Approximation of Differentiable and Analytic Functions by Splines on the Torus

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Abstract

We consider a continuous kernel K on the torus $\mathbb{T}^d$ and we study the rate of convergence in $L^q(\mathbb{T}^d)$, of functions of the type $f = K * \phi$, $\phi \in L^p(\mathbb{T}^d)$, by its interpolating sk-splines. The rate of convergence is obtained for functions in classes of Sobolev, of infinitely differentiable functions and of analytic functions, and it provides optimal error estimates of the same order as best trigonometric approximation, in several cases.

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1 Introduction

In this paper we continue our studies on convergence of sk-splines on the torus initiated in [11]. We consider a continuous kernel K on the torus $\mathbb{T}^d$ and we study the rate of convergence of a function of the type $f = K * \phi$, $\phi \in L^p(\mathbb{T}^d)$, in the norm of $L^q(\mathbb{T}^d)$, by its interpolating sk-splines. In our applications we consider functions in Sobolev classes, in classes of infinitely differentiable functions and in classes of analytic functions. The rate of convergence provides optimal error estimates of the same order as best trigonometric approximation, in several cases.

Studies about convergence of sk-splines on the circle can be found in [5, 6, 7, 10] and on the torus or on the sphere in [1, 2, 3, 4, 8].

Given $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d$ and $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{Z}^d$ let $\mathbf{x_k} = (x_{k_1}, \ldots, x_{k_d})$, $x_{k_l} = \pi k_l / n_l$. We denote by $sk_{\mathbf{n}}(f, \cdot)$ the unique interpolating sk-spline of a function f with set of knots and interpolating points $\Lambda_{\mathbf{n}} = \{\mathbf{x_k} : 0 \leq k_l \leq 2n_l - 1, 1 \leq l \leq d\}$. Let U_p be the closed unit ball of $L^p(\mathbb{T}^d)$ and let $K * U_p = \{K * \phi : \ \phi \in U_p\}$.

The following two theorems are the result of our applications to Sobolev classes and to classes of infinitely differentiable functions and analytic functions.

Theorem 1.1. *For $\gamma \in \mathbb{R}$, $\gamma > d$, let*

$$K_1(\mathbf{x}) = \sum_{\mathbf{l} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}} |\mathbf{l}|^{-\gamma} e^{i \mathbf{l} \cdot \mathbf{x}}, \quad \mathbf{x} \in \mathbb{T}^d,$$

where $|\cdot| = |\cdot|_2$ or $|\cdot| = |\cdot|_\infty$. For $n \in \mathbb{N}$, let $\mathbf{n} = (n, \ldots, n) \in \mathbb{N}^d$. Then for $1 \leq p \leq 2 \leq q \leq \infty$ with $1/p - 1/q \geq 1/2$, there exists a positive constant $C_{p,q}$, independent of $n \in \mathbb{N}$, such that

$$\sup_{f \in K_1 * U_p} ||f - sk_{\mathbf{n}}(f, \cdot)||_q \leq C_{p,q} n^{-\gamma + d(1/p - 1/q)}, \tag{1}$$

and for $1 \le q \le 2 \le p \le \infty$, there exists a positive constant $\overline{C}$, independent of n, p and q, such that

$$\sup_{f \in K_1 * U_p} ||f - sk_{\mathbf{n}}(f, \cdot)||_q \le \overline{C} n^{-\gamma}. \tag{2}$$

Given $p, q \in \mathbb{R}$ satisfying only the condition $1 \le p \le 2 \le q \le \infty$, there exists a positive constant C, independent of n, p and q such that

$$\sup_{f \in K_1 * U_p} ||f - sk_{\mathbf{n}}(f, \cdot)||_2 \le C n^{-\gamma + d(1/p - 1/2)}, \tag{3}$$

$$\sup_{f \in K_1 * U_2} ||f - sk_{\mathbf{n}}(f, \cdot)||_q \le C n^{-\gamma + d(1/2 - 1/q)}, \tag{4}$$

$$\sup_{f \in K_1 * U_p} ||f - sk_{\mathbf{n}}(f, \cdot)||_{p'} \le C n^{-\gamma + d(2/p - 1)}. \tag{5}$$

Theorem 1.2. *Let r, α be two positive real numbers and let*

$$K_2(\mathbf{x}) = \sum_{\mathbf{l} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}} e^{-\alpha |\mathbf{l}|_\infty^r} e^{i \mathbf{l} \cdot \mathbf{x}}, \quad \mathbf{x} \in \mathbb{T}^d.$$

For $n \in \mathbb{N}$, let $\mathbf{n} = (n, \ldots, n) \in \mathbb{N}^d$. Then for $1 \le p \le 2 \le q \le \infty$ and $r \ge 1$, with $1/p - 1/q \ge 1/2$, there exists a positive constant $C_{p,q}$, independent of $n \in \mathbb{N}$, such that

$$\sup_{f \in K_2 * U_p} ||f - sk_{\mathbf{n}}(f, \cdot)||_q \le C_{p,q} e^{-\alpha n^r} n^{(d-1)(1/p - 1/q)}, \tag{6}$$

and for $1 \le q \le 2 \le p \le \infty$ and $r > 0$, there exists a positive constant $\overline{C}$, independent of n, p and q, such that

$$\sup_{f \in K_2 * U_p} ||f - sk_{\mathbf{n}}(f, \cdot)||_q \le \overline{C} e^{-\alpha n^r}. \tag{7}$$

Given $p, q \in \mathbb{R}$ satisfying only the condition $1 \le p \le 2 \le q \le \infty$ and $r \ge 1$, there exists a positive constant C, independent of n, p and q, such that

$$\sup_{f \in K_2 * U_p} ||f - sk_{\mathbf{n}}(f, \cdot)||_2 \le C e^{-\alpha n^r} n^{(d-1)(1/p - 1/2)}, \tag{8}$$

$$\sup_{f\in K_2*U_2} ||f - sk_{\mathbf{n}}(f,\cdot)||_q \le Ce^{-\alpha n^r} n^{(d-1)(1/2-1/q)}, \qquad (9)$$

$$\sup_{f\in K_2*U_p} ||f - sk_{\mathbf{n}}(f,\cdot)||_{p'} \le Ce^{-\alpha n^r} n^{(d-1)(2/p-1)}. \qquad (10)$$

The estimate (1) was proved in [11]. We remark that the set $K_1 * U_p$ is a class of Sobolev type on $\mathbb{T}^d$. The n-width of Kolmogorov of $K * U_p$ in $L^q(\mathbb{T}^d)$ is given by

$$d_n(K * U_p, L^q) = \inf_{T_n} \sup_{f\in K*U_p} \inf_{g\in T_n} ||f-g||_q,$$

where T_n varies in the family of n-dimensional subspaces of $L^q(\mathbb{T}^d)$. It follows from [9] and [12] that

$$d_{(2n)^d}(K_1 * U_p, L^q) \asymp n^{-\gamma+d(1/p-1/2)}, \quad 1 \le p \le 2 \le q \le \infty, \qquad (11)$$

$$d_{(2n)^d}(K_1 * U_p, L^q) \asymp n^{-\gamma}, \quad 1 \le q \le 2 \le p \le \infty. \qquad (12)$$

For $\mathbf{n} = (n,\ldots,n) \in \mathbb{N}^d$, the dimension of the space $SK(\Lambda_{\mathbf{n}})$ of interpolating sk-splines on $\Lambda_{\mathbf{n}}$ is $(2n)^d$. For the cases $1 \le p \le 2 = q$ and $1 \le q \le 2 \le p \le \infty$, we verify that the order of convergence for a function $f \in K_1 * U_p$ by sk-splines obtained in Theorem 1.1 coincides with the order of convergence for f by trigonometric polynomials given in (11) and (12), that is, the rate of convergence is optimal in the sense of n-widths. Therefore $SK(\Lambda_{\mathbf{n}})$ is an optimal subspace for the Kolmogorov $(2n)^d$-width of the Sobolev class $K_1 * U_p$ in L^q, in the above cases.

It was proved in [1] an almost optimal estimate, in the sense of n-widths, for functions in anisotropic Sobolev classes on the torus, optimal up to a logarithmic factor.

The set $K_2 * U_p$ is a class of infinitely differentiable functions if $0 < r < 1$ and a class of analytic functions if $r \ge 1$. It was proved in [9] that for $r \ge 1$,

$$d_{(2n+1)^d}(K_2 * U_p, L^q) \ll e^{-\alpha n^r} n^{(d-1)(1/p-1/q)}, \quad 1 \le p \le 2 \le q \le \infty, \qquad (13)$$

$$d_{(2n+1)^d}(K_2 * U_p, L^q) \ll e^{-\alpha n^r}, \quad 1 \le q \le 2 \le p \le \infty, \qquad (14)$$

and for $0 < r \leq 1$,

$$d_{(2n)^d}(K_2 * U_p, L^q) \asymp e^{-\alpha n^r}, \quad 1 \leq q \leq 2 \leq p \leq \infty. \tag{15}$$

We verify that for all cases studied in Theorem 1.2, the order of convergence for a function $f \in K_2 * U_p$ by sk-splines coincides or is better than the order of convergence by trigonometric polynomials given in (13) and (14). When $0 < r \leq 1$, the result in Theorem 1.2 is optimal in the sense of n-widths, for $1 \leq q \leq 2 \leq p \leq \infty$, hence $SK(\Lambda_{\mathbf{n}})$ is an optimal subspace for the Kolmogorov $(2n)^d$-width of the class $K_2 * U_p$, of infinitely differentiable ($0 < r < 1$) or of analytic functions ($r = 1$), in L^q, for $1 \leq q \leq 2 \leq p \leq \infty$.

We do not know similar results to those of Theorem 1.2, on approximation by splines, for infinitely differenciable or analytic functions on the torus. When $d = 1$, the estimates in Theorems 1.1 and 1.2 were studied by A. K. Kushpel (see [3, 5, 6]).

2 Preliminaries

The definitions, remarks, lemmas and theorems, in this section, can be found in [11].

If (a_n) and (b_n) are sequences, we write $a_n \gg b_n$ to indicate that there is a constant $C_1 > 0$ such that $a_n \geq C_1 b_n$ for all $n \in \mathbb{N}$ and we write $a_n \ll b_n$ to indicate that there is a constant $C_2 > 0$ such that $a_n \leq C_2 b_n$ for all $n \in \mathbb{N}$. We write $a_n \asymp b_n$ to indicate that $a_n \ll b_n$ and $a_n \gg b_n$.

We identify the $d-$dimensional torus $\mathbb{T}^d$ with the $d-$dimensional cube $[-\pi, \pi]^d$. We will consider $\mathbb{T}^d$ endowed with the normalized Lebesgue measure $d\nu(\mathbf{x})$.

For $\mathbf{l} = (l_1, \ldots, l_d)$, $\mathbf{k} = (k_1, \ldots, k_d)$, $\mathbf{j} = (j_1, \ldots, j_d) \in \mathbb{Z}^d$ and $\mathbf{x} = (x_1, \ldots, x_d)$, $\mathbf{y} = (y_1, \ldots, y_d) \in \mathbb{R}^d$, we denote $\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \cdots + x_d y_d$; $\mathbf{lk} = (l_1 k_1, \ldots, l_d k_d)$; $\mathbf{l} \equiv \mathbf{k} \bmod(\mathbf{j})$ if there is $\mathbf{p} \in \mathbb{Z}^d$ such that $\mathbf{l} - \mathbf{k} = \mathbf{pj}$;

$\mathbf{0} = (0, 0, \ldots, 0)$; $\mathbf{1} = (1, 1, \ldots, 1)$; $|\mathbf{x}|_p = (|x_1|^p + |x_2|^p + \cdots + |x_d|^p)^{1/p}$ for $1 \le p < \infty$; $|\mathbf{x}|_\infty = \max_{1\le j\le d} |x_j|$.

We denote by $L^p = L^p(\mathbb{T}^d)$, $1 \le p \le \infty$, the vector space of all mensurable functions f defined on $\mathbb{T}^d$ and with values in $\mathbb{C}$, satisfying

$$||f||_p = \left(\int_{\mathbb{T}^d} |f(\mathbf{x})|^p d\nu(\mathbf{x}) \right)^{1/p} < \infty, \ 1 \le p < \infty,$$
$$||f||_\infty = \operatorname*{ess\,sup}_{\mathbf{x}\in\mathbb{T}^d} |f(\mathbf{x})| < \infty.$$

We write $U_p = \{f \in L^p(\mathbb{T}^d) : ||f||_p \le 1\}$.

Given $f \in L^1(\mathbb{T}^d)$ we define the Fourier series of the function f by $\sum_{\mathbf{m}\in\mathbb{Z}^d} \widehat{f}(\mathbf{m}) e^{i\mathbf{m}\cdot\mathbf{x}}$, where $\widehat{f}(\mathbf{m}) = \int_{\mathbb{T}^d} f(\mathbf{x}) e^{-i\mathbf{m}\cdot\mathbf{x}} d\nu(\mathbf{x})$.

The convolution product of two functions f and g in $L^1(\mathbb{T}^d)$, denoted by $f * g$, is defined by $f * g(\mathbf{x}) = \int_{\mathbb{T}^d} f(\mathbf{x}-\mathbf{y}) g(\mathbf{y}) d\nu(\mathbf{y})$.

Let $(a_\mathbf{l})_{\mathbf{l}\in\mathbb{Z}^d}$ be a sequence of real numbers such that $a_\mathbf{l} = a_{-\mathbf{l}}$ for every $\mathbf{l} \in \mathbb{Z}^d$ and $\sum_{\mathbf{l}\in\mathbb{Z}^\mathbf{d}} |a_\mathbf{l}| < \infty$. Consider the kernel $K(\mathbf{x})$ given by $K(\mathbf{x}) = \sum_{\mathbf{l}\in\mathbb{Z}^d} a_\mathbf{l} e^{i\mathbf{l}\cdot\mathbf{x}}$. Then K is a real function, continuous and even. Consider now the convolution operator defined for $f \in L^1(\mathbb{T}^d)$ by $Tf(\mathbf{x}) = K * f(\mathbf{x})$, $\mathbf{x} \in \mathbb{T}^d$. We have that T is a bounded linear operator from $L^p(\mathbb{T}^d)$ to $L^q(\mathbb{T}^d)$, for $1 \le p, q \le \infty$ and for $f \in L^1(\mathbb{T}^d)$ we have $Tf(\mathbf{x}) = \sum_{\mathbf{l}\in\mathbb{Z}^d} a_\mathbf{l} \hat{f}(\mathbf{l}) e^{i\mathbf{l}\cdot\mathbf{x}}$. We denote $K * U_p = \{K * f : f \in U_p\}$.

Let $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d$. For $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{Z}^d$ we denote $x_{k_l} = \pi k_l / n_l$, $1 \le l \le d$ and $\mathbf{x}_\mathbf{k} = (x_{k_1}, \ldots, x_{k_d})$. We also denote

$$\Omega_\mathbf{n} = \{\mathbf{j} = (j_1, \ldots, j_d) \in \mathbb{Z}^d : \ 0 \le j_l \le 2n_l - 1, \ 1 \le l \le d\},$$

and $\Lambda_\mathbf{n} = \{\mathbf{x}_\mathbf{k} : \mathbf{k} \in \Omega_\mathbf{n}\}$, $\qquad N = \#\Omega_\mathbf{n} = \#\Lambda_\mathbf{n} = 2^d n_1 n_2 \cdots n_d$.

The real vector space of all continuous functions $f : \mathbb{T}^d \to \mathbb{R}$ endowed with the norm of the uniform convergence will be denoted by $C(\mathbb{T}^d)$. For a fixed kernel $K \in C(\mathbb{T}^d)$, a sk-spline on $\Lambda_\mathbf{n}$ is a function represented in the form

$$sk_\mathbf{n}(\mathbf{x}) = c + \sum_{\mathbf{k}\in\Omega_\mathbf{n}} c_\mathbf{k} K(\mathbf{x} - \mathbf{x}_\mathbf{k}),$$

where the coefficients $c, c_{\mathbf{k}} \in \mathbb{R},\ \mathbf{k} \in \Omega_{\mathbf{n}}$, satisfy the condition $\sum_{\mathbf{k}\in\Omega_{\mathbf{n}}} c_{\mathbf{k}} = 0$. The points $\mathbf{x_k}$ are the knots of the sk-spline $sk_{\mathbf{n}}(\mathbf{x})$.

Lemma 2.1. *Let* $\mathbf{l} \in \mathbb{Z}^d$*. Then*

$$\sum_{\mathbf{k}\in\Omega_{\mathbf{n}}} e^{i\mathbf{l}\cdot\mathbf{x_k}} = \begin{cases} N, & \mathbf{l} \equiv \mathbf{0} \bmod(2\mathbf{n}), \\ 0, & \text{otherwise}, \end{cases} \qquad \text{and} \qquad \int_{\mathbb{T}^d} e^{i\mathbf{l}\cdot\mathbf{x}} d\nu(\mathbf{x}) = \begin{cases} 0, & \mathbf{l} \neq \mathbf{0}, \\ 1, & \mathbf{l} = \mathbf{0}. \end{cases}$$

Definition 2.2. *For* $K \in C(\mathbb{T}^d)$*,* $\mathbf{j} \in \mathbb{Z}^d$ *and* $\mathbf{x} \in \mathbb{T}^d$*, we define*

$$\lambda_{\mathbf{j}}(\mathbf{x}) = \sum_{\mathbf{k}\in\Omega_{\mathbf{n}}} e^{i\mathbf{j}\cdot\mathbf{x_k}} K(\mathbf{x} - \mathbf{x_k}),$$

$$\rho_{\mathbf{j}}(\mathbf{x}) = \frac{2}{N}\mathrm{Re}(\lambda_{\mathbf{j}}(\mathbf{x})) = \frac{2}{N}\sum_{\mathbf{k}\in\Omega_{\mathbf{n}}} (\cos(\mathbf{j}\cdot\mathbf{x_k}))K(\mathbf{x} - \mathbf{x_k}),$$

$$\sigma_{\mathbf{j}}(\mathbf{x}) = \frac{2}{N}\mathrm{Im}(\lambda_{\mathbf{j}}(\mathbf{x})) = \frac{2}{N}\sum_{\mathbf{k}\in\Omega_{\mathbf{n}}} (\sin(\mathbf{j}\cdot\mathbf{x_k}))K(\mathbf{x} - \mathbf{x_k}).$$

Lemma 2.3. *Let* $\mathbf{p}, \mathbf{j} \in \mathbb{Z}^d$*. Then for every* $\mathbf{x} \in \mathbb{T}^d$*,*

$$\rho_{2\mathbf{np}+\mathbf{j}}(\mathbf{x}) = \rho_{\mathbf{j}}(\mathbf{x}), \quad \rho_{2\mathbf{np}-\mathbf{j}}(\mathbf{x}) = \rho_{\mathbf{j}}(\mathbf{x}), \quad \rho_{-\mathbf{j}}(\mathbf{x}) = \rho_{\mathbf{j}}(\mathbf{x}),$$

$$\sigma_{2\mathbf{np}+\mathbf{j}}(\mathbf{x}) = \sigma_{\mathbf{j}}(\mathbf{x}), \quad \sigma_{2\mathbf{np}-\mathbf{j}}(\mathbf{x}) = -\sigma_{\mathbf{j}}(\mathbf{x}), \quad \sigma_{-\mathbf{j}}(\mathbf{x}) = -\sigma_{\mathbf{j}}(\mathbf{x}).$$

From now on we will consider a kernel K given by $K(\mathbf{x}) = \sum_{\mathbf{l}\in\mathbb{Z}^d} a_{\mathbf{l}} e^{i\mathbf{l}\cdot\mathbf{x}}$, where $(a_{\mathbf{l}})_{\mathbf{l}\in\mathbb{Z}^d}$ is a sequence of real numbers such that $\sum_{\mathbf{l}\in\mathbb{Z}^d} |a_{\mathbf{l}}| < \infty$, $a_{\mathbf{l}} = a_{-\mathbf{l}}$ for every $\mathbf{l} \in \mathbb{Z}^d$ and $\rho_{\mathbf{j}}(0) \neq 0$ for all $\mathbf{n} \in \mathbb{N}^d$ and $\mathbf{j} \in \Omega_{\mathbf{n}} \setminus \{\mathbf{0}\}$.

Theorem 2.4. *We have that*

$$\rho_{\mathbf{j}}(\mathbf{x}) = \sum_{\mathbf{p}\in\mathbb{Z}^d} (a_{2\mathbf{np}+\mathbf{j}}\cos((2\mathbf{np} + \mathbf{j}) \cdot \mathbf{x}) + a_{2\mathbf{np}-\mathbf{j}}\cos((2\mathbf{np} - \mathbf{j}) \cdot \mathbf{x})),$$

$$\sigma_{\mathbf{j}}(\mathbf{x}) = \sum_{\mathbf{p}\in\mathbb{Z}^d} (a_{2\mathbf{np}+\mathbf{j}}\sin((2\mathbf{np} + \mathbf{j}) \cdot \mathbf{x}) - a_{2\mathbf{np}-\mathbf{j}}\sin((2\mathbf{np} - \mathbf{j}) \cdot \mathbf{x})).$$

Definition 2.5. *We define* $\widetilde{sk}_{\mathbf{n}}$ *by*

$$\widetilde{sk}_{\mathbf{n}}(\mathbf{x}) = \frac{1}{N} + \frac{1}{N}\sum_{\mathbf{j}\in\Omega_{\mathbf{n}}^*} \frac{\rho_{\mathbf{j}}(\mathbf{x})}{\rho_{\mathbf{j}}(\mathbf{0})}, \quad \Omega_{\mathbf{n}}^* = \Omega_{\mathbf{n}} \setminus \{\mathbf{0}\}.$$

Definition 2.6. *Let* f *be a function defined on* $\mathbb{T}^d$ *and let* $\{\mathbf{y_j} : \mathbf{j} \in \Omega_{\mathbf{n}}\} \subset \mathbb{T}^d$. *If there are constants* $c^*, c_{\mathbf{k}}^* \in \mathbb{R}$, *such that*

$$sk_{\mathbf{n}}(f, \mathbf{y_j}) = c^* + \sum_{\mathbf{k}\in\Omega_{\mathbf{n}}} c_{\mathbf{k}}^* K(\mathbf{y_j} - \mathbf{x_k}) = f(\mathbf{y_j}), \quad \mathbf{j} \in \Omega_{\mathbf{n}},$$

we say that the sk-spline

$$sk_{\mathbf{n}}(f, \mathbf{x}) = c^* + \sum_{\mathbf{k}\in\Omega_{\mathbf{n}}} c_{\mathbf{k}}^* K(\mathbf{x} - \mathbf{x_k})$$

is an interpolating sk-spline of f *with knots* $\mathbf{x_k}$ *and interpolation points* $\mathbf{y_k}$.

Theorem 2.7. *For any function* f *defined on* $\mathbb{T}^d$, *there is an unique interpolating sk-spline of* f *with knots and interpolation points* $\mathbf{x_k}, \mathbf{k} \in \Omega_{\mathbf{n}}$, *that can be written in the form*

$$sk_{\mathbf{n}}(f, \mathbf{x}) = \sum_{\mathbf{k}\in\Omega_{\mathbf{n}}} f(\mathbf{x_k})\widetilde{sk}_{\mathbf{n}}(\mathbf{x} - \mathbf{x_k}).$$

Remark 2.8. *Suppose* $a_{\mathbf{l}} \geq 0$, *for all* $\mathbf{l} \in \mathbb{Z}^d$ *and* $\sum_{\mathbf{p}\in\mathbb{Z}^d} a_{\mathbf{2np-k}} \leq C a_{\mathbf{2n-k}}$, *for all* $\mathbf{n} = (n_1, n_2, \ldots, n_d) \in \mathbb{N}^d$ *and every* $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{Z}^d$, *with* $0 \leq k_j \leq n_j$, *for* $j = 1, 2, \ldots, d$, *where* C *is a positive constant independent of* $\mathbf{n}$ *and* $\mathbf{k}$. *Then* $\sum_{\mathbf{l}\in\mathbb{Z}^d} a_{\mathbf{l}} < \infty$.

Lemma 2.9. *Let* $a : [0, +\infty) \to \mathbb{R}$ *be a decreasing and positive function and let* $|\cdot| = |\cdot|_p$ *for some* $1 \leq p \leq \infty$. *For each* $\mathbf{p} \in \mathbb{Z}^d$, *let* $a_{\mathbf{p}} = a(|\mathbf{p}|)$. *Suppose that there is a constant* $c_1 > 0$ *such that for each* $\mathbf{n} \in \mathbb{N}^d$,

$$\sum_{\mathbf{p}\in\mathbb{Z}^d} a_{\mathbf{2np}} \leq c_1 a_{\mathbf{2n}}. \tag{16}$$

Then there is a constant $c_2 > 0$ such that for each $\mathbf{n} \in \mathbb{N}^d$ and $\mathbf{k} \in \mathbb{Z}^d$ with $|\mathbf{k}| \le |\mathbf{n}|$, we have

$$\sum_{\mathbf{p}\in\mathbb{Z}^d} a_{2\mathbf{np}-\mathbf{k}} \le c_2 a_{2\mathbf{n}-\mathbf{k}}. \tag{17}$$

Theorem 2.10. *Let $a : [0, +\infty) \to \mathbb{R}$ be a decreasing and positive function and $|\cdot| = |\cdot|_p$ for some $1 \le p \le \infty$. For each $\mathbf{p} \in \mathbb{Z}^d$ let $a_{\mathbf{p}} = a(|\mathbf{p}|)$. Consider the kernel K given by*

$$K(\mathbf{x}) = \sum_{\mathbf{l}\in\mathbb{Z}^d} a_{\mathbf{l}} e^{i\mathbf{l}\cdot\mathbf{x}},$$

such that

$$\sum_{\mathbf{p}\in\mathbb{Z}^d} a_{2\mathbf{np}} \le C a_{2\mathbf{n}},$$

where C is a positive constant independent of $\mathbf{n} \in \mathbb{N}^d$. Then there is a positive constant $\overline{C}$, such that for each $1 \le p \le 2 \le q \le \infty$, with $p^{-1} - q^{-1} \ge 2^{-1}$ and all $\mathbf{n} \in \mathbb{N}^d$, we have

$$\sup_{f\in K*U_p} ||f - sk_{\mathbf{n}}(f,\cdot)||_q \le \overline{C} \left(\sum_{|\mathbf{l}|\ge|\mathbf{n}|} a_{\mathbf{l}}^{qp(q-p)^{-1}} \right)^{p^{-1}-q^{-1}}.$$

3 Approximation by sk-Splines

In this section we will prove the Theorem 3.6, our main result. In our applications we will use Corollary 3.7 since its hypotheses are easier to verify.

Lemma 3.1. *For $\mathbf{l} \in \mathbb{Z}^d$ we have*

$$\widehat{sk}_{\mathbf{n}}(\mathbf{l}) = \begin{cases} 1/N, & \mathbf{l} = \mathbf{0}, \\ 0, & \mathbf{l} \equiv \mathbf{0} \bmod (2\mathbf{n}), \mathbf{l} \ne \mathbf{0}, \\ \dfrac{2a_{\mathbf{l}}}{N\rho_{\mathbf{l}}(\mathbf{0})}, & \mathbf{l} \not\equiv \mathbf{0} \bmod (2\mathbf{n}). \end{cases}$$

Proof: Using Theorem 2.4,

$$\begin{aligned}
\widehat{\widehat{sk}}_{\mathbf{n}}(\mathbf{l}) &= \frac{1}{N}\int_{\mathbb{T}^d} e^{-i\mathbf{l}\cdot\mathbf{x}}d\nu(\mathbf{x}) + \frac{1}{N}\sum_{\mathbf{j}\in\Omega^*_{\mathbf{n}}}\frac{1}{\rho_{\mathbf{j}}(\mathbf{0})}\int_{\mathbb{T}^d}\rho_{\mathbf{j}}(\mathbf{x})e^{-i\mathbf{l}\cdot\mathbf{x}}d\nu(\mathbf{x}) \\
&= \frac{1}{N}\int_{\mathbb{T}^d} e^{-i\mathbf{l}\cdot\mathbf{x}}d\nu(\mathbf{x}) \\
&+ \frac{1}{2N}\sum_{\mathbf{p}\in\mathbb{Z}^d}\sum_{\mathbf{j}\in\Omega^*_{\mathbf{n}}}\frac{a_{2\mathbf{np}+\mathbf{j}}}{\rho_{\mathbf{j}}(\mathbf{0})}\int_{\mathbb{T}^d}(e^{i(2\mathbf{np}+\mathbf{j}-\mathbf{l})\cdot\mathbf{x}} + e^{-i(2\mathbf{np}+\mathbf{j}+\mathbf{l})\cdot\mathbf{x}})d\nu(\mathbf{x}) \\
&+ \frac{1}{2N}\sum_{\mathbf{p}\in\mathbb{Z}^d}\sum_{\mathbf{j}\in\Omega^*_{\mathbf{n}}}\frac{a_{2\mathbf{np}-\mathbf{j}}}{\rho_{\mathbf{j}}(\mathbf{0})}\int_{\mathbb{T}^d}(e^{i(2\mathbf{np}-\mathbf{j}-\mathbf{l})\cdot\mathbf{x}} + e^{-i(2\mathbf{np}-\mathbf{j}+\mathbf{l})\cdot\mathbf{x}})d\nu(\mathbf{x}) \quad (18)
\end{aligned}$$

Case 1: Suppose that $\mathbf{l}=\mathbf{0}$. Then for $\mathbf{j}\in\Omega^*_{\mathbf{n}}$ we have $2\mathbf{np}+\mathbf{j}\neq\mathbf{0}$ and $2\mathbf{np}-\mathbf{j}\neq\mathbf{0}$, for every $\mathbf{p}\in\mathbb{Z}^d$. Then by Lemma 2.1

$$\int_{\mathbb{T}^d} e^{-i(2\mathbf{np}-\mathbf{j})\cdot\mathbf{x}}d\nu(\mathbf{x}) = 0$$

and then by (18)

$$\widehat{\widehat{sk}}_{\mathbf{n}}(\mathbf{l}) = \widehat{\widehat{sk}}_{\mathbf{n}}(\mathbf{0}) = \frac{1}{N}\int_{\mathbb{T}^d} d\nu(\mathbf{x}) = \frac{1}{N}.$$

Case 2: Suppose $\mathbf{l}\equiv\mathbf{0}\,\mathrm{mod}(2\mathbf{n})$, $\mathbf{l}\neq\mathbf{0}$, that is, $\mathbf{l}=2\mathbf{nq}$, $\mathbf{q}\in\mathbb{Z}^d$, $\mathbf{q}\neq\mathbf{0}$. For every $\mathbf{j}\in\Omega^*_{\mathbf{n}}$ and $\mathbf{p}\in\mathbb{Z}^d$ we have

$$2\mathbf{np}+\mathbf{j}-\mathbf{l} = 2\mathbf{n}(\mathbf{p}-\mathbf{q})+\mathbf{j}\neq\mathbf{0}, \qquad 2\mathbf{np}-\mathbf{j}-\mathbf{l} = 2\mathbf{n}(\mathbf{p}-\mathbf{q})-\mathbf{j}\neq\mathbf{0},$$

$$2\mathbf{np}+\mathbf{j}+\mathbf{l} = 2\mathbf{n}(\mathbf{p}+\mathbf{q})+\mathbf{j}\neq\mathbf{0}, \qquad 2\mathbf{np}-\mathbf{j}+\mathbf{l} = 2\mathbf{n}(\mathbf{p}+\mathbf{q})-\mathbf{j}\neq\mathbf{0}$$

and since we also have $\mathbf{l}\neq\mathbf{0}$, it follows from Lemma 2.1 and (18) that $\widehat{\widehat{sk}}_{\mathbf{n}}(\mathbf{l})=0$.

Case 3: Suppose $\mathbf{l}\not\equiv\mathbf{0}\,\mathrm{mod}(2\mathbf{n})$, that is, $\mathbf{l}=2\mathbf{nq}+\mathbf{k}$, $\mathbf{k}\in\Omega^*_{\mathbf{n}}$, $\mathbf{q}\in\mathbb{Z}^d$. Thus $\mathbf{0}=2\mathbf{np}+\mathbf{j}-\mathbf{l} = 2\mathbf{np}+\mathbf{j}-2\mathbf{nq}-\mathbf{k} = 2\mathbf{n}(\mathbf{p}-\mathbf{q})+(\mathbf{j}-\mathbf{k})$ implies $\mathbf{p}=\mathbf{q}$ and $\mathbf{j}=\mathbf{k}$; $\mathbf{0}=2\mathbf{np}+\mathbf{j}+\mathbf{l} = 2\mathbf{np}+\mathbf{j}+2\mathbf{nq}+\mathbf{k} = 2\mathbf{n}(\mathbf{p}+\mathbf{q})+(\mathbf{j}+\mathbf{k})$ implies that $\mathbf{p}=-\mathbf{1}-\mathbf{q}$ and $\mathbf{j}=2\mathbf{n}-\mathbf{k}$, $\mathbf{1}=(1,1,\ldots,1)$; $\mathbf{0}=2\mathbf{np}-\mathbf{j}-\mathbf{l} =$

$\mathbf{2np}-\mathbf{j}-\mathbf{2nq}-\mathbf{k} \;=\; \mathbf{2n}(\mathbf{p}-\mathbf{q})-(\mathbf{j}+\mathbf{k})$ implies $\mathbf{p}=\mathbf{1}+\mathbf{q}$ and $\mathbf{j}=\mathbf{2n}-\mathbf{k}$; and $\mathbf{0}=\mathbf{2np}-\mathbf{j}+\mathbf{l} \;=\; \mathbf{2np}-\mathbf{j}+\mathbf{2nq}+\mathbf{k} \;=\; \mathbf{2n}(\mathbf{p}+\mathbf{q})+(\mathbf{k}-\mathbf{j})$ implies $\mathbf{p}=-\mathbf{q}$ and $\mathbf{j}=\mathbf{k}$. Then using Lemma 2.1 and (18) we obtain that

$$\widehat{\widetilde{sk}}_{\mathbf{n}}(\mathbf{l}) \;=\; \frac{1}{2N}\frac{a_{\mathbf{2nq}+\mathbf{k}}}{\rho_{\mathbf{k}}(\mathbf{0})}+\frac{1}{2N}\frac{a_{\mathbf{2n}(-\mathbf{1}-\mathbf{q})+\mathbf{2n}-\mathbf{k}}}{\rho_{\mathbf{2n}-\mathbf{k}}(\mathbf{0})}+\frac{1}{2N}\frac{a_{\mathbf{2n}(\mathbf{1}+\mathbf{q})-\mathbf{2n}+\mathbf{k}}}{\rho_{\mathbf{2n}-\mathbf{k}}(\mathbf{0})}+\frac{1}{2N}\frac{a_{-\mathbf{2nq}-\mathbf{k}}}{\rho_{\mathbf{k}}(\mathbf{0})}.$$

By Lemma 2.3 we have that $\rho_{\mathbf{2nq}-\mathbf{k}}(\mathbf{0}) = \rho_{\mathbf{k}}(\mathbf{0}) = \rho_{\mathbf{2nq}+\mathbf{k}}(\mathbf{0}) = \rho_{\mathbf{l}}(\mathbf{0})$ and since $a_{\mathbf{2nq}+\mathbf{k}} = a_{\mathbf{l}} = a_{-\mathbf{l}} = a_{-\mathbf{2nq}-\mathbf{k}}$ it follows that

$$\widehat{\widetilde{sk}}_{\mathbf{n}}(\mathbf{l}) = \frac{2a_{\mathbf{l}}}{N\rho_{\mathbf{l}}(\mathbf{0})}.$$

Thus the lemma is proved. □

Lemma 3.2. *Let $\phi \in L^1(\mathbb{T}^d)$ be such that $||\phi||_1 \le 1$ and let*

$$f(\mathbf{x}) = \int_{\mathbb{T}^d} K(\mathbf{x}-\mathbf{y})\phi(\mathbf{y})d\nu(\mathbf{y}) = K*\phi(\mathbf{x}).$$

For every $\mathbf{j}\in\Omega_{\mathbf{n}}$ we have

$$\sigma_{\mathbf{n}}(f,\mathbf{x}) = f(\mathbf{x}) - sk_{\mathbf{n}}(f,\mathbf{x}) = \int_{\mathbb{T}^d} \Phi_{\mathbf{n}}(\mathbf{x},\mathbf{y})\phi(\mathbf{y})d\nu(\mathbf{y})$$

where

$$\Phi_{\mathbf{n}}(\mathbf{x},\mathbf{y}) = K(\mathbf{x},\mathbf{y}) - \sum_{\mathbf{k}\in\Omega_{\mathbf{n}}} K(\mathbf{x}_{\mathbf{k}}-\mathbf{y})\widetilde{sk}_{\mathbf{n}}(\mathbf{x}-\mathbf{x}_{\mathbf{k}}).$$

Moreover, considering the Fourier series of $\Phi_{\mathbf{n}}(\mathbf{x},\mathbf{y})$ given by

$$\Phi_{\mathbf{n}}(\mathbf{x},\mathbf{y}) \sim \sum_{\mathbf{k_1},\mathbf{k_2}\in\mathbb{Z}^d} \widehat{\Phi_{\mathbf{n}}}(\mathbf{k_1},\mathbf{k_2})e^{i\mathbf{k_1}\cdot\mathbf{x}+i\mathbf{k_2}\cdot\mathbf{y}}$$

and coefficients of Fourier

$$\widehat{\Phi_{\mathbf{n}}}(\mathbf{k_1},\mathbf{k_2}) = \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \Phi_{\mathbf{n}}(\mathbf{x},\mathbf{y})e^{-i\mathbf{k_1}\cdot\mathbf{x}-i\mathbf{k_2}\cdot\mathbf{y}}d\nu(\mathbf{x})d\nu(\mathbf{y}),$$

we have for $\mathbf{k},\mathbf{j}\in\mathbb{Z}^d$,

$$\widehat{\Phi_{\mathbf{n}}}(-\mathbf{k}-\mathbf{j},\mathbf{k}) = a_{-\mathbf{k}}\left\{\begin{array}{ll} 1, & \mathbf{j}=\mathbf{0}, \\ 0, & \mathbf{j}\neq\mathbf{0}, \end{array}\right\} - a_{-\mathbf{k}}\,\widehat{\widetilde{sk}}_{\mathbf{n}}(-\mathbf{k}-\mathbf{j})\left\{\begin{array}{ll} N, & \mathbf{j}\equiv\mathbf{0} \bmod (\mathbf{2n}), \\ 0, & \text{otherwise}, \end{array}\right\}$$

where $\widehat{\widetilde{sk}}_{\mathbf{n}}(\mathbf{l})$ *is given in Lemma 3.1.*

Proof: By Theorem 2.7 we have

$$\sigma_{\mathbf{n}}(f,\mathbf{x}) = \int_{\mathbb{T}^d} \Phi_{\mathbf{n}}(\mathbf{x},\mathbf{y})\phi(\mathbf{y})d\nu(\mathbf{y}).$$

Let $\mathbf{k},\mathbf{j} \in \mathbb{Z}^d$. Observe that

$$\begin{aligned}
\widehat{\Phi_{\mathbf{n}}}(-\mathbf{k}-\mathbf{j},\mathbf{k}) &= \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} K(\mathbf{x}-\mathbf{y})e^{i(\mathbf{k}+\mathbf{j})\cdot\mathbf{x}-i\mathbf{k}\cdot\mathbf{y}}d\nu(\mathbf{x})d\nu(\mathbf{y}) \\
&- \int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\left(\sum_{\mathbf{m}\in\Omega_{\mathbf{n}}} K(\mathbf{x}_{\mathbf{m}}-\mathbf{y})\widetilde{sk}_{\mathbf{n}}(\mathbf{x}-\mathbf{x}_{\mathbf{m}})e^{i(\mathbf{k}+\mathbf{j})\cdot\mathbf{x}-i\mathbf{k}\cdot\mathbf{y}}\right)d\nu(\mathbf{x})d\nu(\mathbf{y}).
\end{aligned}$$

Let us verify each of the above integrals separately. We denote the first term of the sum by I and the second by II. Then by Lemma 2.1 we have

$$\begin{aligned}
I &= \int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\left(\sum_{\mathbf{s}\in\mathbb{Z}^d} a_{\mathbf{s}}e^{i\mathbf{s}\cdot(\mathbf{x}-\mathbf{y})}\right)e^{i(\mathbf{k}+\mathbf{j})\cdot\mathbf{x}-i\mathbf{k}\cdot\mathbf{y}}d\nu(\mathbf{x})d\nu(\mathbf{y}) \\
&= \sum_{\mathbf{s}\in\mathbb{Z}^d} a_{\mathbf{s}}\left(\int_{\mathbb{T}^d} e^{i(\mathbf{s}+\mathbf{k}+\mathbf{j})\cdot\mathbf{x}}d\nu(\mathbf{x})\right)\left(\int_{\mathbb{T}^d} e^{i(-\mathbf{s}-\mathbf{k})\cdot\mathbf{y}}d\nu(\mathbf{y})\right) \\
&= \sum_{\mathbf{s}\in\mathbb{Z}^d} a_{\mathbf{s}}\delta_{\mathbf{s},-(\mathbf{k}+\mathbf{j})}\delta_{\mathbf{s},-\mathbf{k}} \\
&= \begin{cases} a_{-\mathbf{k}}, & \mathbf{j}=\mathbf{0}, \\ 0, & \mathbf{j}\neq\mathbf{0}, \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
II &= \sum_{\mathbf{m}\in\Omega_\mathbf{n}} \sum_{\mathbf{s}\in\mathbb{Z}^d} e^{i\mathbf{s}\cdot\mathbf{x}_\mathbf{m}} a_\mathbf{s} \left(\int_{\mathbb{T}^d} e^{i(-\mathbf{s}-\mathbf{k})\cdot\mathbf{y}} d\nu(\mathbf{y})\right) \left(\int_{\mathbb{T}^d} e^{i(\mathbf{k}+\mathbf{j})\cdot\mathbf{x}} \widetilde{sk}_\mathbf{n}(\mathbf{x}-\mathbf{x}_\mathbf{m}) d\nu(\mathbf{x})\right) \\
&= \sum_{\mathbf{m}\in\Omega_\mathbf{n}} \sum_{\mathbf{s}\in\mathbb{Z}^d} e^{i\mathbf{s}\cdot\mathbf{x}_\mathbf{m}} a_\mathbf{s} \delta_{\mathbf{s},-\mathbf{k}} \int_{\mathbb{T}^d} e^{i(\mathbf{k}+\mathbf{j})\cdot\mathbf{x}} \widetilde{sk}_\mathbf{n}(\mathbf{x}-\mathbf{x}_\mathbf{m}) d\nu(\mathbf{x}) \\
&= \sum_{\mathbf{m}\in\Omega_\mathbf{n}} e^{-i\mathbf{k}\cdot\mathbf{x}_\mathbf{m}} a_{-\mathbf{k}} \int_{\mathbb{T}^d} e^{i(\mathbf{k}+\mathbf{j})\cdot\mathbf{x}} \widetilde{sk}_\mathbf{n}(\mathbf{x}-\mathbf{x}_\mathbf{m}) d\nu(\mathbf{x}) \\
&= a_{-\mathbf{k}} \sum_{\mathbf{m}\in\Omega_\mathbf{n}} e^{-i\mathbf{k}\cdot\mathbf{x}_\mathbf{m}} \int_{\mathbb{T}^d} e^{i(\mathbf{k}+\mathbf{j})\cdot\mathbf{x}} \left(\sum_{\mathbf{l}\in\mathbb{Z}^d} \widehat{\widetilde{sk}}_\mathbf{n}(\mathbf{l}) e^{-i\mathbf{l}\cdot\mathbf{x}_\mathbf{m}} e^{i\mathbf{l}\cdot\mathbf{x}}\right) d\nu(\mathbf{x}) \\
&= a_{-\mathbf{k}} \sum_{\mathbf{m}\in\Omega_\mathbf{n}} e^{-i\mathbf{k}\cdot\mathbf{x}_\mathbf{m}} \sum_{\mathbf{l}\in\mathbb{Z}^d} \widehat{\widetilde{sk}}_\mathbf{n}(\mathbf{l}) e^{-i\mathbf{l}\cdot\mathbf{x}_\mathbf{m}} \delta_{\mathbf{l},-(\mathbf{k}+\mathbf{j})} \\
&= a_{-\mathbf{k}} \sum_{\mathbf{m}\in\Omega_\mathbf{n}} e^{-i\mathbf{k}\cdot\mathbf{x}_\mathbf{m}} \widehat{\widetilde{sk}}_\mathbf{n}(-\mathbf{k}-\mathbf{j}) e^{i(\mathbf{k}+\mathbf{j})\cdot\mathbf{x}_\mathbf{m}} \\
&= a_{-\mathbf{k}} \widehat{\widetilde{sk}}_\mathbf{n}(-\mathbf{k}-\mathbf{j}) \sum_{\mathbf{m}\in\Omega_\mathbf{n}} e^{-i\mathbf{j}\cdot\mathbf{x}_\mathbf{m}} \\
&= a_{-\mathbf{k}} \widehat{\widetilde{sk}}_\mathbf{n}(-\mathbf{k}-\mathbf{j}) \begin{cases} N, & \mathbf{j}\equiv\mathbf{0} \bmod (2\mathbf{n}), \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

Thus we obtain the expression of $\widehat{\Phi}_\mathbf{n}(-\mathbf{k}-\mathbf{j},\mathbf{k})$ as a consequence of integrals I and II. □

Lemma 3.3. *Let $\Phi_\mathbf{n}$ as in Lemma 3.2. Suppose that $a_\mathbf{k} > 0$ for every $\mathbf{k}\in\mathbb{Z}^d$. Then*

$$\sum_{\mathbf{l},\mathbf{m}\in\mathbb{Z}^d} |\widehat{\Phi_\mathbf{n}}(\mathbf{l},\mathbf{m})| < \infty.$$

Proof: Let us show that $\sum_{\mathbf{k},\mathbf{j}\in\mathbb{Z}^d} |\widehat{\Phi_\mathbf{n}}(-\mathbf{k}-\mathbf{j},\mathbf{k})| < \infty$. Note that

$$\begin{aligned}
\sum_{\mathbf{k},\mathbf{j}\in\mathbb{Z}^d} |\widehat{\Phi_\mathbf{n}}(-\mathbf{k}-\mathbf{j},\mathbf{k})| &= \sum_{\mathbf{k}\in\mathbb{Z}^d} \sum_{\mathbf{j}\in\mathbb{Z}^d} |\widehat{\Phi_\mathbf{n}}(-\mathbf{k}-\mathbf{j},\mathbf{k})| \\
&= \sum_{\mathbf{k}\in\mathbb{Z}^d} |\widehat{\Phi_\mathbf{n}}(-\mathbf{k},\mathbf{k})| + \sum_{\mathbf{k}\in\mathbb{Z}^d} \sum_{\mathbf{j}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}} |\widehat{\Phi_\mathbf{n}}(-\mathbf{k}-\mathbf{j},\mathbf{k})| \\
&= S_1 + S_2. \qquad (19)
\end{aligned}$$

Applying Lemma 3.1, Lemma 3.2 and Lemma 2.3, we can estimate S_2 in the following form:

$$\begin{aligned} S_2 &= \sum_{\mathbf{k}\in\mathbb{Z}^d}\sum_{\mathbf{j}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}}\left| a_{-\mathbf{k}}\left\{\begin{array}{ccc} 1 & , & \mathbf{j}=\mathbf{0}, \\ 0 & , & \mathbf{j}\neq\mathbf{0}, \end{array}\right\} - a_{-\mathbf{k}}\widehat{\widetilde{sk}}_{\mathbf{n}}(-\mathbf{k}-\mathbf{j})\left\{\begin{array}{ccc} N & , & \mathbf{j}\equiv\mathbf{0} \bmod (2\mathbf{n}), \\ 0 & , & \mathbf{j}\not\equiv\mathbf{0} \bmod (2\mathbf{n}), \end{array}\right\}\right| \\ &= N\sum_{\mathbf{k}\in\mathbb{Z}^d} a_{-\mathbf{k}} \sum_{\mathbf{s}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}} \widehat{\widetilde{sk}}_{\mathbf{n}}(-(\mathbf{k}+2\mathbf{ns})) \\ &\leq \sum_{\mathbf{s}\in\mathbb{Z}^d} a_{2\mathbf{ns}} + 2\sum_{\mathbf{k}\in\mathbb{Z}^d}\frac{a_{-\mathbf{k}}}{\rho_{\mathbf{k}}(\mathbf{0})}\sum_{\mathbf{s}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}} a_{-(\mathbf{k}+2\mathbf{ns})}. \end{aligned} \tag{20}$$

By Theorem 2.4, $\rho_{\mathbf{k}}(\mathbf{0}) > \sum_{\mathbf{p}\in\mathbb{Z}^d} a_{2\mathbf{np}+\mathbf{k}}$, then using (20) we obtain that

$$S_2 < \sum_{\mathbf{s}\in\mathbb{Z}^d} a_{2\mathbf{ns}} + 2\sum_{\mathbf{k}\in\mathbb{Z}^d} a_{\mathbf{k}} \leq 3\sum_{\mathbf{s}\in\mathbb{Z}^d} a_k < \infty.$$

By Lemmas 3.1 and 3.2 we can estimate S_1 in the following way:

$$\begin{aligned} S_1 &= \sum_{\mathbf{k}\in\mathbb{Z}^d}\left| a_{-\mathbf{k}}\left(1 - N\widehat{\widetilde{sk}}_{\mathbf{n}}(-\mathbf{k})\right)\right| \\ &= \sum_{\mathbf{k}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}} a_{-2\mathbf{nk}} + \sum_{\mathbf{k}\not\equiv\mathbf{0} \bmod (2\mathbf{n})} a_{-\mathbf{k}}\left|1 - \frac{2a_{-\mathbf{k}}}{\rho_{\mathbf{k}}(\mathbf{0})}\right|. \end{aligned}$$

Since $0 < 2a_{\mathbf{k}}/\rho_{\mathbf{k}}(\mathbf{0}) = 2a_{-\mathbf{k}}/\rho_{\mathbf{k}}(\mathbf{0}) < 1$, we have $0 < 1 - 2a_{-\mathbf{k}}/\rho_{\mathbf{k}}(\mathbf{0}) < 1$ and thus $S_1 < \sum_{\mathbf{k}\in\mathbb{Z}^d} a_{\mathbf{k}} < \infty$. Since S_1 and S_2 are bounded, the result follows by (19). □

Lemma 3.4. *Let $\Phi_{\mathbf{n}}$ as in Lemma 3.2. Consider the operator T defined on $L^p(\mathbb{T}^d)$, $1 \leq p \leq \infty$, by*

$$T\phi(\mathbf{x}) = \int_{\mathbb{T}^d}\Phi_{\mathbf{n}}(\mathbf{x},\mathbf{y})\phi(\mathbf{y})\,d\nu(\mathbf{y}).$$

Then

$$||T||_{p,p} \leq \sum_{\mathbf{j}\in\mathbb{Z}^d} ||\Lambda_{\mathbf{j}}||_{p,p},$$

where $\Lambda_{\mathbf{j}}$ is the multiplier operator generated by the sequence $\Lambda_j = \{\widehat{\Phi_{\mathbf{n}}}(-\mathbf{k}-\mathbf{j},\mathbf{k})\}_{\mathbf{k}\in\mathbb{Z}^d}$.

Proof: Applying Lemma 3.3 we have

$$\begin{aligned} T\phi(\mathbf{x}) &= \int_{\mathbb{T}^d} \left(\sum_{\mathbf{k}_1\in\mathbb{Z}^d} \sum_{\mathbf{k}_2\in\mathbb{Z}^d} \widehat{\Phi_{\mathbf{n}}}(\mathbf{k}_1,\mathbf{k}_2) e^{i\mathbf{k}_1\cdot\mathbf{x}+i\mathbf{k}_2\cdot\mathbf{y}} \right) \phi(\mathbf{y})\, d\nu(\mathbf{y}) \\ &= \int_{\mathbb{T}^d} \left(\sum_{\mathbf{j}\in\mathbb{Z}^d} \sum_{\mathbf{k}\in\mathbb{Z}^d} \widehat{\Phi_{\mathbf{n}}}(-\mathbf{k}+\mathbf{j},\mathbf{k}) e^{-i(\mathbf{k}-\mathbf{j})\cdot\mathbf{x}+i\mathbf{k}\cdot\mathbf{y}} \right) \phi(\mathbf{y})\, d\nu(\mathbf{y}) \\ &= \sum_{\mathbf{j}\in\mathbb{Z}^d} e^{i\mathbf{j}\cdot\mathbf{x}} \left(\sum_{\mathbf{k}\in\mathbb{Z}^d} \widehat{\Phi_{\mathbf{n}}}(-\mathbf{k}+\mathbf{j},\mathbf{k}) \widehat{\phi}(-\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} \right) \\ &= \sum_{\mathbf{j}\in\mathbb{Z}^d} e^{i\mathbf{j}\cdot\mathbf{x}} (\Lambda_{\mathbf{j}}^*\phi)(x), \end{aligned}$$

where $\Lambda_{\mathbf{j}}^* = \{\widehat{\Phi_{\mathbf{n}}}(\mathbf{k}+\mathbf{j},-\mathbf{k})\}_{\mathbf{k}\in\mathbb{Z}^d}$. Using Lemma 2.3 we have that $\rho_{\mathbf{j}}(\mathbf{0}) = \rho_{-\mathbf{j}}(\mathbf{0})$ and then by Lemma 3.1 $\widehat{\widehat{sk}}_{\mathbf{n}}(\mathbf{l}) = \widehat{\widehat{sk}}_{\mathbf{n}}(-\mathbf{l})$, for every $\mathbf{l} \in \mathbb{Z}^d$. In particular $\widehat{\widehat{sk}}_{\mathbf{n}}(-\mathbf{k}-\mathbf{j}) = \widehat{\widehat{sk}}_{\mathbf{n}}(\mathbf{k}+\mathbf{j})$ for every $\mathbf{k},\mathbf{j} \in \mathbb{Z}^d$. Therefore, since $\mathbf{a}_{\mathbf{k}} = \mathbf{a}_{-\mathbf{k}}$, by Lemma 3.2 we have $\widehat{\Phi_{\mathbf{n}}}(-\mathbf{k}-\mathbf{j},\mathbf{k}) = \widehat{\Phi_{\mathbf{n}}}(\mathbf{k}+\mathbf{j},-\mathbf{k})$. Then

$$||T\phi||_p \leq \sum_{\mathbf{j}\in\mathbb{Z}^d} ||\Lambda_{\mathbf{j}}^*\phi||_p = \sum_{\mathbf{j}\in\mathbb{Z}^d} ||\Lambda_{\mathbf{j}}\phi||_p,$$

what proves the lemma. □

Lemma 3.5. *Let $\Phi_{\mathbf{n}}$ be as in Lemma 3.2 and $\Lambda_{\mathbf{j}}$ as in Lemma 3.4. Then for $1 \leq p \leq \infty$ and $\mathbf{n} \in \mathbb{Z}^d$, we have*

$$\sup_{f\in K*U_p} ||f - sk_{\mathbf{n}}(f,\cdot)||_p \leq \sum_{\mathbf{s}\in\mathbb{Z}^d} ||\Lambda_{2\mathbf{n}\mathbf{s}}||_{p,p}$$

where $U_p = \{\varphi \in L^p(\mathbb{T}^d) : ||\varphi||_p \leq 1\}$ and $K * U_p = \{K * \varphi : \varphi \in U_p\}$.

Proof: Applying Lemma 3.2 we have that $\widehat{\Phi_{\mathbf{n}}}(-\mathbf{k}-\mathbf{j},\mathbf{k}) = 0$ for every $\mathbf{j} \not\equiv \mathbf{0}$ mod $(2\mathbf{n})$, for all $\mathbf{k} \in \mathbb{Z}^d$. By Lemma 3.2 and Lemma 3.4 we have

$$\begin{aligned} \sup_{f\in K*U_p} ||f - sk_{\mathbf{n}}(f,\cdot)||_p &= \sup_{\phi\in U_p} \left|\left| \int_{\mathbb{T}^d} \Phi_{\mathbf{n}}(\cdot,\mathbf{y})\phi(\mathbf{y})d\nu(\mathbf{y}) \right|\right|_p \\ &\leq \sum_{\mathbf{j}\in\mathbb{Z}^d} ||\Lambda_{\mathbf{j}}||_{p,p} = \sum_{\mathbf{s}\in\mathbb{Z}^d} ||\Lambda_{2\mathbf{n}\mathbf{s}}||_{p,p}, \end{aligned}$$

concluding the proof. □

Theorem 3.6. *Let $|\cdot|$ be a norm on $\mathbb{R}^d$ and let K be the kernel given by*

$$K(\mathbf{x}) = \sum_{\mathbf{l}\in\mathbb{Z}^d} a_{\mathbf{l}} e^{i\mathbf{l}\cdot\mathbf{x}},$$

where $(a_{\mathbf{l}})_{\mathbf{l}\in\mathbb{Z}^d}$ is a sequence of real numbers such that $a_{\mathbf{l}} = a_{-\mathbf{l}}$ for every $\mathbf{l} \in \mathbb{Z}^d$ and $a_{\mathbf{l}} \geq a_{\mathbf{k}} > \mathbf{0}$ if $|\mathbf{k}| \geq |\mathbf{l}|$, $\mathbf{k}, \mathbf{l} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$. Suppose that there exists a positive constant C such that for every $\mathbf{n} \in \mathbb{N}^d$ and all $\mathbf{k} \in \mathbb{Z}^d$ with $|\mathbf{k}| \leq |\mathbf{n}|$,

$$\sum_{\mathbf{p}\in\mathbb{Z}^d} a_{2\mathbf{np}-\mathbf{k}} \leq C a_{2\mathbf{n}-\mathbf{k}}.$$

Then there exists a positive constant $\overline{C}$ independent of $\mathbf{n}$ such that for $1 \leq q \leq 2 \leq p \leq \infty$,

$$\sup_{f\in K*U_p} ||f - sk_{\mathbf{n}}(f,\cdot)||_q \leq \overline{C} a_{\mathbf{n}}.$$

Proof: It follows from Remark 2.8 that the kernel K satisfies the conditions in Lemma 2.3. Let us apply the Lemma 3.5 for $p = 2$ and then we have to bound $||\Lambda_{\mathbf{j}}||_{2,2}$ for $\mathbf{j} = 2\mathbf{ns}$, where $\Lambda_{\mathbf{j}} = \{\widehat{\Phi_{\mathbf{n}}}(-\mathbf{k}-\mathbf{j},\mathbf{k})\}_{\mathbf{k}\in\mathbb{Z}^d}$. Let $\mathbf{j} = \mathbf{0}$. By Lemmas 3.1 and 3.2, since $a_{\mathbf{k}} = a_{-\mathbf{k}}$ and $\widetilde{sk}_{\mathbf{n}}(-\mathbf{k}) = \widetilde{sk}_{\mathbf{n}}(\mathbf{k})$, we have that

$$\widehat{\Phi_{\mathbf{n}}}(-\mathbf{k},\mathbf{k}) = a_{\mathbf{k}}\left(1 - \left\{\begin{array}{cc} 1, & \mathbf{k} = \mathbf{0}, \\ 0, & \mathbf{k} \equiv \mathbf{0} \bmod (2\mathbf{n}), \mathbf{k} \neq \mathbf{0}, \\ \dfrac{2a_{\mathbf{k}}}{\rho_{\mathbf{k}}(\mathbf{0})}, & \mathbf{k} \not\equiv \mathbf{0} \bmod (2\mathbf{n}). \end{array}\right\}\right) \quad (21)$$

For $\mathbf{k} = \mathbf{0}$ we have $\widehat{\Phi_{\mathbf{n}}}(-\mathbf{k},\mathbf{k}) = 0$. For $\mathbf{k} \equiv \mathbf{0} \bmod (2\mathbf{n})$, $\mathbf{k} \neq \mathbf{0}$, let $\mathbf{q} \in \mathbb{Z}^d$, $\mathbf{q} \neq \mathbf{0}$ be such that $\mathbf{k} = 2\mathbf{nq}$. Applying the hypothesis we obtain that $\widehat{\Phi_{\mathbf{n}}}(-\mathbf{k},\mathbf{k}) = a_{2\mathbf{nq}} \leq C a_{2\mathbf{n}} \leq C a_{\mathbf{n}}$.

By Theorem 2.4

$$\frac{2a_{\mathbf{k}}}{\rho_{\mathbf{k}}(\mathbf{0})} = 1 - \frac{\sum_{\mathbf{p}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}} (a_{2\mathbf{np}+\mathbf{k}} + a_{2\mathbf{np}-\mathbf{k}})}{2a_{\mathbf{k}} + \sum_{\mathbf{p}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}} (a_{2\mathbf{np}+\mathbf{k}} + a_{2\mathbf{np}-\mathbf{k}})}. \quad (22)$$

Suppose now that $\mathbf{k} \not\equiv \mathbf{0} \bmod(2\mathbf{n})$. For $|\mathbf{k}| \leq |\mathbf{n}|$, we have $2|\mathbf{n}| \leq |2\mathbf{n} - \mathbf{k}| + |\mathbf{k}| \leq |2\mathbf{n} - \mathbf{k}| + |\mathbf{n}|$ and thus $|\mathbf{n}| \leq |2\mathbf{n} - \mathbf{k}|$. In the same way we obtain $|\mathbf{n}| \leq |2\mathbf{n} + \mathbf{k}|$. Thus by the hypothesis, (21) and (22)

$$\begin{aligned} \widehat{\Phi_{\mathbf{n}}}(-\mathbf{k}, \mathbf{k}) &= \frac{a_{\mathbf{k}} \sum_{\mathbf{p} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}} (a_{2\mathbf{np}+\mathbf{k}} + a_{2\mathbf{np}-\mathbf{k}})}{2a_{\mathbf{k}} + \sum_{\mathbf{p} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}} (a_{2\mathbf{np}+\mathbf{k}} + a_{2\mathbf{np}-\mathbf{k}})} \\ &\leq \frac{1}{2} \sum_{\mathbf{p} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}} (a_{2\mathbf{np}+\mathbf{k}} + a_{2\mathbf{np}-\mathbf{k}}) \\ &\leq \frac{C}{2} (a_{2\mathbf{n}+\mathbf{k}} + a_{2\mathbf{n}-\mathbf{k}}) \\ &\leq C a_{\mathbf{n}}. \end{aligned} \tag{23}$$

For $|\mathbf{k}| > |\mathbf{n}|$, using (23) we have

$$\widehat{\Phi_{\mathbf{n}}}(-\mathbf{k}, \mathbf{k}) \leq \frac{a_{\mathbf{k}} \sum_{\mathbf{p} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}} (a_{2\mathbf{np}+\mathbf{k}} + a_{2\mathbf{np}-\mathbf{k}})}{\sum_{\mathbf{p} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}} (a_{2\mathbf{np}+\mathbf{k}} + a_{2\mathbf{np}-\mathbf{k}})} = a_{\mathbf{k}} \leq a_{\mathbf{n}}.$$

Therefore we show that for every $\mathbf{k} \in \mathbb{Z}^d$ we have that

$$|\widehat{\Phi_{\mathbf{n}}}(-\mathbf{k}, \mathbf{k})| \leq C a_{\mathbf{n}}. \tag{24}$$

Let $\mathbf{j} = 2\mathbf{ns},\ \mathbf{s} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$. For $|\mathbf{k}| \leq |\mathbf{ns}|$ we have $2|\mathbf{ns}| \leq |2\mathbf{ns} + \mathbf{k}| + |\mathbf{ns}|$ and then $|\mathbf{ns}| \leq |2\mathbf{ns} + \mathbf{k}|$. By Lemmas 3.1 and 3.2

$$\left|\widehat{\Phi_{\mathbf{n}}}(-\mathbf{k} - 2\mathbf{ns}, \mathbf{k})\right| = a_{-\mathbf{k}} \widehat{\widehat{Nsk}_{\mathbf{n}}}(-2\mathbf{ns} - \mathbf{k}) \leq 2a_{2\mathbf{ns}+\mathbf{k}} \frac{a_{\mathbf{k}}}{\rho_{\mathbf{k}}(\mathbf{0})} \leq 2a_{2\mathbf{ns}+\mathbf{k}} \leq 2a_{\mathbf{ns}}$$

since $a_{\mathbf{k}}/\rho_{\mathbf{k}}(\mathbf{0}) < 1$ for every $\mathbf{k} \in \mathbb{Z}^d$. For $|\mathbf{k}| \geq |\mathbf{ns}|$, by Lemmas 3.1 and 3.2

$$\left|\widehat{\Phi_{\mathbf{n}}}(-\mathbf{k} - 2\mathbf{ns}, \mathbf{k})\right| = a_{-\mathbf{k}} \widehat{\widehat{Nsk}_{\mathbf{n}}}(-\mathbf{k} - 2\mathbf{ns}) = 2a_{\mathbf{k}} \frac{a_{2\mathbf{ns}+\mathbf{k}}}{\rho_{\mathbf{k}}(\mathbf{0})} \leq 2a_{\mathbf{k}} \leq 2a_{\mathbf{ns}}$$

because $a_{2\mathbf{ns}+\mathbf{k}}/\rho_{\mathbf{k}}(\mathbf{0}) < 1$ for every $\mathbf{k} \in \mathbb{Z}^d$. So we proved that for any $\mathbf{s}, \mathbf{k} \in \mathbb{Z}^d$, $\mathbf{s} \neq \mathbf{0}$ we have that

$$|\widehat{\Phi_{\mathbf{n}}}(-\mathbf{k} - 2\mathbf{ns}, \mathbf{k})| \leq 2a_{\mathbf{ns}}. \tag{25}$$

Since for every bounded sequence of multipliers $\Lambda = \{\lambda_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}^d}$ we have $||\Lambda_{\mathbf{j}}||_{2,2} = \sup_{\mathbf{k}\in\mathbb{Z}^d} |\lambda_{\mathbf{k}}|$, then by hypothesis, Lemma 3.5, (24) and (25), we have that

$$\begin{aligned}
\sup_{f\in K*U_2} ||f - sk_{\mathbf{n}}(f,\cdot)||_2 &\leq \sum_{\mathbf{s}\in\mathbb{Z}^d} ||\Lambda_{\mathbf{2ns}}||_{2,2} \\
&= \sum_{\mathbf{s}\in\mathbb{Z}^d} \sup_{\mathbf{k}\in\mathbb{Z}^d} |\widehat{\Phi_{\mathbf{n}}}(-\mathbf{k} - 2\mathbf{ns}, \mathbf{k})| \\
&= \sup_{\mathbf{k}\in\mathbb{Z}^d} |\widehat{\Phi_{\mathbf{n}}}(-\mathbf{k}, \mathbf{k})| + \sum_{\mathbf{s}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}} \sup_{\mathbf{k}\in\mathbb{Z}^d} |\widehat{\Phi_{\mathbf{n}}}(-\mathbf{k} - 2\mathbf{ns}, \mathbf{k})| \\
&\leq Ca_{\mathbf{n}} + \sum_{\mathbf{s}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}} 2a_{\mathbf{ns}}. \qquad (26)
\end{aligned}$$

Given $\mathbf{p} \in \mathbb{Z}^d$, there exists $\mathbf{q} \in \mathbb{Z}^d$ such that $\mathbf{np} = 2\mathbf{nq} + \mathbf{k}$, where $k_j = 0$ or $k_j = n_j$. Let $A = \{\mathbf{k} = (k_1, \ldots, k_d) : k_j = 0 \ or \ k_j = n_j, \ 1 \leq j \leq d\}$. If $\mathbf{k} \in A$ we have $|\mathbf{k}| \leq |\mathbf{n}|$. Then using the hypothesis we obtain

$$\sum_{\mathbf{p}\in\mathbb{Z}^d} a_{\mathbf{np}} \leq \sum_{\mathbf{k}\in A} \sum_{\mathbf{q}\in\mathbb{Z}^d} a_{\mathbf{2nq+k}} \leq C \sum_{\mathbf{k}\in A} a_{\mathbf{2n+k}}.$$

But $|2\mathbf{n} + \mathbf{k}| \geq |2\mathbf{n}| \geq |\mathbf{n}|$, then $a_{\mathbf{2n+k}} \leq a_{\mathbf{n}}$ and therefore

$$\sum_{\mathbf{p}\in\mathbb{Z}^d} a_{\mathbf{np}} \leq C \sum_{\mathbf{k}\in A} a_{\mathbf{n}} \leq 2^d C a_{\mathbf{n}}.$$

By (26) we conclude that

$$\sup_{f\in K*U_2} ||f - sk_{\mathbf{n}}(f,\cdot)||_2 \leq Ca_{\mathbf{n}} + 2^{d+1}Ca_{\mathbf{n}} = \bar{C}a_{\mathbf{n}}.$$

Then we proved the result for $p = q = 2$.

If $1 \leq q \leq 2 \leq p \leq \infty$, using that $||f||_q \leq ||f||_2 \leq ||f||_p$ we conclude the proof of the theorem. □

The following result is a direct consequence of Theorem 3.6 and Lemma 2.9.

Corollary 3.7. *Let $a : [0, +\infty) \to \mathbb{R}$ be a decreasing and positive function and $|\cdot| = |\cdot|_p$ for some $1 \le p \le \infty$. For $\mathbf{p} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$ let $a_{\mathbf{p}} = a(|\mathbf{p}|)$ and let $a_{\mathbf{0}} = \mathbf{0}$. Consider a kernel K given by*

$$K(\mathbf{x}) = \sum_{\mathbf{l} \in \mathbb{Z}^d} a_{\mathbf{l}} e^{i\mathbf{l}\cdot\mathbf{x}},$$

such that

$$\sum_{\mathbf{p} \in \mathbb{Z}^d} a_{2\mathbf{np}} \le C a_{2\mathbf{n}},$$

where C is a positive constant that is independent of $\mathbf{n} \in \mathbb{N}^d$. Then there exists a positive constant $\overline{C}$ such that for $1 \le q \le 2 \le p \le \infty$ and all $\mathbf{n} \in \mathbb{N}^d$, we have

$$\sup_{f \in K * U_p} ||f - sk_{\mathbf{n}}(f, \cdot)||_q \le \overline{C} a_{\mathbf{n}}.$$

Corollary 3.8. *Let K be a kernel as in Corollary 3.7 and let $1 \le p \le 2 \le q \le \infty$. Then there exists a positive constant C, independent of $\mathbf{n} \in \mathbb{N}^d$, such that*

$$\sup_{f \in K * U_p} ||f - sk_{\mathbf{n}}(f, \cdot)||_2 \le C \left(\sum_{|\mathbf{l}| \ge |\mathbf{n}|} a_{\mathbf{l}}^2 \right)^{(2-p)/2p} a_{\mathbf{n}}^{2-2/p}, \tag{27}$$

$$\sup_{f \in K * U_2} ||f - sk_{\mathbf{n}}(f, \cdot)||_q \le C \left(\sum_{|\mathbf{l}| \ge |\mathbf{n}|} a_{\mathbf{l}}^2 \right)^{(q-2)/2q} a_{\mathbf{n}}^{2/q}, \tag{28}$$

$$\sup_{f \in K * U_p} ||f - sk_{\mathbf{n}}(f, \cdot)||_{p'} \le C \left(\sum_{|\mathbf{l}| \ge |\mathbf{n}|} a_{\mathbf{l}} \right)^{2/p-1} a_{\mathbf{n}}^{2-2/p}. \tag{29}$$

Proof: Fix $\mathbf{n} \in \mathbb{N}^d$. Let $\Phi_{\mathbf{n}}(\mathbf{x}, \mathbf{y})$ be as in Lemma 3.2 and let T be the operator of Lemma 3.4. It follows from the Corollaries 2.10 and 3.7 that there exists a positive constant C such that

$$||T||_{1,2} \le C \left(\sum_{|\mathbf{l}| \ge |\mathbf{n}|} a_{\mathbf{l}}^2 \right)^{1/2} = M_0,$$

$$||T||_{2,2} \leq Ca_{\mathbf{n}} = M_1,$$

$$||T||_{2,\infty} \leq M_0.$$

$$||T||_{1,\infty} \leq C \sum_{|\mathbf{l}| \geq |\mathbf{n}|} a_{\mathbf{l}} = M_2.$$

As $||T||_{1,2} \leq M_0$ and $||T||_{2,2} \leq M_1$, applying the Riesz-Thorin Interpolation Theorem we obtain (27). Now using that $||T||_{2,\infty} \leq M_0$ and $||T||_{2,2} \leq M_1$, applying the Riesz-Thorin Interpolation Theorem we obtain (28). In an analogous way, since $||T||_{1,\infty} \leq M_2$ and $||T||_{2,2} \leq M_1$, we obtain (29).

□

4 Proofs of the Theorens 1.1 and 1.2

Proof of Theorem 1.1: The proof of (1) and the verification that the hypothesis of Corollary 3.7 is satisfied can be found in [11]. Applying the Corollary 3.7 we obtain

$$\sup_{f \in K * U_p} ||f - sk_{\mathbf{n}}(f, \cdot)||_q \leq C_5 |\mathbf{n}|^{-\gamma},$$

for $1 \leq q \leq 2 \leq p \leq \infty$. For $\mathbf{n} = (n, \ldots, n)$ we have $|\mathbf{n}|_2 = \sqrt{d}n$ and $|\mathbf{n}|_\infty = n$, therefore the estimates (1) and (2) are proved.

From (1) and (2) we have

$$\left(\sum_{|\mathbf{l}| \geq |\mathbf{n}|} a_{\mathbf{l}}^2 \right)^{1/2} \leq C n^{-\gamma + d/2}, \quad \sum_{|\mathbf{l}| \geq |\mathbf{n}|} a_{\mathbf{l}} \leq C n^{-\gamma + d} \ \text{ e } \ a_{\mathbf{n}} = n^{-\gamma}.$$

Then the estimates (3), (4) and (5) follow from Corollary 3.8. □

Proof of Theorem 1.2: Consider $r > 0$ and let us fix $\mathbf{n} = (n, n, \ldots, n)$. For each $s \in \mathbb{N}$ let $B_s = \{\mathbf{l} \in \mathbb{Z}^d : s - 1 \leq |\mathbf{l}|_\infty < s\}$. Then we have $\mathbb{Z}^d = \bigcup_{s=1}^\infty B_s$. If $\mathbf{p} \in B_s$, then $-s^r < -|\mathbf{p}|_\infty^r \leq -(s-1)^r$. Let $a_{\mathbf{l}} = e^{-\alpha |\mathbf{l}|_\infty^r}$ for $\mathbf{l} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$

and $a_{\mathbf{0}} = 0$. Since $2\mathbf{np} = 2n\mathbf{p}$ and $\#B_s \leq Cs^{d-1}$, $s \in \mathbb{N}$, we have

$$
\begin{aligned}
\sum_{\mathbf{p}\in\mathbb{Z}^d} a_{2\mathbf{np}} &= \sum_{s=2}^{\infty}\sum_{\mathbf{p}\in B_s} a_{2n\mathbf{p}} \\
&\leq \sum_{s=2}^{\infty}\sum_{\mathbf{p}\in B_s} e^{-\alpha(2n)^r(s-1)^r} \\
&\leq C\sum_{s=2}^{\infty} s^{d-1}e^{-\alpha(2n)^r(s-1)^r} \\
&= Ca_{2\mathbf{n}}\sum_{s=2}^{\infty} e^{(d-1)\ln s-\alpha(2n)^r((s-1)^r-1)} \\
&\leq Ca_{2\mathbf{n}}\sum_{s=2}^{\infty} e^{As^{r/4}-\alpha(2n)^r((s-1)^r-1)}, \qquad (30)
\end{aligned}
$$

where $A = 4(d-1)/r$, considering that $g(x) = \frac{4(d-1)}{r}x^{r/4} - (d-1)\ln x \geq 0$ if $x \geq 1$. Let $a \in \mathbb{N}$ such that $(x-1)^r - 1 \geq x^{r/2},\ x \geq a$. Then there exists a constant $M_1 > 0$ such that

$$
\sum_{p\in\mathbb{Z}^d} a_{2\mathbf{np}} \leq Ca_{2\mathbf{n}}\left(M_1 + \sum_{s=a}^{\infty} e^{As^{r/4}-\alpha(2n)^r s^{r/2}}\right).
$$

Let $b \in \mathbb{N}, b \geq a$, such that $(A+1)x^{r/4} \leq \alpha 4^r x^{r/2}, x \geq b$. Then there exists a constant $M_2 > 0$ such that

$$
\sum_{p\in\mathbb{Z}^d} a_{2\mathbf{np}} \leq Ca_{2\mathbf{n}}\left(M_1 + M_2 + \sum_{s=b}^{\infty} e^{-s^{r/4}}\right).
$$

Now let $c \in \mathbb{N},\ c \geq b$ be such that $e^{-x^{r/4}} \leq x^{-2},\ x \geq c$. Then there exist positive constants M_3 and M_4 such that

$$
\sum_{\mathbf{p}\in\mathbb{Z}^d} a_{2\mathbf{np}} \leq Ca_{2\mathbf{n}}\left(M_1 + M_2 + M_3 + \sum_{s=c}^{\infty} s^{-2}\right) \leq Ca_{2\mathbf{n}}\left(M_1 + M_2 + M_3 + M_4\right) = \bar{C}a_{2\mathbf{n}}.
$$

Thus we showed that the hypotheses of Theorem 2.10 and Corollary 3.7 are satisfied.

Now consider $r \geq 1$ and $1 \leq p \leq 2 \leq q \leq \infty$ such that $p^{-1} - q^{-1} \geq 1/2$ and let $s = (p^{-1} - q^{-1})^{-1}$. Then since $\# B_s \leq C_1 s^{d-1}$, $s \in \mathbb{N}$ and $(a-b)^r \leq a^r - b^r$ if $a \geq b > 0$, we have

$$\begin{aligned}
\sum_{|\mathbf{p}|_\infty \geq |\mathbf{n}|_\infty} (a_{\mathbf{p}})^s &\leq \sum_{j=n+1}^{\infty} \sum_{\mathbf{p} \in B_j} e^{-\alpha s |\mathbf{p}|_\infty^r} \\
&\leq \sum_{j=n+1}^{\infty} C_1 j^{d-1} e^{-\alpha s (j-1)^r} \\
&= C_1 \sum_{l=n}^{\infty} (l+1)^{d-1} e^{-\alpha s |\mathbf{n}|_\infty^r} e^{-\alpha s (l^r - |\mathbf{n}|_\infty^r)} \\
&\leq C_1 e^{-\alpha s n^r} \sum_{j=0}^{\infty} (j+n+1)^{d-1} e^{-\alpha s j^r} \\
&\leq C_3 n^{d-1} e^{-\alpha s n^r} \sum_{j=1}^{\infty} j^{d-1} e^{-\alpha s j^r} \\
&\leq C_3 n^{d-1} e^{-\alpha s n^r} \sum_{j=1}^{\infty} j^{d-1} e^{-\alpha s j},
\end{aligned}$$

because $(j+1+n)^{d-1} \leq 2^{d-1} C_2 j^{d-1} n^{d-1}$. Let $k \in \mathbb{N}$ be such that $e^{-\alpha s t} \leq t^{-2d}, t \geq k$. Then

$$\begin{aligned}
\sum_{|\mathbf{p}|_\infty \geq |\mathbf{n}|_\infty} (a_{\mathbf{p}})^s &\leq C_3 n^{d-1} e^{-\alpha s n^r} \left(\sum_{j=1}^{k-1} j^{d-1} e^{-\alpha s j} + \sum_{j=k}^{\infty} j^{d-1} j^{-2d} \right) \\
&\leq C_3 n^{d-1} e^{-\alpha s n^r} \left(K_5 + \sum_{j=k}^{\infty} j^{-2} \right) \\
&= C_4 n^{d-1} e^{-\alpha s n^r}.
\end{aligned}$$

Thus

$$\left(\sum_{|\mathbf{p}|_\infty \geq |\mathbf{n}|_\infty}^{\infty} (a_{\mathbf{p}})^s \right)^{s^{-1}} \leq C_5 e^{-\alpha n^r} n^{(d-1)(1/p-1/q)}.$$

By Theorem 2.10 we have that

$$\sup_{f\in K*U_p} ||f - sk_{\mathbf{n}}(f,\cdot)||_q \le C_6 e^{-\alpha n^r} n^{(d-1)(1/p-1/q)},$$

for $1 \le p \le 2 \le q \le \infty$ and $r \ge 1$, and by Corollary 3.7 we have that

$$\sup_{f\in K*U_p} ||f - sk_{\mathbf{n}}(f,\cdot)||_q \le C_7 e^{-\alpha n^r},$$

for $1 \le q \le 2 \le p \le \infty$ and $r > 0$. This proves (6) and (7).

From (6) and (7) we have

$$\left(\sum_{|\mathbf{l}|_\infty \ge |\mathbf{n}|_\infty} a_{\mathbf{l}}^2\right)^{1/2} \le C_8 e^{-\alpha n^r} n^{(d-1)/2}, \quad \sum_{|\mathbf{l}|_\infty \ge |\mathbf{n}|_\infty} a^l \le C_8 e^{-\alpha n^r} n^{(d-1)} \text{ e } a_{\mathbf{n}} = e^{-\alpha n^r}.$$

Therefore the estimates (8), (9) and (10) follow from Corollary 3.8. □

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