{d\Gamma(d/2)}\right) N^d \leq n \leq \left(\frac{2\pi^{d/2}}{d\Gamma(d/2)}\right) N^d + C_3 N^{d-1}. \tag{4.15}$$

Hence

$$n^{r/d} \leq \left(\frac{2\pi^{d/2}}{d\Gamma(d/2)}\right)^{r/d} N^r \left(1 + \frac{C_3}{\left(\frac{2\pi^{d/2}}{d\Gamma(d/2)}\right) N}\right)^{r/d}.$$

If $A_d = 2\pi^{d/2}/(d\Gamma(d/2))$, then there exists a constant C_4 such that

$$\begin{aligned}\left(1+\frac{C_3}{A_d N}\right)^{r/d} &= 1 + \frac{r}{d}\frac{C_3}{A_d N}\left(1 + \frac{r/d-1}{2}\frac{C_3}{A_d N} + \frac{(r/d-1)(r/d-2)}{3!}\left(\frac{C_3}{A_d N}\right)^2 + \cdots\right)\\ &= 1 + \frac{r}{d}\frac{C_3}{A_d N} S_N \leq 1 + \frac{r}{d}\frac{C_3 C_4}{A_d N},\end{aligned}$$

Thus, for $r > 0$,

$$n^{r/d} \leq \left(\frac{2\pi^{d/2}}{d\Gamma(d/2)}\right)^{r/d} N^r + C_4 N^{r-1}, \tag{4.16}$$

and therefore from (4.15) and (4.16), since $n \asymp N^d$, if we write $\mathcal{R} = \gamma(d\Gamma(d/2)/(2\pi^{d/2}))^{r/d}$, we get

$$-\mathcal{R} n^{r/d} \leq -\gamma N^r \leq -\mathcal{R} n^{r/d} + C_5 n^{(r-1)/d}, \quad r > 0. \tag{4.17}$$

From the fact that $d_l \asymp l^{d-1}$ and $n \asymp N^d$, if we take $\rho = 1 - n^{-r/d}$, we obtain

$$(1-\rho)^{1/2} n^{1/2} \left(\sum_{l=1}^{N} |\lambda(l)|^{-2} d_l\right)^{-1/2} \gg n^{-r/2d+1/2} (N^d)^{(r-d)/2d} e^{-\gamma N^r} \gg e^{-\gamma N^r},$$

and hence by Corollary 3.3 and (4.17) we get

$$\begin{aligned} d_{[\omega_N]}(\Lambda^{(2)} U_p, L^q) &\gg (1-\rho)^{1/2} n^{1/2} \left(\sum_{l=1}^{N} |\lambda(l)|^{-2} d_l\right)^{-1/2} K_n \\ &\gg e^{-\gamma N^r} K_n \gg e^{-\mathcal{R} n^{r/d}} K_n. \end{aligned} \tag{4.18}$$

But $n = \phi_N \asymp N^d$, then $(\log_2 \phi_N)^{-1/2} \asymp (\log_2 N)^{-1/2}$, therefore $K_{\phi_N} \asymp K_N$ and consequently

$$d_{[\omega_N]}(\Lambda^{(2)} U_p, L^q) \gg e^{-\mathcal{R}\phi_N^{r/d}} K_N,$$

and hence we obtain (1.3). If $r > d$, then $[\omega_N] = \phi_N - 1$ and thus by (1.3) we get (1.5).

Now consider $0 < r \leq d$. We have that

$$\begin{aligned}(1-n^{-r/d})^{r/d} &= 1 - \frac{r}{d} n^{-r/d} + \frac{r(r-d)}{2!d^2}(n^{-r/d})^2 - \frac{r(r-d)(r-2d)}{3!d^3}(n^{-r/d})^3 + \cdots \\ &= 1 - n^{-r/d}\frac{r}{d}\left(1 + \frac{(d-r)}{2!d} n^{-r/d} + \frac{(d-r)(2d-r)}{3!d^2}(n^{-r/d})^2 + \cdots\right) \\ &= 1 - n^{-r/d}\frac{r}{d}\frac{1}{1-n^{-r/d}} \geq 1 - C_6 n^{-r/d},\end{aligned}$$

and then $(\omega_N+1)^{r/d} = n^{r/d}(1-n^{-r/d})^{r/d} \geq \phi_N^{r/d} - C_6$. Therefore

$$e^{-\mathcal{R}(\omega_N+1)^{r/d}} \leq e^{-\mathcal{R}(\phi_N^{r/d}-C_6)} \ll e^{-\mathcal{R}\phi_N^{r/d}} \tag{4.19}$$

and hence from (1.3) we get

$$d_{[\omega_N]}(\Lambda^{(2)}U_p, L^q) \gg e^{-\mathcal{R}(\omega_N+1)^{r/d}} K_N. \tag{4.20}$$

Let us consider now $0 < r \leq 1$. We have that

$$1 < \frac{e^{-\gamma N^r}}{e^{-\gamma(N+1)^r}} \leq e^{\gamma r}. \tag{4.21}$$

Using (1.3), (4.17), (4.20) and (4.21), we can conclude that

$$\begin{aligned} d_{[\omega_{N+1}]}(\Lambda^{(2)}U_p, L^q) &\gg e^{-\gamma(N+1)^r} K_{N+1} \gg e^{-\gamma N^r} K_N \geq e^{-\mathcal{R}\phi_N^{r/d}} K_N \\ &\gg e^{-\mathcal{R}(\omega_N+1)^{r/d}} K_N. \end{aligned} \tag{4.22}$$

For $k \in \mathbb{N}$ such that $[\omega_N] < k \leq [\omega_{N+1}]$, it follows by (4.21) and (4.22) that

$$\begin{aligned} d_k(\Lambda^{(2)}U_p, L^q) &\gg d_{[\omega_{N+1}]}(\Lambda^{(2)}U_p, L^q) \gg e^{-\mathcal{R}(\omega_N+1)^{r/d}} K_N \gg e^{-\mathcal{R}k^{r/d}} K_N \\ &\gg e^{-\mathcal{R}k^{r/d}} K_k, \end{aligned}$$

and thus we obtain (1.6). Finally, if $0 < r \leq d$, by (4.21) we obtain $e^{-\mathcal{R}l^{r/d}} \asymp e^{-\mathcal{R}(l+1)^{r/d}}$, $l \in \mathbb{N}$ and hence we get by (4.20) that

$$d_{[\omega_N]}(\Lambda^{(2)}U_p, L^q) \gg e^{-\mathcal{R}(\omega_N+1)^{r/d}} K_N \gg e^{-\mathcal{R}(\omega_N)^{r/d}} K_N,$$

which demonstrates (1.4).

Now we will prove (1.7) and (1.8). Given $k \in \mathbb{N}$, let $x_k \in \mathbb{R}$ such that $2\lambda(x_k) = \lambda(N_k)$, that is, $x_k = \left(N_k^r + \ln 2/\gamma\right)^{1/r}$. We have that $N_k < x_k \leq N_{k+1} < x_k + 1$, and thus

$$(N_k^r + (\ln 2)/\gamma)^{1/r} \leq N_{k+1} < (N_k^r + (\ln 2)/\gamma)^{1/r} + 1. \tag{4.23}$$

Hence

$$\begin{aligned} N_{k+1}^r &\leq \left(N_k^r + \frac{\ln 2}{\gamma}\right)\left[1 + r\left(N_k^r + \frac{\ln 2}{\gamma}\right)^{-1/r} + \frac{r(r-1)}{2!}\left(N_k^r + \frac{\ln 2}{\gamma}\right)^{-2/r} + \cdots\right] \\ &\leq \left(N_k^r + \frac{\ln 2}{\gamma}\right) + C_1\left(N_k^r + \frac{\ln 2}{\gamma}\right)^{1-1/r}. \end{aligned}$$

Suppose $0 < r \leq 1$. We have that $(N_k^r + (\ln 2)/\gamma)^{1-1/r} \leq 1$, hence $N_{k+1}^r \leq N_k^r + C_2$ and consequently for all $k \in \mathbb{N}$,

$$N_k^r + \frac{\ln 2}{\gamma} \leq N_{k+1}^r \leq N_k^r + C_2. \tag{4.24}$$

Proceeding by induction on k, for the inequality in (4.24), since $N_1 = N$, we get

$$N^r + k\frac{\ln 2}{\gamma} \leq N_{k+1}^r \leq N^r + C_2 k. \tag{4.25}$$

From (4.23), it follows that $N_{k+1} \asymp N_k(1 + ((\ln 2)/\gamma)N_k^{-r})^{1/r}$, then

$$N_{k+1} \asymp N_k\left(1 + \frac{\ln 2}{r\gamma}N_k^{-r} + \frac{(1-r)(\ln 2)^2}{2r^2\gamma^2}N_k^{-2r} + \frac{(1-r)(1-2r)(\ln 2)^3}{6r^3\gamma^3}N_k^{-3r} + \cdots\right)$$

and therefore $N_{k+1}-N_k \asymp N_k^{1-r}$. In an analogous way we can show that $N_{k+1}^d-N_k^d \asymp N_k^{d-r}$ and $N_{k+1}^{d-1}-N_k^{d-1} \asymp N_k^{d-r-1}$. Thus, we get

$$\theta_{N_k,N_{k+1}} \asymp \sum_{s=N_k+1}^{N_{k+1}} l^{d-1} \leq \int_{N_k}^{N_{k+1}} x^{d-1}dx + (N_{k+1}^{d-1} - N_k^{d-1}) \asymp N_k^{d-r}(1+N_k^{-1}) \asymp N_k^{d-r},$$

$$\theta_{N_k,N_{k+1}} \geq \int_{N_k}^{N_{k+1}} x^{d-1}dx \asymp N_{k+1}^d - N_k^d \asymp N_k^{d-r}$$

and then

$$\theta_{N_k,N_{k+1}} \asymp N_k^{d-r}. \tag{4.26}$$

Therefore by (4.25) and (4.26) it follows that $(\theta_{N_k,N_{k+1}}/\theta_{N_1,N_2})^{r/(d-r)} \asymp N^{-r}N_k^r \leq 1 + C_2 kN^{-r}$ for k sufficiently large. Thus, since for any $\delta > 0$, there exists $k_\delta \in \mathbb{N}$ such that $2^{\delta k} \geq 1 + C_2 kN^{-r}$, if $k \geq k_\delta$, taking $\delta' = \delta(d-r)/r$, we obtain

$$\frac{\theta_{N_k,N_{k+1}}}{\theta_{N_1,N_2}} \leq C_3 2^{\delta' k}, < \quad k \geq k_\delta. \tag{4.27}$$

Suppose $1 \leq p \leq 2$, and consider ϵ, p and δ satisfying $\epsilon/2 + \delta'/p < 1$. Then

$$\sum_{k=1}^{M} 2^{-k(1-\epsilon/2)} \frac{\theta_{N_k,N_{k+1}}^{1/p}}{\theta_{N_1,N_2}^{1/2}} = \sum_{k=1}^{M} 2^{-k(1-\epsilon/2)} \left(\frac{\theta_{N_k,N_{k+1}}}{\theta_{N_1,N_2}}\right)^{1/p} \theta_{N_1,N_2}^{1/p-1/2} \leq C_4 \theta_{N_1,N_2}^{1/p-1/2},$$

and therefore $\Lambda^{(2)} \in K_{\epsilon,p}$. Hence applying Corollary 3.7 we get

$$\begin{aligned} d_\beta(\Lambda^{(2)}U_p, L^q) &\leq C_{\epsilon,p} |\lambda(N)| \, \theta_{N_1,N_2}^{1/p-1/2} \begin{cases} q^{1/2}, & 2 \leq q < \infty, \\ \sup_{1\leq k\leq M}(\log_2 \theta_{N_k,N_{k+1}})^{1/2}, & q = \infty, \end{cases} \\ &+ C_{\epsilon,p} |\lambda(N)| \sum_{k=M+1}^{\infty} 2^{-k} \theta_{N_k,N_{k+1}}^{1/p-1/q}. \end{aligned} \tag{4.28}$$

Now, from (4.26) and (4.27) we obtain

$$\begin{aligned} |\lambda(N)| \sum_{k=M+1}^{\infty} 2^{-k} \theta_{N_k,N_{k+1}}^{1/p-1/q} &= |\lambda(N)| \, \theta_{N_1,N_2}^{1/p-1/q} \sum_{k=M+1}^{\infty} 2^{-k} \left(\frac{\theta_{N_k,N_{k+1}}}{\theta_{N_1,N_2}}\right)^{1/p-1/q} \\ &\leq C_5 |\lambda(N)| \, N^{(d-r)(1/p-1/q)} (2^{-(1-\delta'')})^{M+1} \sum_{j=0}^{\infty} (2^{-(1-\delta'')})^j \\ &\leq C_6 |\lambda(N)| \, N^{(d-r)(1/p-1/q)} (2^{-(1-\delta'')})^{M+1}, \end{aligned}$$

where $\delta'' = \delta'(1/p-1/q)$, $\delta'' < 1$. Since 3.5, $M = \left[\epsilon^{-1}\log_2 \theta_{N_1,N_2}\right] \asymp \epsilon^{-1}\log_2 \theta_{N_1,N_2}$ and $\theta_{N_1,N_2} \asymp N^{d-r}$, we get

$$|\lambda(N)| \sum_{k=M+1}^{\infty} 2^{-k} \theta_{N_k,N_{k+1}}^{1/p-1/q} \leq C_8 |\lambda(N)| \, N^{((1/p-1/q)-C_7(1-\delta'')\epsilon))(d-r)} \leq C_8 |\lambda(N)|,$$

for $0 < \epsilon < C_7(1-\delta'')(p^{-1}-q^{-1})$. Then, it follows by (4.26) and (4.28) that

$$d_\beta(\Lambda^{(2)}U_p, L^q) \ll e^{-\gamma N^r} N^{(d-r)(1/p-1/2)} \begin{cases} q^{1/2}, & 2 \leq q < \infty, \\ \sup_{1\leq k\leq M}(\log_2 \theta_{N_k,N_{k+1}})^{1/2}, & q = \infty. \end{cases} \tag{4.29}$$

Using (4.25) and (4.26), we obtain $\theta_{N_k,N_{k+1}} \leq (N^r + C_2(k-1))^{(d-r)/r}$, and using that $M \leq \epsilon^{-1}\log_2 \theta_{N_1,N_2}$, $\theta_{N_1,N_2} \asymp N^{d-r}$, $N^d \asymp n$, $n = \phi_N = \dim \mathcal{T}_N$ and that $1 \leq k \leq M$, we obtain that $\log_2 \theta_{N_k,N_{k+1}} \ll \log_2 n$. Therefore by (4.29)

$$d_\beta(\Lambda^{(2)}U_p, L^q) \ll e^{-\gamma N^r} N^{(d-r)(1/p-1/2)} \begin{cases} q^{1/2}, & 2 \leq q < \infty, \\ (\log_2 n)^{1/2}, & q = \infty. \end{cases} \tag{4.30}$$

It follows from Remark 3.5, (3.11) and (4.26), that $\beta = m_0 + \sum_{k=1}^{M} m_k \leq n + C_9 N^{d-r} \leq n + C_{10} n^{1-r/d}$. Thus

$$d_{[n+C_{10}n^{1-r/d}]}(\Lambda^{(2)}U_p, L^q) \ll e^{-\gamma N^r} N^{(d-r)(1/p-1/2)} \begin{cases} q^{1/2}, & 2 \leq q < \infty, \\ (\log_2 n)^{1/2}, & q = \infty. \end{cases} \tag{4.31}$$

Let $\tau_N = n + C_{10} n^{1-r/d}$. Since $0 < r \leq 1$, we get from (4.17) that $-\gamma N^r \leq -\mathcal{R} n^{r/d} + C_{11}$ and hence

$$\begin{aligned} -\gamma N^r + \mathcal{R}\tau_N^{r/d} &\leq -\mathcal{R} n^{r/d} + \mathcal{R}\tau_N^{r/d} + C_{11} \\ &= \mathcal{R}\phi_N^{r/d}(-1 + (1 + C_{10}\phi_N^{-r/d})^{r/d}) + C_{11}. \end{aligned}$$

If N is large enough, we have that

$$\begin{aligned} (1 + C_{10}\phi_N^{-r/d})^{r/d} &= 1 + \sum_{k=1}^{\infty} \frac{(r/d)(r/d-1)\dots(r/d-k+1)}{k!}(C_{10}\phi_N^{-r/d})^k \\ &\leq 1 + C_{10}\frac{r}{d}\phi_N^{-r/d} S_N \leq 1 + C_{12}\phi_N^{-r/d} \end{aligned}$$

and then

$$-\gamma N^r + \mathcal{R}\tau_N^{r/d} \leq C_{13}. \tag{4.32}$$

Consider $l \in \mathbb{N}$, such that $[\tau_N] \leq l \leq [\tau_{N+1}]$. From (4.21) it follows that $1 < e^{-\gamma N^r}/(e^{-\gamma (N+1)^r}) \leq e^{\gamma r}$, and therefore using (4.31), (4.32) and that $n \asymp N^d$, we obtain

$$\begin{aligned} d_l(\Lambda^{(2)}U_p, L^q) &\leq d_{[\tau_N]}(\Lambda^{(2)}U_p, L^q) \\ &\ll e^{-\gamma N^r} N^{(d-r)(1/p-1/2)} \begin{cases} q^{1/2}, & 2 \leq q < \infty, \\ (\log_2 \phi_N)^{1/2}, & q = \infty, \end{cases} \\ &\ll e^{-\gamma (N+1)^r} N^{(d-r)(1/p-1/2)} \begin{cases} q^{1/2}, & 2 \leq q < \infty, \\ (\log_2 \phi_N)^{1/2}, & q = \infty, \end{cases} \\ &\ll e^{-\mathcal{R}(\tau_{N+1})^{r/d}} (\tau_N)^{(1-r/d)(1/p-1/2)} \begin{cases} q^{1/2}, & 2 \leq q < \infty, \\ (\log_2 \tau_N)^{1/2}, & q = \infty, \end{cases} \\ &\ll e^{-\mathcal{R} l^{r/d}} l^{(1-r/d)(1/p-1/2)} \begin{cases} q^{1/2}, & 2 \leq q < \infty, \\ (\log_2 l)^{1/2}, & q = \infty, \end{cases} \end{aligned}$$

which proves (1.7) for $1 \leq p \leq 2 \leq q \leq \infty$. If $2 \leq p \leq \infty$, (1.7) follows since $\Lambda U_p \subset \Lambda U_2$.

Now, consider $r > 1$. In this case we will apply Theorem 3.4. Since $-\gamma(k+1)^r + \gamma k^r \leq -\gamma r k^{r-1}$ for $k \geq 1$ then $e\lambda(k+1) \leq e^{-\gamma r k^{r-1}+1}\lambda(k)$. Fix $a \in \mathbb{N}$ such that $e^{-\gamma r k^{r-1}} \leq 1$ for $k \geq a$ and let $N \geq a$, $N_0 = 0$, $N_1 = N$, $N_{k+1} = N + k$, $M = 0$, $\beta = m_0 = n = \phi_N$. Applying Theorem 3.4 for $1 \leq p \leq 2 \leq q \leq \infty$, we get

$$d_{\phi_N}(\Lambda^{(2)}U_p, L^q) \ll \sum_{k=1}^{\infty} \lambda(N_k)\theta_{N_k,N_{k+1}}^{1/p-1/q} = \sum_{k=1}^{\infty} \lambda(N+k-1)\theta_{N_k,N_{k+1}}^{1/p-1/q}.$$

But $\lambda(N+k) \le e^{-k}\lambda(N)$ for $k \ge 1$, and thus

$$d_{\phi_N}(\Lambda^{(2)}U_p, L^q) \ll \lambda(N) \sum_{k=1}^{\infty} e^{-(k-1)} \theta_{N_k,N_{k+1}}^{1/p-1/q} \ll \lambda(N) \sum_{k=1}^{\infty} e^{-k} \theta_{N_k,N_{k+1}}^{1/p-1/q}.$$

Since $\theta_{N_k,N_{k+1}} = \sum_{s=N_k+1}^{N_{k+1}} \dim \mathcal{H}_s = \sum_{s=N+k}^{N+k} \dim \mathcal{H}_s = \dim \mathcal{H}_{N+k}$, for $1 \le p \le 2 \le q \le \infty$ it follows that

$$\begin{aligned} d_{\phi_N}(\Lambda^{(2)}U_p, L^q) &\ll \lambda(N) \sum_{k=1}^{\infty} e^{-k} (\dim \mathcal{H}_{N+k})^{1/p-1/q} \ll \lambda(N) \sum_{k=1}^{\infty} e^{-k} ((N+k)^{d-1})^{1/p-1/q} \\ &\ll \lambda(N) \sum_{k=1}^{\infty} e^{-k} (Nk)^{(d-1)(1/p-1/q)} = \lambda(N) N^{(d-1)(1/p-1/q)} \sum_{k=1}^{\infty} e^{-k} k^{(d-1)(1/p-1/q)} \\ &\ll \lambda(N) N^{(d-1)(1/p-1/q)} = e^{-\gamma N^r} N^{(d-1)(1/p-1/q)}. \end{aligned}$$

Finally if $2 \le p, q \le \infty$, since $\Lambda^{(2)}U_p \subset \Lambda^{(2)}U_2$, we obtain

$$d_{\phi_N}(\Lambda^{(2)}U_p, L^q) \le d_{\phi_N}(\Lambda^{(2)}U_2, L^q) \ll e^{-\gamma N^r} N^{(d-1)(1/2-1/q)},$$

concluding the proof of (1.8). □

Remark 4.5. *Now consider $|\mathbf{k}|_* = \max_{1\le j\le d} |k_j|$ for $\mathbf{k} \in \mathbb{N}_0^d$ and let*

$$A_l^* = \{\, \mathbf{k} \in \mathbb{N}_0^d : |\mathbf{k}|_* \le l\}, \quad \mathcal{H}_l^* = [\psi_{\mathbf{k}} : \mathbf{k} \in A_l^* \backslash A_{l-1}^*],$$

$$d_l^* = \dim \mathcal{H}_l^*, \quad \mathcal{T}_N^* = \bigoplus_{l=0}^{N} \mathcal{H}_l^*.$$

We can see that $d_l^* = (2l+1)^d - (2(l-1)+1)^d \asymp l^{d-1}$ and $2^d N^d \le \dim \mathcal{T}_N^* \le 2^d N^d + C N^{d-1}$. Considerer the multiplier operators $\Lambda_*^{(1)} = \{|\mathbf{k}|_*^{-\gamma}(\log_2|\mathbf{k}|_*)^{-\xi}\}_{\mathbf{k}\in\mathbb{N}_0^d}$ and $\Lambda_*^{(2)} = \{e^{-\gamma|\mathbf{k}|_*^r}\}_{\mathbf{k}\in\mathbb{N}_0^d}$ for $\gamma, r > 0$ and $\xi \ge 0$. Through minor modifications in the proofs in this paper we can show that Theorem 2.4, Theorem 3.2, Corollary 3.3, Theorem 3.4 and Corollary 3.7 also hold if we change $\mathcal{H}_l$, $\mathcal{T}_N$, d_l and $\lambda_{\mathbf{k}}$ by $\mathcal{H}_l^*$, $\mathcal{T}_N^*$, d_l^* and $\lambda_{\mathbf{k}}^*$. Theorem 1.1 also hold if we change $\Lambda^{(1)}$ by $\Lambda_*^{(1)}$ and Theorem 1.2 hold if we change $\Lambda^{(2)}$ by $\Lambda_*^{(2)}$ and the constant $\mathcal{R}$ by the constant $\mathcal{R}_* = \gamma 2^{-r}$.

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