

ON SPACE MAXIMAL CURVES

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ABSTRACT. Any maximal curve \mathcal{X} is equipped with an intrinsic embedding $\pi : \mathcal{X} \rightarrow \mathbf{P}^r$ which reveal outstanding properties of the curve. By dealing with the contact divisors of the curve $\pi(\mathcal{X})$ and tangent lines, in this paper we investigate the first positive element that the Weierstrass semigroup at rational points can have whenever $r = 3$ and $\pi(\mathcal{X})$ is contained in a cubic surface.

1. INTRODUCTION

Throughout this paper, \mathbf{F} stands for the finite field \mathbb{F}_{q^2} of order q^2 . A projective, geometrically irreducible, non-singular algebraic curve \mathcal{X} defined over \mathbf{F} of genus $g = g(\mathcal{X})$ is said to be \mathbf{F} -*maximal* if the number of its \mathbf{F} -rational points attains the Hasse-Weil upper bound; that is,

$$\#\mathcal{X}(\mathbf{F}) = q^2 + 1 + 2q \cdot g.$$

Apart from being interesting mathematical objects by their own, these curves have been extensively studied as they are of great interest in Coding Theory, Cryptography and related areas; see for example the books [24], [14], [16].

Let \mathcal{X} be an \mathbf{F} -maximal curve of genus g . Then the numerator of the Zeta function of \mathcal{X} is the polynomial $L(t) = (1+qt)^{2g}$ and hence $h(t) = t^{2g}L(t^{-1}) = (t+q)^{2g}$ is the characteristic polynomial of certain endomorphism $\tilde{\Phi}$ on the Jacobian \mathcal{J} of \mathcal{X} . This map is uniquely determined by the \mathbf{F} -Frobenius morphism $\Phi : \mathcal{X} \rightarrow \mathcal{X}$ in such a way that $\iota \circ \Phi = \tilde{\Phi} \circ \iota$, where $\iota : \mathcal{X} \rightarrow \mathcal{J}$ is the natural embedding given by $P \mapsto [P - P_0]$ with $P_0 \in \mathcal{X}(\mathbf{F})$. It turns out that $\tilde{\Phi}$ is semisimple and so the following linear equivalence (sometimes called the *fundamental equivalence*) on \mathcal{X} arises (see [14, Thm. 10.1, Thm. 9.79]):

$$(1.1) \quad (q+1)P_0 \sim qP + \Phi(P), \quad P \in \mathcal{X}.$$

This suggests the study of the (complete) linear series $\mathcal{D}_{\mathcal{X}} := |(q+1)P_0|$ (sometimes called the *Frobenius linear series* of \mathcal{X}) whose definition clearly does not depend on the choice of the \mathbf{F} -rational point P_0 . As a matter of fact, several arithmetical and geometrical properties of maximal curves are revealed through this linear series (loc. cit.). In particular, $\mathcal{D}_{\mathcal{X}}$

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is very ample [8, Prop. 1.9], [18, Thm. 2.5] which means that the morphism associated to $\mathcal{D}_{\mathcal{X}}$

$$(1.2) \quad \pi_{\mathcal{D}_{\mathcal{X}}} : \mathcal{X} \rightarrow \mathbf{P}^r$$

is an embedding, where $r = r(\mathcal{X}) \geq 2$ is the projective dimension of $\mathcal{D}_{\mathcal{X}}$ (sometimes called the *Frobenius dimension* of \mathcal{X}), and \mathbf{P}^r is the projective r -space over the algebraic closure of \mathbf{F} .

It is well-known that the Hermitian curve \mathcal{H} defined by $v^{q+1} = u^{q+1} + 1$ is \mathbf{F} -maximal of genus $g(\mathcal{H}) = g_0 := q(q-1)/2$; see [24, Ex. 6.3.6]. Indeed, among \mathbf{F} -maximal curves, the curve \mathcal{H} admits the following characterization.

Proposition 1.1. ([23], [27], [9]) *If \mathcal{X} is \mathbf{F} -maximal, the following sentences are equivalent:*

- (1) \mathcal{X} is \mathbf{F} -isomorphic to the Hermitian curve \mathcal{H} ;
- (2) $g(\mathcal{X}) = g_0$;
- (3) $g(\mathcal{X}) > (q-1)^2/4$;
- (4) $r(\mathcal{X}) = 2$;
- (5) *There exists $P \in \mathcal{X}(\mathbf{F})$ such that the first positive element of $H(P)$, the Weierstrass semigroup at P , equals q .*

From (1.2) any \mathbf{F} -maximal curve \mathcal{X} is \mathbf{F} -isomorphic to a non-degenerate curve of degree $q+1$ in \mathbf{P}^r , $r = r(\mathcal{X})$. Thus the classical Castelnuovo's genus bound can be used to explain partially Proposition 1.1 as it gives the first general constrain between $g(\mathcal{X})$ and the pair (q, r) (see [14, Cor. 10.25]):

$$(1.3) \quad g(\mathcal{X}) \leq F(q, r) := \begin{cases} [(q - (r-1)/2)^2 - 1/4]/2(r-1), & \text{if } r \text{ is even,} \\ [(q - (r-1)/2)^2]/2(r-1), & \text{if } r \text{ is odd.} \end{cases}$$

Notice that the function $F(q, r)$ satisfies $F(q, r) \leq F(q, s)$ for $r \geq s$; in particular, $g(\mathcal{X}) \leq F(q, 3) = (q-1)^2/4$ provided that $r(\mathcal{X}) \geq 3$. Then, with $g_1 := \lfloor (q-1)^2/4 \rfloor$, by Proposition 1.1 the *spectrum for the genera* of \mathbf{F} -maximal curves, namely the set

$$\mathbf{M}(q^2) := \{g \in \mathbb{N}_0 : \text{there is an } \mathbf{F}\text{-maximal curve of genus } g\},$$

satisfies

$$(1.4) \quad \mathbf{M}(q^2) \subseteq [0, g_1] \cup \{g_0\}.$$

We recall that g_0 is the well-known Ihara's bound on the genus of \mathbf{F} -maximal curves [17]. One of the main problems in Curve Theory Over Finite Fields is the computation of $\mathbf{M}(q^2)$; in general one cannot expect to give a full answer to this matter but improvements on (1.4) can be expected as far as improvements on Castelnuovo's genus bound of curves in \mathbf{P}^r are known.

In view of Proposition 1.1 it is natural to investigate *space \mathbf{F} -maximal curves* with respect to $\mathcal{D}_{\mathcal{X}}$; that is, those with $r(\mathcal{X}) = 3$. Here a natural way of bounding $g(\mathcal{X})$, which generalizes Castelnuovo's method, is by looking at the degree $d \geq 2$ of surfaces $S \subseteq \mathbf{P}^3$ such that $\pi(\mathcal{X}) \subseteq S$ where $\pi = \pi_{\mathcal{X}}$ is as in (1.2); cf. [13], [21]. We have the following Halphen-Ballico result (see [3]) which deals with the case of quadrics. Let g_1 be as in (1.4) and set $g_2 := \lfloor (q^2 - q + 4)/6 \rfloor$; then

$$(1.5) \quad \pi(\mathcal{X}) \text{ is contained in a quadric in } \mathbf{P}^3 \text{ provided that } g_2 < g(\mathcal{X}) \leq g_1.$$

Now the \mathbf{F} -maximal property of \mathcal{X} implies certain constraints on the first positive element $m_1(P)$ of the Weierstrass semigroup $H(P)$ at some $P \in \mathcal{X}(\mathbf{F})$, and (1.4) admits the following improvement [18]:

$$(1.6) \quad \mathbf{M}(q^2) \subseteq [0, g_2] \cup \{g_1\} \cup \{g_0\}.$$

An analogue of Proposition 1.1 emerges, namely

Proposition 1.2. ([8], [1], [19], [18]) *Let \mathcal{X} be an \mathbf{F} -maximal curve. The following sentences are equivalent:*

- (1) \mathcal{X} is isomorphic to a quotient of \mathcal{H} by certain involution;
- (2) $g(\mathcal{X}) = g_1$;
- (3) $\pi(\mathcal{X})$ is contained in a quadric;
- (4) There exists $P \in \mathcal{X}(\mathbf{F})$ such that the first positive element of $H(P)$, the Weierstrass semigroup at P , equals $\lfloor (q+1)/2 \rfloor$.

The starting points of our result are in fact Propositions 1.1, 1.2 above. Under condition (2.1) below, the main result in this paper is Corollary 2.6, where a hypothesis on a cubic surface is considered; in this way a weak version of the aforementioned propositions is obtained. We always assume $q > 7$; cf. [2].

We do point out that our approach follows closely the works by Cossidente-Korchmáros-Torres [5, Sect. 3], [4, Sect. 5], Korchmáros-Torres [18], Fanali-Giulietti [7] and Arakelian-Tafazolian-Torres [2].

Conventions. \mathbf{P}^s stands for the projective s -space over the algebraic closure of the base field. For a point P in a curve, $H(P)$ denotes the Weierstrass semigroup at P ; $m_1(P)$ is the first positive element of $H(P)$.

2. MAXIMAL CURVES AND CUBIC SURFACES

Let \mathcal{X} be an \mathbf{F} -maximal curve, $P_0 \in \mathcal{X}(\mathbf{F})$ and $\mathcal{D} = \mathcal{D}_{\mathcal{X}} = |(q+1)P_0|$ the linear series introduced in Section 1; i.e., it is the set of effective divisors on \mathcal{X} which are linearly equivalent to the divisor $(q+1)P_0$. We always assume $g(\mathcal{X}) > 0$; taking into consideration (1.6) and Propositions 1.1, 1.2 above, we also assume:

$$(2.1) \quad r(\mathcal{X}) = 3 \quad \text{and} \quad g(\mathcal{X}) \leq g_2 = \lfloor (q^2 - q + 4)^2/6 \rfloor.$$

Remark 2.1. Let \mathcal{X} be an \mathbf{F} -maximal curve. From (1.3) and Proposition 1.1, a sufficient condition to have $r(\mathcal{X}) = 3$ is that $(q-1)(q-2)/6 < g(\mathcal{X}) \leq g_1 = \lfloor (q-1)^2/4 \rfloor$.

Let $\pi = \pi_{\mathcal{D}} : \mathcal{X} \rightarrow \mathbf{P}^3$ be the morphism associated to \mathcal{D} .

Definition 2.2. For $P \in \mathcal{X}$, a non-negative integer j is called an (\mathcal{D}, P) -order if there is $D \in \mathcal{D}$ such that the coefficient $v_P(D)$ of P in D equals j .

Now let $P \in \mathcal{X}(\mathbf{F})$. Relation (1.1) implies the following behaviour for elements of $H(P)$:

$$m_0(P) = 0 < m_1(P) < m_2(P) < m_3(P) = q + 1.$$

Thus for each $i = 0, 1, 2, 3$ there are rational functions on \mathcal{X} , $h_i : \mathcal{X} \rightarrow \mathbf{P}^1$ such that $\text{div}(h_i) = D_i - m_i(P)P$, $P \notin \text{Supp}(D_i)$ with $\text{div}(h_3) = (q+1)P - (q+1)P_0$, $P \neq P_0$. For $P = P_0$ we put $h_3 = 1$. Then

$$\text{div}(h_i h_3) + (q+1)P_0 = D_i + (q+1 - m_i(P))P \in \mathcal{D}$$

and the (\mathcal{D}, P) -orders do satisfy (cf. [8, Prop. 1.5(iii)])

$$(2.2) \quad j_i(P) = q + 1 - m_{3-i}(P), \quad i = 0, 1, 2, 3;$$

therefore at $P \in \mathcal{X}(\mathbf{F})$, $j_3(P) = q + 1$ and the first positive element $m_1(P)$ of $H(P)$ and $j_2(P)$ are related to each other by the equation

$$(2.3) \quad m_1(P) = q + 1 - j_2(P).$$

Remark 2.3. For the linear system \mathcal{D} above and any $P \in \mathcal{X}$, the (\mathcal{D}, P) orders can be ordered as a sequence $j_0(P) < j_1(P) < j_2(P) < j_3(P) \leq q + 1$ with $j_0(P) = 0$ as \mathcal{D} is base-point-free. Relation (1.1) shows that 1 and q are (\mathcal{D}, P) -orders for $P \notin \mathcal{X}(\mathbf{F})$. Thus for such points $j_1(P) = 1$ and $j_3(P) = q$ (as $g(\mathcal{X}) > 0$).

Now $j_3(P)$ is the intersection multiplicity of the curve $\pi(\mathcal{X}) \subseteq \mathbf{P}^3$ and the osculating hyperplane at $\pi(P)$ (cf. [25]); in addition, (1.1) also shows that $\pi(\Phi(P))$ belongs to this hyperplane and we have the following key observation due to Stöhr and Voloch [25, Cor. 2.6]: Let $\nu_2 := q$ and $P \in \mathcal{X}(\mathbf{F})$. Then $j_3(P) - j_1(P) \geq \nu_2$; in particular, for $P \in \mathcal{X}(\mathbf{F})$, $j_1(P) = 1$, and so $m_2(P) = q$ by (2.2).

Lemma 2.4. *Let \mathcal{X} be an \mathbf{F} -maximal curve satisfying (2.1) and let $P \in \mathcal{X}(\mathbf{F})$.*

- (1) *If $q > 3$, then $j_2(P) \notin \{(q+3)/2, (2q+3)/3, (2q+2)/3, q-1, q\}$;*
- (2) *$j_2(P) \notin \{(q+1)/2, (q+2)/2\}$.*
- (3) *If q is even and $j_2(P) = q/2$, then $g(\mathcal{X}) \leq q^2/8$.*

Proof. We have $m_1(P) = q + 1 - j_2(P)$; see (2.3).

(1) Since $2m_1(P) \geq m_2(P)$ and $m_2 = q$ by Remark 2.3, then $j_2(P) \leq (q+2)/2$. If any of the values in (1) were allowed, then $q \leq 3$.

(2) Suppose $j_2 = (q + 1)/2$ (resp. $j_2 = (q + 2)/2$). Then $m_1(P) = (q + 1)/2$ (resp. $m_1(P) = q/2$) and hence $g(\mathcal{X}) = \lfloor (q - 1)^2/4 \rfloor$ by [18, Thm. 1].

(3) If $j_2(P) = q/2$, $m_1(P) = (q + 2)/2$ by Remark 2.3; then $g(\mathcal{X}) \leq g(H)$ where H is the semigroup generated by $(q + 2)/2, q, q + 1$ and $g(H) = \#(\mathbb{N}_0 \setminus S)$ is the genus of H . This number can be computed by the method of Rosales and García-Sánchez in [22]; i.e., $g(H) = q^2/8$ and the result follows. \square

Theorem 2.5. *Let \mathcal{X} be an \mathbf{F} -maximal curve satisfying (2.1). Suppose that $\pi(\mathcal{X})$ is contained in a cubic surface S .*

- (1) For $P \in \mathcal{X}(\mathbf{F})$, $j_2(P) \in \{2, 3, q/2, (q + 1)/3, (q + 2)/3, (q + 3)/3\}$;
- (2) If q is even and $g(\mathcal{X}) > q^2/8$, then $j_2(P) \neq q/2$.

Proof. Let $j_0 = 0 < j_1 = 1 < j_2 < j_3 = q + 1$ be the (\mathcal{D}, P) -orders with $j_2 = j_2(P)$ and $v = v_P$ the valuation at P . Then π can be defined by $(f_0 : f_1 : f_2 : f_3)$ such that $v(f_i) = j_i$; in particular, $\pi(P) = (1 : 0 : 0 : 0)$ and throughout we assume $f_0 = 1$. Let the cubic surface S be defined by

$$\begin{aligned} F(X_0, X_1, X_2, X_3) &= a_{000}X_0^3 + a_{001}X_0^2X_1 + a_{002}X_0^2X_2 + a_{003}X_0^2X_3 + a_{111}X_1^3 \\ &+ a_{110}X_1^2X_0 + a_{112}X_1^2X_2 + a_{113}X_1^2X_3 + a_{222}X_2^3 + a_{220}X_2^2X_0 \\ &+ a_{221}X_2^2X_1 + a_{223}X_2^2X_3 + a_{333}X_3^3 + a_{330}X_3^2X_0 + a_{331}X_3^2X_1 \\ &+ a_{332}X_3^2X_2 + a_{012}X_0X_1X_2 + a_{013}X_0X_1X_3 + a_{023}X_0X_2X_3 \\ &+ a_{123}X_1X_2X_3. \end{aligned}$$

Then $F(1, f_1, f_2, f_3) = 0$ and $a_{000} = 0$. Now the valuation at P of the functions

$$\begin{aligned} &f_1, f_2, f_3, f_1^3, f_1^2, f_1^2f_2, f_1^2f_3, f_2^3, f_2^2, f_2^2f_1, f_2^2f_3, f_3^3, \\ &f_3^2, f_3^2f_1, f_3^2f_2, f_1f_2, f_1f_3, f_2f_3, f_1f_2f_3 \end{aligned}$$

are respectively

$$\begin{aligned} &1, j_2, j_3, 3, 2, 2 + j_2, 2 + j_3, 3j_2, 2j_2, 1 + 2j_2, 2j_2 + j_3, 3j_3, \\ &2j_3, 1 + 2j_3, j_2 + 2j_3, 1 + j_2, 1 + j_3, j_2 + j_3, 1 + j_2 + j_3. \end{aligned}$$

Then the valuation property of v implies $a_{001} = 0$. Let $j_2 > 3$ so that $a_{111} = a_{110} = 0$ (recall that $q > 7$). We have $j_2 + 2 < j_3$, otherwise $j_2 \in \{q, q - 1\}$ which is not possible by Lemma 2.4. Thus

$$j_2 < j_2 + 1 < j_2 + 2 < j_3 < j_3 + 1 < j_3 + 2 < j_3 + j_2 < j_3 + j_2 + 1 < 2j_3 < 2j_3 + 1 < 2j_3 + j_2 < 3j_3.$$

Since $2j_2 < 2j_2 + 1 < 3j_2 < 2j_2 + j_3$, the valuation property of v implies the following cases:

- (1) Either $2j_2 \in \{j_3, j_3 + 1, j_3 + 2\}$, or
- (2) $2j_2 + 1 = j_3$, or
- (3) $3j_2 \in \{j_3, j_3 + 1, j_3 + 2, 2j_3, 2j_3 + 1\}$.

By Lemma 2.4 $2j_2 \neq j_3, j_3 + 1, j_3 + 2$, $3j_2 \neq 2j_3$, $3j_2 \neq 2j_3 + 1$, and $2j_2 + 1 \neq j_3$ whenever $g > q^2/8$.

Therefore $j_2 \in \{2, 3, (q+1)/3, (q+2)/3, (q+1)/3\}$ and the proof follows. \square

Now we can state the main result in this paper.

Corollary 2.6. *Let \mathcal{X} be an \mathbf{F} -maximal curve as in Theorem 2.5. Then the multiplicity $m_1(P)$ of the Weierstrass semigroup $H(P)$ at $P \in \mathcal{X}(\mathbf{F})$ do satisfy*

$$m_1(P) \in \{(q+2)/2, (2q+2)/3, (2q+1)/3, 2q/3, q-2, q-1\}.$$

In addition, if q is even and $g(\mathcal{X}) > q^2/8$, then $m_1(P) \neq (q+2)/2$.

Proof. It follows from (2.3) and the theorem above. \square

Remark 2.7. Notation as in Remark 2.3. For the linear series \mathcal{D} , a basic result is that for almost $P \in \mathcal{X}$, the sequence $j_0(P) < j_1(P) < j_2(P) < j_3(P)$ is constant (so called *order sequence of \mathcal{D}*) cf. [25, p. 5]. In Remark 2.3 we noticed that $j_0(P) = 0$, $j_1(P) = 1$, $j_3(P) = q$ for $P \notin \mathcal{X}(\mathbf{F})$ and thus the order sequence of \mathcal{D} is of type $0 < 1 < \epsilon_2 < q$.

By the proof of [2, Prop. 3.1], $\epsilon_2 = 2$ provided that

$$g(\mathcal{X}) > \begin{cases} (q^2 + 1)(q - 4)/2(4q - 1), & \text{whenever } q \not\equiv 0 \pmod{3}, \\ g > (q^2 + 1)(q - 3)/2(3q - 1), & \text{otherwise.} \end{cases}$$

This forces $g(\mathcal{X}) \geq (q^2 - 2q + 3)/6$ (*) (see [5, Remark 3.3], [2, Prop. 3.1]).

Now for $P \in \mathcal{X}(\mathbf{F})$ the Weierstrass semigroup $H(P)$ contains the semigroup generated by $m, q, q+1$, where $m = m_1(P) = q+1 - j_2(P)$ (cf. 2.3); hence $g(\mathcal{X}) \leq g(H)$ (the genus of H). Then by using heavy arithmetical computations from [10, Sect. 2] and by taking into consideration restriction (*) above, Corollary 2.6 was already proved in [5, Cor. 3.5] without the hypothesis regarding the cubic surface.

Remark 2.8. Let \mathcal{X} be an \mathbf{F} -maximal curve such that (2.1) holds; in particular, we identify \mathcal{X} with a non-degenerate projective curve in \mathbf{P}^3 and we can apply the aforementioned Castelnuovo and Halphen-Ballico results as they are true in positive characteristic [3]. We look forward a result of type: There exists a polynomial (of one indeterminate) $A(x) \in \mathbb{Q}[x]$ such that

$$(2.4) \quad g(\mathcal{X}) > A(q+1) \quad \text{implies} \quad \pi(\mathcal{X}) \subseteq S,$$

where S is a surface of degree $d \leq 3$ (Then we shall assume $d = 3$ by Propositions 1.1, 1.2.)

Remark 2.9. In the literature, for a non-degenerate projective space curve \mathcal{C} of degree $q+1$ over the complex numbers, there are available results of type (2.4) which in fact appear as particular cases of a vast theory that generalize the aforementioned Castelnuovo and Halphen results; see Eisenbud-Harris book [13, Thm. 3.22, p. 117].

Let q be large, says $q \geq 107$. If

$$g(\mathcal{C}) > B(q+1) := \begin{cases} \frac{q(q+2)}{8}, & \text{if } q \equiv 0, 2 \pmod{4}, \\ \frac{q^2 + 2q - 3}{8}, & \text{if } q \equiv 1 \pmod{4}, \\ \frac{q^2 + 2q + 9}{8}, & \text{otherwise,} \end{cases}$$

then there exists a surface S of degree 2 or 3 such that $\mathcal{C} \subseteq S$.

Question 2.10. Is Remark 2.9 true in positive characteristic?

3. EXAMPLES

In this section we illustrate Corollary 2.6. Notation as above; in particular, \mathcal{H} is the Hermitian curve over $\mathbf{F} = \mathbf{F}_{q^2}$ defined by $v^{q+1} = u^{q+1} + 1$. Let $\pi : \mathcal{H} \rightarrow \mathbf{P}^2$ be a non-trivial morphism over \mathbf{F} and \mathcal{X} the non-singular model of the plane curve $\pi(\mathcal{H})$; then π can be lifted to a morphism $\mathcal{H} \rightarrow \mathcal{X}$, which we still denote by π . In this case, the curve \mathcal{X} is also \mathbf{F} -maximal (see e.g. [20]).

Example 3.1. (cf. [7, Sect. 5]) Let $q \equiv 2 \pmod{3}$ and $\pi : \mathcal{H} \rightarrow \mathbf{P}^2$ be the morphism given by $\pi = (x : y : 1) := (u^3 : uv : 1)$. Then the plane curve $\pi(\mathcal{H})$ is defined by

$$y^{q+1} = x^{(q+1)/3}(x^{(q+1)/3} + 1),$$

and by applying the Riemann-Hurwitz formula to the function $x : \mathcal{X} \rightarrow \mathbf{P}^1$, where \mathcal{X} is the non-singular model of $\pi(\mathcal{H})$, we find that \mathcal{X} is \mathbf{F} -maximal of genus $g(\mathcal{X}) = g_2 = (q^2 - q + 4)/6$ (cf. [11], [4, Prop. 2.1]). We notice that $r(\mathcal{X}) = 3$ by Remark 2.1 above.

Next we shall compute the Weierstrass semigroup $H(P)$ at certain points of \mathcal{X} ; we start by computing some principal divisors on \mathcal{X} via tools from [24].

- (a) There are $(q+1)/3$ points in $x^{-1}(\infty)$, say P_i , $i = 1, \dots, (q+1)/3$.
Set $D_\infty := P_1 + \dots + P_{(q+1)/3}$.
- (b) There are $(q+1)/3$ points in $x^{-1}(0)$, say Q_i , $i = 1, \dots, (q+1)/3$.
Set $D_0 := Q_1 + \dots + Q_{(q+1)/3}$. Then $\text{div}(x) = 3D_0 - 3D_\infty$.
- (c) Let $a \in \mathbf{F}$ such that $a^{(q+1)/3} = -1$ (*). There is just point R_a over $x^{-1}(a)$ and $\text{div}(x - a) = (q+1)R_a - 3D_\infty$. Set $D := \sum_{i/i^{(q+1)/3} = -1} R_i$.
Then $\text{div}(x^{(q+1)/3} + 1) = (q+1)D - (q+1)D_\infty$.

From (a), (b), (c), $\text{div}(y) = D_0 + D - 2D_\infty$, and for $a \in \mathbf{F}$ as in (*) above

$$\begin{aligned} \text{div}((x - a)^{-1}) &= 3D_\infty - (q+1)R_a, & \text{div}(y(x - a)^{-1}) &= D_0 + D' + D_\infty - qR_a \quad \text{and} \\ \text{div}(y^3 x^{-1} (x - a)^{-1}) &= 3D' - (q-2)R_a, \end{aligned}$$

where $D' = D - R_a$. It follows that $H(R_a) \supseteq H := \langle q-2, q, q+1 \rangle$ so that $g(\mathcal{X}) \leq g(H)$. We have that the sequence $q-2, q+1, q$ is telescopic and so $g(H) = (q^2 - q + 4)/6$ (see e.g. [15, Prop. 5.35]). Therefore

Claim. $H(R_a) = H$ and $m_1(R_a) = q - 2$ (this also shows that $r(\mathcal{X}) = 3$).

Moreover by Remark 2.7 the order sequence of \mathcal{X} is $0 < 1 < 2 < q$ and thus there is also a point $P \in \mathcal{X}(\mathbf{F})$ with $m_1(P) = q - 1$ (see [5, Lemma 3.7]).

Remark 3.2. We can construct explicit and outstanding AG one-point codes based on the curve in Example 3.1 by taking into consideration the telescopic property of $H(R_a)$; cf. [15, Sect. 5], [26, Sect. 5].

Example 3.3. Let $q \equiv 2 \pmod{3}$. Here we point out properties of an arbitrary \mathbf{F} -maximal curve \mathcal{X} of genus $g(\mathcal{X}) = g_2 = (q^2 - q + 4)/6$. We have $r(\mathcal{X}) = 3$ by Remark 2.1, and that $0 < 1 < 2 < q$ is the order sequence of $\mathcal{D} = \mathcal{D}_{\mathcal{X}}$ by Remark 2.7. Then by [5, Lemma 3.7] there is $\bar{P} \in \mathcal{X}(\mathbf{F})$ with $m_1(\bar{P}) = q - 1$, or $j_2(\bar{P}) = 2$ by (2.3).

Claim. There is $P \in \mathcal{X}(\mathbf{F})$ such that $j_2(P) > 2$.

Indeed, otherwise [18, Lemma 7] would imply $g = (q^2 - 2q + 3)/6$, a contradiction.

Let $\pi : \mathcal{X} \rightarrow \mathbf{P}^3$ be the morphism associated to \mathcal{D} . We are led to the following questions.

- (A) Is $\pi(\mathcal{X})$ contained in a cubic surface? (This would be true if the answer to Question 2.10 is affirmative)
- (B) Let \mathcal{X} be an \mathbf{F} -maximal curve. Then $g(\mathcal{X}) = g_2$ if and only if $\pi(\mathcal{X})$ is contained in a cubic surface and there is $\bar{P} \in \mathcal{X}(\mathbf{F})$ with $j_2(\bar{P}) > 2$?

Question (B) above is related to the following result which is a consequence of the proof of [18, Thm. 1] and [Lemma 7]KT.

Remark 3.4. With g_1 as in (1.4), for an \mathbf{F} -maximal curve \mathcal{X} we have that $g(\mathcal{X}) = g_1$ if and only if $\pi(\mathcal{X})$ is contained in a quadric and there is $\bar{P} \in \mathcal{X}(\mathbf{F})$ with $j_2(\bar{P}) > 2$.

Example 3.5. Let $q \equiv 2 \pmod{3}$. We investigate \mathbf{F} -maximal curves of genus $g(\mathcal{X}) = g_3 = g_2 - 1 = (q^2 - q - 2)/6$ which were constructed in [5]. To start with, $r(\mathcal{X}) = 3$ by Remark 2.1 and the order sequence of \mathcal{D} is $0 < 1 < 2 < q$ by Remark 2.7. In particular, there is $P \in \mathcal{X}(\mathbf{F})$ such that $m_1(P) = q - 1$ by [5, Lemma 3.7].

We further assume the following properties:

- (a) $\pi(\mathcal{X})$ is contained in a cubic surface;
- (b) $\pi : \mathcal{H} \rightarrow \mathcal{X}$ is Galois of degree three.

(The aforementioned curves in [5] satisfy these properties)

Claim. There is $\bar{P} \in \mathcal{X}(\mathbf{F})$ with $m_1(\bar{P}) = (2q + 2)/3$.

Proof of the Claim. By the Riemann-Hurwitz relation there is $\bar{P} \in \mathcal{X}(\mathbf{F})$ which is totally ramified for π . Let $Q = \pi^{-1}(\bar{P}) \in \mathcal{H}$. The first six positive elements of the Weierstrass semigroup at Q are $q, q + 1, 2q, 2q + 1, 2q + 2, 3q$. Now let $m = m_1(\bar{P}) < q < q + 1$ be the first three positive elements of $H(\bar{P})$. Then $3m \in \{q, q + 1, 2q, 2q + 1, 2q + 2\}$ and so

$m \in \{(q+1)/3, (2q+2)/3\}$. We eliminate the case $m = (q+1)/3$ by Corollary 2.6 and the Claim follows.

Example 3.6. Let $q \not\equiv 2 \pmod{3}$ and \mathcal{X} be an \mathbf{F} -maximal curve of genus $g(\mathcal{X}) = g_2 = (q^2 - q)/6$; hence $r(\mathcal{X}) = 3$ by Remark 2.1 and the order sequence of \mathcal{D} is $0 < 1 < 2 < q$ by Remark 2.7. In particular, there is $P \in \mathcal{X}(\mathbf{F})$ such that $m_1(P) = q - 1$ by [5, Lemma 3.7]. We notice that examples of such curves do exist: see e.g. [11], [4, Prop. 2.1].

Let us assume properties (a) and (b) in Example 3.5 (indeed, the aforementioned examples satisfy these hypotheses).

Claim. If $q \equiv 1 \pmod{3}$ (resp. $q \equiv 0 \pmod{3}$), then there exists $\bar{P} \in \mathcal{X}(\mathbf{F})$ with $m_1(\bar{P}) = (2q+1)/3$ (resp. $m_1(\bar{P}) = 2q/3$).

Arguing as in Example 3.5 there is $\bar{P} \in \mathcal{X}(\mathbf{F})$ such that $3m \in \{q, q+1, 2q, 2q+1, 2q+2\}$ with $m = m_1(\bar{P})$.

If $q \equiv 1 \pmod{3}$, $m = (2q+1)/3$.

If $q \equiv 0 \pmod{3}$, either $m = q/3$ or $m = 2q/3$. The former case is eliminated by Corollary 2.6.

Example 3.7. Here we present an \mathbf{F} -maximal curve \mathcal{X} with $r(\mathcal{X}) = 3$ such that $\pi(\mathcal{X})$ cannot be contained in a cubic surface, where π is the morphism associated to \mathcal{D} . Indeed, we consider the so-called GK-curve [12] whose Weierstrass semigroups at rational points were computed in [7]. This curve is defined over $\mathbf{F} = \mathbb{F}_{q^2}$ with $q = \ell^3$. For $\ell > 2$ this is the first example of an \mathbf{F} -maximal curve that cannot be dominated by \mathcal{H} (loc. cit.)

On this curve there is $\bar{P} \in \mathcal{X}(\mathbf{F})$ such that $m_1(\bar{P}) = \ell^3 - \ell^2 + \ell$ [12, Sect. 4], and therefore, according to Corollary 2.6, $\pi(\mathcal{X})$ cannot be contained in a cubic. We notice that the genus of \mathcal{X} is $g(\mathcal{X}) = \frac{1}{2}(\ell^5 - 2\ell^3 + \ell^2)/2$ and so it does not satisfies Remark 2.9. Further examples can be found in [26].

We end this paper with the following:

Question 3.8. Let \mathcal{X} be an \mathbf{F} -maximal curve with $r(\mathcal{X}) = 3$. Suppose that $\pi(\mathcal{X}) \subseteq S$, where S is a surface of degree $d \geq 2$. Let $P \in \mathcal{X}(\mathbf{F})$ and suppose $g(\mathcal{X})$ large enough. Then $m_1(P) = (q+1) - \frac{q+i}{d}$ or $m_1(P) = q - j$ for some $i = 1, \dots, d$, $j = 2, \dots, d$. Are all these cases possible?

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