Estimates for Entropy Numbers of Sets of Smooth Functions on the Torus $\mathbb{T}^d$

R. L. B. Stabile*and S. A. Tozoni†

Instituto de Matemática, Universidade Estadual de Campinas, Rua Sérgio Buarque de Holanda 651, 13083-859, Campinas, SP, Brazil

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Abstract

In this paper, we investigate entropy numbers of multiplier operators $\Lambda = \{\lambda_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}^d}$ and $\Lambda_* = \{\lambda^*_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}^d}$, $\Lambda, \Lambda_* : L^p(\mathbb{T}^d) \to L^q(\mathbb{T}^d)$ on the d-dimensional torus $\mathbb{T}^d$, where $\lambda_{\mathbf{k}} = \lambda(|\mathbf{k}|)$ and $\lambda^*_{\mathbf{k}} = \lambda(|\mathbf{k}|_*)$ for a function λ defined on the interval $[0,\infty)$, with $|\mathbf{k}| = \left(k_1^2 + \cdots + k_d^2\right)^{1/2}$ and $|\mathbf{k}|_* = \max_{1\le j\le d} |k_j|$. In the first part, upper and lower bounds are established for entropy numbers of general multiplier operators. In the second part, we apply these results to the specific multiplier operators $\Lambda^{(1)} = \left\{|\mathbf{k}|^{-\gamma}(\ln|\mathbf{k}|)^{-\xi}\right\}_{\mathbf{k}\in\mathbb{Z}^d}$, $\Lambda^{(1)}_* = \left\{|\mathbf{k}|_*^{-\gamma}(\ln|\mathbf{k}|_*)^{-\xi}\right\}_{\mathbf{k}\in\mathbb{Z}^d}$, $\Lambda^{(2)} = \left\{e^{-\gamma|\mathbf{k}|^r}\right\}_{\mathbf{k}\in\mathbb{Z}^d}$ and $\Lambda^{(2)}_* = \left\{e^{-\gamma|\mathbf{k}|_*^r}\right\}_{\mathbf{k}\in\mathbb{Z}^d}$ for $\gamma > 0$, $0 < r \le 1$ and $\xi \ge 0$. We have that $\Lambda^{(1)}U_p$ and $\Lambda^{(1)}_*U_p$ are sets of finitely differentiable functions on $\mathbb{T}^d$, in particular, $\Lambda^{(1)}U_p$ and $\Lambda^{(1)}_*U_p$ are Sobolev-type classes if $\xi = 0$, and $\Lambda^{(2)}U_p$ and $\Lambda^{(2)}_*U_p$ are sets of infinitely differentiable ($0 < r < 1$) or analytic ($r = 1$) functions on $\mathbb{T}^d$, where U_p denotes the closed unit ball of $L^p(\mathbb{T}^d)$. In particular, we prove that, the estimates for the entropy numbers $e_n(\Lambda^{(1)}U_p, L^q(\mathbb{T}^d))$, $e_n(\Lambda^{(1)}_*U_p, L^q(\mathbb{T}^d))$, $e_n(\Lambda^{(2)}U_p, L^q(\mathbb{T}^d))$ and $e_n(\Lambda^{(2)}_*U_p, L^q(\mathbb{T}^d))$ are order sharp in various important situations.

1 Introduction

In [8], [9], [10] techniques were developed to obtain estimates for entropy numbers of multiplier operators defined for functions on the sphere $\mathbb{S}^d$ and on two-points homogeneous spaces. In this paper, we obtain estimates for entropy numbers of multiplier operators $\Lambda = \{\lambda_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}^d}$ and $\Lambda_* = \{\lambda^*_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}^d}$ defined for functions on the d-dimensional torus $\mathbb{T}^d$, where $\lambda_{\mathbf{k}} = \lambda(|\mathbf{k}|)$ and $\lambda^*_{\mathbf{k}} = \lambda(|\mathbf{k}|_*)$, for a function λ defined on $[0,\infty)$, with $|\mathbf{k}| = \left(k_1^2 + \cdots + k_d^2\right)^{1/2}$ and $|\mathbf{k}|_* = \max_{1\le j\le d} |k_j|$.

The entropy numbers measure a kind of degree of compactness of an operator and have many applications in theory of functions and spectral theory ([4] and [14]), signals and image processing ([2] and [3]), probability theory ([2]), among others.

In the first part of this paper we give an unified treatment for entropy numbers of sets of functions determined by multiplier operators. Upper and lower bounds are established for entropy numbers of general multiplier operators. Among the tools used in the proofs of these results, the main is a theorem proved in [7], which provides estimates for Levy Means of norms defined on $\mathbb{R}^n$. In the second part, we apply these results in the study of estimates for entropy numbers of sets of finitely and infinitely differentiable functions on $\mathbb{T}^d$. We show, in particular, that

*E-mail address: registabile@ifsp.udu.br
†E-mail address: tozoni@ime.unicamp.br

in various important situations the estimates are order sharp. An important tool used in the second part is the estimate for the number of points with integer coordinates, contained in a closed ball centered at the origin of $\mathbb{R}^d$, for a given norm on $\mathbb{R}^d$.

Consider two Banach spaces X and Y. The norm of X will be denoted by $\|\cdot\|$ or $\|\cdot\|_X$ and the closed unit ball $\{x \in X : \|x\| \leq 1\}$ by B_X. Let K be a compact subset of X and let $\epsilon > 0$. A finite subset $S = \{x_1, x_2, \ldots, x_m\}$ of X is called an ϵ-net for K in X, if for each $x \in K$, there is at least one point $x_k \in S$ such that $\|x_k - x\| \leq \epsilon$, that is, $K \subset \cup_{k=1}^m (x_k + \epsilon B_X)$. The set $S = \{x_1, x_2, \ldots, x_m\}$ is called an ϵ-distinguishable subset of K in X, if $S \subset K$ and $\|x_i - x_j\| > \epsilon$ for all $1 \leq i, j \leq m$, $i \neq j$. If every ϵ-distinguishable subset of K has at most m elements, we say that S is a maximal ϵ-distinguishable subset of K in X. A maximal ϵ-distinguishable subset of K in X is a ϵ-net for K in X.

Let K be a compact subset of X. The nth entropy number of K in X, denoted by $e_n(K, X)$, is defined as the infimum of all positive ϵ such that there exist $x_1, ..., x_{2^{n-1}}$ in X satisfying

$$K \subset \bigcup_{k=1}^{2^{n-1}} (x_k + \epsilon B_X).$$

If $T \in \mathcal{L}(X, Y)$ is a compact operator, the nth entropy number $e_n(T)$ is defined as

$$e_n(T) = e_n(T(B_X), Y).$$

Assume that Y is isometric to a subspace of a Banach space Y_1 and denote by $i : Y \to Y_1$ the isometric embedding. Then for any $T \in \mathcal{L}(X, Y)$ (see [13])

$$2^{-1} e_k(T) \leq e_k(i \circ T) \leq e_k(T), \;\; k \in \mathbb{N}. \tag{1.1}$$

If X and Y are Banach spaces and $T, S \in \mathcal{L}(X, Y)$, then

$$e_{k+l-1}(T + S) \;\leq\; e_k(T) + e_l(S), \quad k, l \in \mathbb{N}. \tag{1.2}$$

The d-dimensional torus $\mathbb{T}^d$, is defined as the cartesian product of d-times the quotient group $\mathbb{R}/2\pi\mathbb{Z}$, namely, $\mathbb{T}^d = \mathbb{R}/2\pi\mathbb{Z} \times \cdots \times \mathbb{R}/2\pi\mathbb{Z}$. We can identify $\mathbb{T}^d$ with the d-dimensional cube $[-\pi, \pi]^d$ and also with the cartesian product $\mathbb{S}^1 \times \cdots \times \mathbb{S}^1$, where $\mathbb{S}^1$ denotes the unitary circle $\{e^{it} : t \in [-\pi, \pi]\}$. We consider $\mathbb{T}^d$ endowed with the normalized Lebesgue measure $d\nu(\mathbf{x}) = \left(1/(2\pi)^d\right) dx_1 dx_2 \cdots dx_d$, where $(1/2\pi)dt$ is the normalized Lebesgue measure on $\mathbb{S}^1$. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ we denote by $\mathbf{x} \cdot \mathbf{y}$ the usual inner product $x_1 y_1 + x_2 y_2 + \cdots + x_d y_d$.

We denote by $L^p = L^p(\mathbb{T}^d)$, $1 \leq p \leq \infty$, the vector space consisting of all measurable functions φ defined on $\mathbb{T}^d$ and with values in $\mathbb{C}$, satisfying

$$\begin{aligned}
\|\varphi\|_p \;&=\; \|\varphi\|_{L^p(\mathbb{T}^d)} \;=\; \left(\int_{\mathbb{T}^d} |\varphi(\mathbf{x})|^p \, d\nu(\mathbf{x}) \right)^{1/p} \;<\; \infty, \quad 1 \leq p < \infty, \\
\|\varphi\|_\infty \;&=\; \|\varphi\|_{L^\infty(\mathbb{T}^d)} \;=\; \operatorname*{ess\,sup}_{\mathbf{x} \in \mathbb{T}^d} |\varphi(\mathbf{x})| \;<\; \infty.
\end{aligned}$$

We write $U_p = B_{L^p} = \{\varphi \in L^p : \|\varphi\|_p \leq 1\}$.

For $\mathbf{k} = (k_1, k_2, \ldots, k_d) \in \mathbb{Z}^d$ and $l, N \in \mathbb{N}$, we will denote

$$A_l = \{\mathbf{k} \in \mathbb{Z}^d : |\mathbf{k}| \leq l\}, \quad \mathcal{H}_l = \left[e^{i\mathbf{k}\cdot\mathbf{x}} : \mathbf{k} \in A_l \backslash A_{l-1} \right], \quad d_l = \dim \mathcal{H}_l \quad , \quad \mathcal{T}_N = \bigoplus_{l=0}^N \mathcal{H}_l,$$

and

$$A_l^* = \{\mathbf{k} \in \mathbb{Z}^d : |\mathbf{k}|_* \leq l\}, \quad \mathcal{H}_l^* = \left[e^{i\mathbf{k}\cdot\mathbf{x}} : \mathbf{k} \in A_l^* \backslash A_{l-1}^* \right], \quad d_l^* = \dim \mathcal{H}_l^* \quad , \quad \mathcal{T}_N^* = \bigoplus_{l=0}^N \mathcal{H}_l^*,$$

where $A_{-1} = A^*_{-1} = \emptyset$ and $[f_j : j \in \Gamma]$ denotes the vector space generated by the functions f_j, with $j \in \Gamma$.

As a consequence of the estimates for the number of points with integer coordinates contained in a closed Euclidean ball centered at the origin of $\mathbb{R}^d$ (see [5], [6] and [11]) we have that

$$\frac{2\pi^{d/2}}{d\Gamma(d/2)} l^{d-1} - C_1 l^{d-2} \leq d_l \leq \frac{2\pi^{d/2}}{d\Gamma(d/2)} l^{d-1} + C_2 l^{d-2} \quad \text{and} \quad \frac{2\pi^{d/2}}{d\Gamma(d/2)} N^d \leq \dim \mathcal{T}_N \leq \frac{2\pi^{d/2}}{d\Gamma(d/2)} N^d + C_3 N^{d-1}. \quad (1.3)$$

It is easy to see that

$$d^*_l = (2l+1)^d - (2l-1)^d \quad \text{and} \quad 2^d N^d \leq \dim \mathcal{T}^*_N \leq 2^d N^d + C_4 N^{d-1}.$$

In particular $d_l \asymp d^*_l \asymp l^{d-1}$ and $\dim \mathcal{T}_N \asymp \dim \mathcal{T}^*_N \asymp N^d$. Aplying the above inequalities we get

$$\frac{1}{\dim \mathcal{T}_N} \geq \frac{1}{FN^d} - \frac{C_5}{F^2 N^{d+1}}, \quad F = \frac{2\pi^{d/2}}{d\Gamma(d/2)} \quad (1.4)$$

and

$$\frac{1}{\dim \mathcal{T}^*_N} \geq 2^{-d} N^{-d} - C_6 N^{-d+1}.$$

Let $\Lambda = \{\lambda_{\mathbf{k}}\}_{\mathbf{k} \in \mathbb{Z}^d}$, $\lambda_{\mathbf{k}} \in \mathbb{C}$, and $1 \leq p, q \leq \infty$. If for any $\varphi \in L^p(\mathbb{T}^d)$ there is a function $f = \Lambda\varphi \in L^q(\mathbb{T}^d)$ with formal Fourier expansion given by

$$f \sim \sum_{\mathbf{k} \in \mathbb{Z}^d} \lambda_{\mathbf{k}} \, \widehat{f}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}},$$

such that $\|\Lambda\|_{p,q} = \sup\{\|\Lambda\varphi\|_q : \varphi \in U_p\} < \infty$, we say that the multiplier operator Λ is bounded from L^p into L^q, with norm $\|\Lambda\|_{p,q}$.

In this paper, we consider multipliers operators $\Lambda = \{\lambda_{\mathbf{k}}\}_{\mathbf{k} \in \mathbb{Z}^d}$ and $\Lambda_* = \{\lambda^*_{\mathbf{k}}\}_{\mathbf{k} \in \mathbb{Z}^d}$, where $\lambda_{\mathbf{k}} = \lambda(|\mathbf{k}|)$ and $\lambda^*_{\mathbf{k}} = \lambda(|\mathbf{k}|_*)$ for a complex function λ defined on the interval $[0, \infty)$. Let

$$\Lambda^{(1)} = \left\{|\mathbf{k}|^{-\gamma}(\ln|\mathbf{k}|)^{-\xi}\right\}_{\mathbf{k} \in \mathbb{Z}^d}, \ \Lambda^{(1)}_* = \left\{|\mathbf{k}|_*^{-\gamma}(\ln|\mathbf{k}|_*)^{-\xi}\right\}_{\mathbf{k} \in \mathbb{Z}^d}, \ \Lambda^{(2)} = \left\{e^{-\gamma|\mathbf{k}|^r}\right\}_{\mathbf{k} \in \mathbb{Z}^d} \quad \text{and} \quad \Lambda^{(2)}_* = \left\{e^{-\gamma|\mathbf{k}|_*^r}\right\}_{\mathbf{k} \in \mathbb{Z}^d},$$

where $\gamma,\ r > 0$, $\xi \geq 0$. We prove, in particular, that for $2 \leq p < \infty$, $1 < q < \infty$, $\xi \geq 0$ and $0 < r \leq 1$,

$$e_k(\Lambda^{(1)} U_p, L^q) \asymp e_k(\Lambda^{(1)}_* U_p, L^q) \asymp k^{-\gamma/d}(\ln k)^{-\xi}, \quad \gamma > d/2$$

and

$$e_k(\Lambda^{(2)} U_p, L^q) \asymp e^{-\mathcal{C} k^{r/(d+r)}}, \qquad e_k(\Lambda^{(2)}_* U_p, L^q) \asymp e^{-\mathcal{C}_* k^{r/(d+r)}}, \quad \gamma > 0,$$

where

$$\mathcal{C} = \gamma^{d/(d+r)} \left(\frac{(d+r)d\Gamma(d/2)(\ln 2)}{2r\pi^{d/2}}\right)^{r/(d+r)} \quad \text{and} \quad \mathcal{C}_* = \gamma^{d/(d+r)} \left(\frac{(d+r)(\ln 2)}{2^d r}\right)^{r/(d+r)}.$$

$\Lambda^{(1)} U_p$ and $\Lambda^{(1)}_* U_p$ are classes of finitely differentiable functions on $\mathbb{T}^d$. In particular, if $\xi = 0$, are classes of Sobolev type. The sets $\Lambda^{(2)} U_p$ and $\Lambda^{(2)}_* U_p$ are classes of infinitely differentiable functions if $0 < r < 1$ or analytic functions if $r = 1$.

In general, when we work with harmonic analysis on the torus, it is indifferent to use square or spherical partial sums. However, we verified that the technique used here is sensitive to the type of sum that we used to work with sets of infinitely differentiable or analytic functions on the torus.

Let A be a convex, compact, centrally symmetric subset of X. The Kolmogorov n-width of A in X is defined by

$$d_n(A, X) = \inf_{X_n} \sup_{f \in A} \inf_{g \in X_n} \|f - g\|,$$

where X_n runs over all subspaces of X of dimension n. It was proved in [7] that, for $2 \leq p, q < \infty$, $\xi \geq 0$ and $0 < r \leq 1$,

$$d_n(\Lambda^{(1)} U_p, L^q) = d_n(\Lambda^{(1)}_* U_p, L^q) \asymp n^{-\gamma/d}(\ln n)^{-\xi}, \quad \gamma > d/2,$$

and

$$d_n(\Lambda^{(2)}U_p, L^q) \asymp e^{-\mathcal{R}n^{r/d}} \quad , \quad d_n(\Lambda_*^{(2)}U_p, L^q) \asymp e^{-\mathcal{R}_* n^{r/d}}, \quad \gamma > 0,$$

where $\mathcal{R} = \gamma\left(d\Gamma(d/2)/2\pi^{d/2}\right)^{r/d}$ and $\mathcal{R}_* = \gamma 2^{-r}$. We remark that on the circle $\mathbb{S}^1 = \mathbb{T}^1$, unlike the Kolmogorov n-widths, the entropy numbers of the Sobolev (or Sobolev type) classes $W_p^\gamma = W_p^\gamma(\mathbb{T}^1)$, have essentially the same asymptotic behavior for all $1 \le p, q \le \infty$. This fact was discovered in [1] and we have that

$$d_n(W_p^\gamma, L^q) \gg e_n(W_p^\gamma, L^q)$$

as $n \to \infty$. We show that for $\mathbb{T}^d$ and the multiplier operators $\Lambda^{(2)}$ and $\Lambda_*^{(2)}$, the n-widths and the entropy numbers are essentially different. We can verify that for $2 \le p, q < \infty$, $\gamma > d/2$ and $\xi \ge 0$,

$$d_n(\Lambda^{(1)}U_p, L^q) \gg e_n(\Lambda^{(1)}U_p, L^q) \asymp n^{-\gamma/d}(\ln n)^{-\xi},$$

but for $2 \le p, q < \infty$, $\gamma > 0$ and $0 < r \le 1$,

$$d_n(\Lambda^{(2)}U_p, L^q) \ll e_n(\Lambda^{(2)}U_p, L^q).$$

In this article there are several universal constants which enter into the estimates. These positive constants are mostly denoted by the letters $C, C_1, C_2 \ldots$. We did not carefully distinguish between the different constants, neither did we try to get good estimates for them. The same letter will be used to denote different universal constants in different parts of the paper. For ease of notation we will write $a_n \gg b_n$ for two sequences if $a_n \ge C b_n$ for $n \in \mathbb{N}$, $a_n \ll b_n$ if $a_n \le C b_n$ for $n \in \mathbb{N}$, and $a_n \asymp b_n$ if $a_n \ll b_n$ and $a_n \gg b_n$.

2 Estimates for Levy Means

The results in this section were proved in [7].

Let us denote by $|||x|||$ the euclidean norm $\left(\sum_{k=1}^n |x_k|^2\right)^{1/2}$ of the element $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and by S^{n-1} the unit euclidean sphere $\{x \in \mathbb{R}^n : |||x||| = 1\}$ in $\mathbb{R}^n$. The Levy mean for a norm $\|\cdot\|$ on $\mathbb{R}^n$ is defined by

$$M(\|\cdot\|) \;=\; \left(\int_{S^{n-1}} \|x\|^2 \, d\mu(x)\right)^{1/2},$$

where μ denotes the normalized Lebesgue measure on S^{n-1}.

Given $M_1, M_2 \in \mathbb{N}$, with $M_1 < M_2$, we will use the following notations

$$\mathcal{T}_{M_1,M_2} \;=\; \bigoplus_{l=M_1+1}^{M_2} \mathcal{H}_l, \qquad D_{M_1,M_2}(\mathbf{x}) \;=\; D_{M_2}(\mathbf{x}) - D_{M_1}(\mathbf{x}) \quad \text{and} \quad n \;=\; \dim \mathcal{T}_{M_1,M_2}.$$

Let $\{\xi_k\}_{k=1}^n$ be a basis of $\mathcal{T}_{M_1,M_2}$, orthonormal in $L^2(\mathbb{T}^d)$ and let $J : \mathbb{R}^n \to \mathcal{T}_{M_1,M_2}$ be the coordinate isomorphism that assigns to $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$ the function

$$J(\alpha) \;=\; \sum_{k=1}^n \alpha_k \xi_k \;\in\; \mathcal{T}_{M_1,M_2}.$$

Let B_l be a subset of $A_l \setminus A_{l-1}$ with exactly $d_l/2$ elements and such that, if $\mathbf{k} \in B_l$, then $-\mathbf{k} \notin B_l$. Take $B_l = \{\mathbf{m}_j^l : 1 \le j \le d_l/2\}$ where the elements are chosen satisfying $|\mathbf{m}_j^l| \le |\mathbf{m}_{j+1}^l|$ for $1 \le j \le d_l/2$. For each $1 \le j \le d_l/2$ we define $\xi_{2j-1}^l(\mathbf{x}) \;=\; \sqrt{2}\cos(\mathbf{m}_j^l \cdot \mathbf{x})$ and $\xi_{2j}^l(\mathbf{x}) \;=\; \sqrt{2}\,\mathrm{sen}(\mathbf{m}_j^l \cdot \mathbf{x})$. Thus $\{\xi_j^l\}$ is a ordered and orthonormal basis of $\mathcal{H}_l$, constituted only by real functions. We consider the orthonormal basis

$$\Theta_{M_1}^{M_2} \;=\; \{\xi_k\}_{k=1}^n \;=\; \{\xi_j^l : M_1 + 1 \le l \le M_2,\ 1 \le j \le d_l\}$$

of $\mathcal{T}_{M_1,M_2}$ endowed with the order $\xi_1^{M_1+1},\ldots,\xi_{d_{M_1+1}}^{M_1+1},\ldots,\xi_1^{M_2},\ldots,\xi_{d_{M_2}}^{M_2}$.

Consider a function $\lambda:[0,\infty)\to\mathbb{R}$ and let $\Lambda=\{\lambda_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}^d}$ be the sequence of multipliers defined by $\lambda_{\mathbf{k}}=\lambda(|\mathbf{k}|)$. For $M_1+1\le l\le M_2$ e $1\le j\le d_l/2$, we define $\lambda_{2j-1}^l=\lambda_{2j}^l=\lambda_{\mathbf{m}_j^l}=\lambda(|\mathbf{m}_j^l|)$ and consider the numerical sequence

$$\Lambda_n=\{\widetilde{\lambda}_k\}_{k=1}^n \;=\; \{\lambda_j^l: M_1+1\le l\le M_2,\ 1\le j\le d_l\}$$

endowed with the order $\lambda_1^{M_1+1},\ldots,\lambda_{d_{M_1+1}}^{M_1+1},\ldots,\lambda_1^{M_2},\ldots,\lambda_{d_{M_2}}^{M_2}$. Let Λ_n be the multiplier operator defined on $\mathcal{T}_{M_1,M_2}$ by

$$\Lambda_n\left(\sum_{k=1}^n\alpha_k\xi_k\right) \;=\; \sum_{k=1}^n\widetilde{\lambda}_k\alpha_k\xi_k. \tag{2.5}$$

We denote also by Λ_n te multiplier operator on $\mathbb{R}^n$, defined by

$$\Lambda_n(\alpha_1,\ldots,\alpha_n) \;=\; (\widetilde{\lambda}_1\alpha_1,\ldots,\widetilde{\lambda}_n\alpha_n).$$

We suppose that $\lambda(|\mathbf{k}|)\neq 0$ for $M_1<|\mathbf{k}|\le M_2,\ \mathbf{k}\in\mathbb{Z}^d$. Given $\xi\in\mathcal{T}_{M_1,M_2}$ e $1\le p\le\infty$, we define

$$\|\xi\|_{\Lambda_n,p} \;=\; \|\Lambda_n\xi\|_p.$$

The application $\mathcal{T}_{M_1,M_2}\ni\xi\longmapsto\|\xi\|_{\Lambda_n,p}$ is a norm on $\mathcal{T}_{M_1,M_2}$.For $\alpha\in\mathbb{R}^n$, we define

$$\|\alpha\|_{(\Lambda_n,p)} \;=\; \|\Lambda_n J(\alpha)\|_{\Lambda_n,p}$$

and we have that the application $\mathbb{R}^n\ni\alpha\longmapsto\|\alpha\|_{(\Lambda_n,p)}$ is a norm on $\mathbb{R}^n$. We will denote

$$\begin{aligned} B^n_{\Lambda_n,p} &= B^n_{\Lambda,p} \;=\; \{\xi\in\mathcal{T}_{M_1,M_2}:\|\xi\|_{\Lambda_n,p}\le 1\},\\ B^n_{(\Lambda_n,p)} &= B^n_{(\Lambda,p)} \;=\; \{\alpha\in\mathbb{R}^n:\|\alpha\|_{(\Lambda_n,p)}\le 1\}.\end{aligned}$$

If Λ_n is the identity operator I, we will write $\|\cdot\|_{I,p} = \|\cdot\|_p$, $\|\cdot\|_{(I,p)} = \|\cdot\|_{(p)}$, $B^n_{I,p} = B^n_p$ and $B^n_{(I,p)} = B^n_{(p)}$. Now, let $\varphi\in\mathcal{T}_{M_1,M_2}$, φ real. We have that

$$\varphi \;=\; \sum_{l=M_1+1}^{M_2}\sum_{\mathbf{k}\in A_l\setminus A_{l-1}}\widehat{\varphi}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} \;=\; \sum_{l=M_1+1}^{M_2}\sum_{j=1}^{d_l/2}\left(\alpha_{2j-1}^l\xi_{2j-1}^l(\mathbf{x})+\alpha_{2j}^l\xi_{2j}^l(\mathbf{x})\right) \;=\; \sum_{k=1}^n\alpha_k\xi_k(\mathbf{x}),$$

where $\alpha_{2j-1}^l=\left(\widehat{\varphi}(\mathbf{m}_j^l)+\widehat{\varphi}(-\mathbf{m}_j^l)\right)/\sqrt{2}$ and $\alpha_{2j}^l=\left(-\widehat{\varphi}(\mathbf{m}_j^l)+\widehat{\varphi}(-\mathbf{m}_j^l)\right)/\sqrt{2}i$. Hence

$$\Lambda\varphi(\mathbf{x}) \;=\; \sum_{l=M_1+1}^{M_2}\sum_{\mathbf{k}\in A_l\setminus A_{l-1}}\lambda_{\mathbf{k}}\widehat{\varphi}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} \;=\; \sum_{k=1}^n\widetilde{\lambda}_k\alpha_k\xi_k(\mathbf{x}) \;=\; \Lambda_n\varphi(\mathbf{x}).$$

Thus, since $B_p^n=U_p\cap\mathcal{T}_{M_1,M_2}$, then

$$\Lambda_n B_p^n \;=\; \Lambda B_p^n \;\subseteq\; \Lambda U_p. \tag{2.6}$$

Theorem 2.1. ([7], p. 51) *Let $\lambda:[0,\infty)\to\mathbb{R}$ such that $t\mapsto|\lambda(t)|$ is a monotonic function, let $n=\dim\mathcal{T}_{M_1,M_2}$ and consider the orthonormal system $\{\xi_k\}_{k=1}^n$ of $\mathcal{T}_{M_1,M_2}$ and the multiplier operator Λ_n on $\mathcal{T}_{M_1,M_2}$ defined in* (2.5)*. If $t\mapsto|\lambda(t)|$ is decreasing, then there is an absolute constant $C>0$ such that:*

(i) If $2\le p<\infty$, then

$$n^{-1/2}\left(\sum_{l=M_1+1}^{M_2}|\lambda(l)|^2d_l\right)^{1/2} \;\le\; M(\|\cdot\|_{(\Lambda_n,p)}) \;\le\; Cp^{1/2}n^{-1/2}\left(\sum_{l=M_1+1}^{M_2}|\lambda(l-1)|^2d_l\right)^{1/2};$$

(ii) If $p=\infty$, then

$$n^{-1/2}\left(\sum_{l=M_1+1}^{M_2}|\lambda(l)|^2 d_l\right)^{1/2} \;\leq\; M(\|\cdot\|_{(\Lambda_n,\infty)}) \;\leq\; C(\ln\, n)^{1/2}n^{-1/2}\left(\sum_{l=M_1+1}^{M_2}|\lambda(l-1)|^2 d_l\right)^{1/2};$$

(iii) If $1\leq p\leq 2$, then

$$\frac{1}{2}n^{-1/2}\left(\sum_{l=M_1+1}^{M_2}|\lambda(l)|^2 d_l\right)^{1/2} \;\leq\; M(\|\cdot\|_{(\Lambda_n,p)}) \;\leq\; n^{-1/2}\left(\sum_{l=M_1+1}^{M_2}|\lambda(l-1)|^2 d_l\right)^{1/2};$$

(iv) If $p=2$, then

$$n^{-1/2}\left(\sum_{l=M_1+1}^{M_2}|\lambda(l)|^2 d_l\right)^{1/2} \;\leq\; M(\|\cdot\|_{(\Lambda_n,2)}) \;\leq\; n^{-1/2}\left(\sum_{l=M_{1+1}}^{M_2}|\lambda(l-1)|^2 d_l\right)^{1/2}.$$

If $t\mapsto|\lambda(t)|$ is increasing, then we obtain the estimates in (i), (ii), (iii) e (iv), permuting $\lambda(l)$ for $\lambda(l-1)$.

3 Estimates for entropy numbers of general multiplier operators

The polar set of a compact subset K of $\mathbb{R}^n$ is defined by

$$K^\circ \;=\; \left\{x\in\mathbb{R}^n : \sup_{y\in K}|\langle x,y\rangle|\leq 1\right\}$$

and the norm $\|\cdot\|_{K^\circ}$ is defined by

$$\|x\|_{K^\circ} \;=\; \sup\left\{|\langle x,y\rangle| : y\in K\right\},\; x\in\mathbb{R}^n.$$

Theorem 3.1. (Urysohn Inequality, [13]). *Let K be a compact subset of $\mathbb{R}^n$. We have that*

$$\left(\frac{\mathrm{Vol}_n(K)}{\mathrm{Vol}_n(B^n_{(2)})}\right)^{1/n} \;\leq\; \int_{S^{n-1}}\|x\|_{K^\circ}\,d\mu(x),$$

where $Vol_n(A)$ denotes the volume of a measurable subset A of $\mathbb{R}^n$.

Proposition 3.1. ([13]) *Let V be a convex, centrally symmetric, limited and absorbent subset. Then there is a constant $C>0$ such that for all $n\in\mathbb{N}$,*

$$\left(\frac{\mathrm{Vol}_n(V)\mathrm{Vol}_n(V^\circ)}{\left(\mathrm{Vol}_n\left(B^n_{(2)}\right)\right)^2}\right)^{1/n} \;\geq\; C.$$

Theorem 3.2. *Let $\Lambda=\{\lambda_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}^d}$ be a multiplier operator, where $\lambda_{\mathbf{k}}=\lambda(|\,\mathbf{k}\,|)$, for some function $\lambda:[0,\infty)\to\mathbb{R}$, such that $t\mapsto|\lambda(t)|$ is decreasing. Then there is a constant $C>0$, depending only p and q, such that, for all $N,k\in\mathbb{N}$ and $n=\sum_{l=1}^N d_l$,*

$$e_k(\Lambda U_p,L^q) \;\geq\; C2^{-k/n}\left(\prod_{l=1}^N|\lambda(l)|^{d_l}\right)\vartheta_n,$$

where $d_l = \dim \mathcal{H}_l$ *and*

$$\vartheta_n = \begin{cases} 1, & p<\infty,\, q>1, \\ (\ln n)^{-1/2}, & p<\infty, q=1, \\ (\ln n)^{-1/2}, & p=\infty,\, q>1, \\ (\ln n)^{-1}, & p=\infty,\, q=1. \end{cases}$$

In particular, if $k = n$*, then*

$$e_n(\Lambda U_p, L^q) \;\geq\; C|\lambda(N)|\vartheta_n. \tag{3.7}$$

Proof. By the definitions of the norms $\|\cdot\|^\circ$ e $\|\cdot\|_{(q')}$ on $\mathbb{R}^n$, where $n = \dim \mathcal{T}_{0,N}$, $\mathcal{T}_{0,N} = \bigoplus_{l=1}^N \mathcal{H}_l$ and by the Holder inequality, we have that for all $x \in \mathbb{R}^n$

$$\|x\|^\circ_{(q)} \;\leq\; \|x\|_{(q')}, \quad 1/q + 1/q^{'} = 1.$$

Thus, if $1 \leq q \leq 2$, it follows by Theorem 3.1 and Theorem 2.1 that

$$\begin{aligned} \left(\frac{\mathrm{Vol}_n\left(B^n_{(q)}\right)}{\mathrm{Vol}_n\left(B^n_{(2)}\right)}\right)^{1/n} &\leq \int_{S^{n-1}} \|x\|^\circ_{(q)}\, d\mu(x) \;\leq\; \int_{S^{n-1}} \|x\|_{(q')}\, d\mu(x) \\ &\leq M(\|\cdot\|_{(q')}) \;\leq\; C_1 \begin{cases} (q')^{1/2}, & 2 \leq q' < \infty, \\ (\ln n)^{1/2}, & q' = \infty. \end{cases} \end{aligned} \tag{3.8}$$

Analogously, for all $2 \leq p \leq \infty$

$$\left(\frac{\mathrm{Vol}_n\left((B^n_{(p)})^\circ\right)}{\mathrm{Vol}_n\left(B^n_{(2)}\right)}\right)^{1/n} \;\leq\; C_2 \begin{cases} p^{1/2}, & 2 \leq p < \infty, \\ (\ln n)^{1/2}, & p = \infty. \end{cases} \tag{3.9}$$

Hence, we obtain by Proposition 3.1 and by (3.9)

$$\left(\frac{\mathrm{Vol}_n(B^n_{(p)})}{\mathrm{Vol}_n(B^n_{(2)})}\right)^{1/n} \;\geq\; C_3 \begin{cases} p^{-1/2}, & 2 \leq p < \infty, \\ (\ln n)^{-1/2}, & p = \infty. \end{cases} \tag{3.10}$$

Now, let $x_1, \ldots, x_{N(\epsilon)}$ a minimal ϵ-net for $\Lambda_n B^n_{(p)}$ in $(\mathbb{R}^n, \|\cdot\|_{(q)})$. Then $\Lambda_n B^n_{(p)} \subseteq \bigcup_{k=1}^{N(\epsilon)} (x_k + \epsilon B^n_{(q)})$, from where $\mathrm{Vol}_n(\Lambda_n B^n_{(p)}) \leq \epsilon^n N(\epsilon) \mathrm{Vol}_n(B^n_{(q)})$. Thus, remembering that for $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ and $X \subseteq \mathbb{R}^n$, $\mathrm{Vol}_n(T(X)) = (\det T)\mathrm{Vol}_n(X)$, where $\det T$ denotes the determinant of the matrix of the operator T, we obtain

$$(\det\Lambda_n)^{1/n} \frac{\left(\mathrm{Vol}_n(B^n_{(p)})\right)^{1/n}}{\left(\mathrm{Vol}_n(B^n_{(2)})\right)^{1/n}} \;=\; \frac{\left(\mathrm{Vol}_n(\Lambda_n B^n_{(p)})\right)^{1/n}}{\left(\mathrm{Vol}_n(B^n_{(2)})\right)^{1/n}} \;\leq\; \epsilon (N(\epsilon))^{1/n} \left(\frac{\mathrm{Vol}_n(B^n_{(q)})}{\mathrm{Vol}_n(B^n_{(2)})}\right)^{1/n}. \tag{3.11}$$

Hence, by (3.10), (3.11) and (3.8), it follows that

$$\begin{aligned} C_3 (\det\Lambda_n)^{1/n} \begin{cases} p^{-1/2}, & 2 \leq p < \infty, \\ (\ln n)^{-1/2}, & p = \infty, \end{cases} &\leq (\det\Lambda_n)^{1/n} \left(\frac{\mathrm{Vol}_n(B^n_{(p)})}{\mathrm{Vol}_n(B^n_{(2)})}\right)^{1/n} \\ \leq\; \epsilon (N(\epsilon))^{1/n} \left(\frac{\mathrm{Vol}_n(B^n_{(q)})}{\mathrm{Vol}_n(B^n_{(2)})}\right)^{1/n} &\leq\; C_1 \epsilon (N(\epsilon))^{1/n} \begin{cases} (q')^{1/2}, & 2 \leq q' < \infty, \\ (\ln n)^{1/2}, & q' = \infty. \end{cases} \end{aligned} \tag{3.12}$$

Taking $N(\epsilon) = 2^{k-1}$, we obtain by basic properties of entropy numbers, by (**??**), by entropy numbers definition and by (3.12), for $2 \leq p < \infty$ and $1 < q \leq 2$

$$e_k(\Lambda U_p, L^q) \;\;\geq\;\; 2^{-1} e_k(\Lambda U_p \cap \mathcal{T}_{0,N}, L^q \cap \mathcal{T}_{0,N}) = 2^{-1}\epsilon \;\geq\; C_4 2^{-k/n} (\det \Lambda_n)^{1/n} \;\geq\; C_5 2^{-k/n} \left(\prod_{l=1}^N \lambda(l)^{d_l}\right)^{1/n}.$$

Now, if $1 \le p < 2$ e $1 < q \le 2$, observing that $U_2 \subseteq U_p$, we find by basic properties of entropy numbers and by the previous inequality, that

$$e_k(\Lambda U_p, L^q) \;\ge\; e_k(\Lambda U_2, L^q) \;\ge\; C_6 2^{-k/n} \left(\prod_{l=1}^{N} \lambda(l)^{d_l} \right)^{1/n}.$$

For $1 \le p < \infty$ and $2 < q \le \infty$, we have that $U_q \subseteq U_2$ and thus, using basic properties of entropy numbers and the two previous inequalities, we get

$$e_k(\Lambda U_p, L^q) \;\ge\; e_k(\Lambda U_p, L^2) \;\ge\; C_7 2^{-k/n} \left(\prod_{l=1}^{N} \lambda(l)^{d_l} \right)^{1/n}.$$

We therefore conclude that, for $p < \infty$ and $q > 1$,

$$e_k(\Lambda U_p, L^q) \;\gg\; 2^{-k/n} \left(\prod_{l=1}^{N} \lambda(l)^{d_l} \right)^{1/n}.$$

The remaining cases follow with similar analysis.

To prove (3.7), we assume $k = n$. Since $t \mapsto |\lambda(t)|$ is decreasing, then $|\lambda(1)| \;\ge\; |\lambda(2)| \;\ge\; \cdots \;\ge\; |\lambda(N)|$ and thus

$$\prod_{k=1}^{N} |\lambda(k)|^{d_k} \;\ge\; \prod_{k=1}^{N} |\lambda(N)|^{d_k} \;=\; |\lambda(N)|^{\sum_{k=1}^{N} d_k} \;=\; |\lambda(N)|^n,$$

whence $e_n(\Lambda U_p, L^q) \;\ge\; C|\lambda(N)|\vartheta_n$.

□

Remark 3.1. *Let $N \in \mathbb{N}$. We define $N_0 = 0,\ N_1 = N$ and*

$$N_{k+1} = \min\{M \in \mathbb{N} : e|\lambda(M)| \le |\lambda(N_k)|\}.$$

Let

$$\theta_{N_k, N_{k+1}} = \dim \mathcal{T}_{N_k, N_{k+1}} = \sum_{l=N_k+1}^{N_{k+1}} d_l.$$

For $\epsilon > 0$, we put

$$M \;=\; \left[\frac{\ln]\theta_{N_1,N_2}}{\epsilon} \right],$$

$$m_k = [e^{-\epsilon k}\theta_{N_1,N_2}] + 1, \quad k = 1, \ldots, M$$

and $m_0 = \theta_{N_0,N_1} = \theta_{0,N}$. We have that

$$\sum_{j=1}^{M} m_j \le C_\epsilon \theta_{N_1,N_2}.$$

We say that $\Lambda = \{\lambda_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}^{\mathbf{d}}} \in K_{\epsilon,p},\ \epsilon > 0,\ 1 \le p \le 2$, if $|\lambda(k+1)| < |\lambda(k)|,\ N_{k+1} > N_k$, for all $k \in \mathbb{N}$ and if for all $N \in \mathbb{N}$ we have

$$\sum_{k=1}^{M} e^{-k(1-\epsilon)} \frac{\theta_{N_k,N_{k+1}}^{1/p}}{\theta_{N_1,N_2}^{1/2}} \;\le\; C_{\epsilon,p}\theta_{N_1,N_2}^{1/p-1/2}.$$

Remark 3.2. *For $\Lambda = \{\lambda(|\mathbf{k}|)\}_{\mathbf{k}\in\mathbb{Z}^\mathbf{d}}$ and for $k \in \mathbb{N}$ and $1 \le q \le \infty$ fixed, we define*

$$\chi_k^{(q)} = \chi_k = 3 \sup_{N\ge 1} \left(\frac{2^{-k+1}\mathrm{Vol}_n(B^n_{(2)})}{\mathrm{Vol}_n(B^n_{(q)})} \prod_{j=1}^{N} |\lambda(j)|^{d_j} \right)^{1/n},$$

where $n = \dim \mathcal{T}_{0,N}$. Observed that χ_k depends on k, q and the function λ. We have that

$$\left(\prod_{j=1}^{n} |\lambda(j)|^{d_j}\right)^{1/n} \le \left(\prod_{j=1}^{N} \left(\sup_{1\le j\le N} |\lambda(j)|\right)^{d_j}\right)^{1/n} = \sup_{1\le j\le N} |\lambda(j)| \tag{3.13}$$

and consequently

$$\chi_k^{(q)} = \chi_k \le 3 \sup_{N\ge1} (2^{-k+1})^{1/n} \left(\frac{\mathrm{Vol}_n(B^n_{(2)})}{\mathrm{Vol}_n(B^n_{(q)})}\right)^{1/n} \sup_{1\le j\le N} |\lambda(j)|$$

for all $k \in \mathbb{N}$. Since $B^n_{(q)} \subseteq B^n_{(2)}$ for $2 \le q \le \infty$, we get by (3.10)

$$1 \le \left(\frac{\mathrm{Vol}_n(B^n_{(2)})}{\mathrm{Vol}_n(B^n_{(q)})}\right)^{1/n} \le C \begin{cases} q^{1/2}, & 2 \le q < \infty, \\ (\ln n)^{1/2}, & q = \infty. \end{cases} \tag{3.14}$$

Lemma 3.1. ([12]) *Let $E = (\mathbb{R}^n, \|\cdot\|)$ a Banach Space n-dimensional and let*

$$M^* = M^*(E) = \int_{S^{n-1}} \|x\|\, d\mu(x).$$

Then, there is a positive constant C, such that, for all $m \in \mathbb{N}$

$$e_m\left(B^n_{(2)}, E\right) \le C \begin{cases} (n/m)^{1/2} M^*, & m \le n, \\ e^{-m/n} M^*, & m > n. \end{cases}$$

Theorem 3.3. *Let $\lambda : [0,\infty) \to \mathbb{R}$ such that $t \mapsto |\lambda(t)|$ is a decreasing function and let $\Lambda = \{\lambda_\mathbf{k}\}_{\mathbf{k}\in\mathbb{Z}^d}$, $\lambda_\mathbf{k} = \lambda(|\mathbf{k}|)$. We suppose that $\Lambda \in K_{\epsilon,2}$ for any $\epsilon > 0$. Let χ_k as in Remark 3.2, M, N_l, $\theta_{N_l,N_{l+1}}$ and m_l as in Remark 3.1 and let $\eta = k + \sum_{l=1}^{M} m_l$, $k \in \mathbb{N}$. Then, there is an absolute constant $C > 0$ such that, for $2 \le p \le \infty$, $1 \le q \le 2$, we have*

$$e_\eta(\Lambda U_p, L^q) \le C\chi_k,$$

and for $2 \le p, q \le \infty$ we have that

$$\begin{aligned} & e_\eta(\Lambda U_p, L^q) \\ \le\ & C\chi_k \left(\left\{ \begin{array}{ll} q^{1/2}, & 2 \le q < \infty, \\ \sup_{1\le j\le M} (\ln \theta_{N_j,N_{j+1}})^{1/2}, & q = \infty, \end{array} \right\} + \sum_{j=M+1}^{\infty} e^{-j} \theta_{N_j,N_{j+1}}^{1/2-1/q} \right) \end{aligned} \tag{3.15}$$

Proof. By ([7], p. 59), for $p = 2$, we get

$$\Lambda U_2 \subseteq \Lambda U_2 \cap \mathcal{T}_N + \bigoplus_{j=1}^{M} |\lambda(N_j)| B_2^{N_j,N_{j+1}} + \bigoplus_{j=M+1}^{\infty} |\lambda(N_j)| \theta_{N_j,N_{j+1}}^{1/2-1/q} B_q^{N_j,N_{j+1}},$$

where $\mathcal{T}_N = \bigoplus_{j=0}^{N} \mathcal{H}_j$. Using basic properties of entropy numbers and (1), it follows that

$$\begin{aligned} e_\eta(\Lambda U_2, L^q) &\leq e_k(\Lambda U_2 \cap \mathcal{T}_N, L^q \cap \mathcal{T}_N) + \sum_{l=1}^{M} |\lambda(N_l)| e_{m_l}\left(B_2^{N_l,N_{l+1}}, L^q \cap \mathcal{T}_{N_l,N_{l+1}}\right) \\ &+ e_1\left(\bigoplus_{l=M+1}^{\infty} |\lambda(N_l)| \theta_{N_l,N_{l+1}}^{1/2-1/q} B_q^{N_l,N_{l+1}}, L^q \cap \overline{\mathcal{T}}_{N_{M+1}}\right), \end{aligned} \tag{3.16}$$

where $\overline{\mathcal{T}}_s = \{\varphi \in L^1(\mathbb{T}^d) : \widehat{\varphi}(\mathbf{k}) = 0, \ |\, \mathbf{k} \,| \leq s\}$. Using basic properties of entropy numbers again and (1), we have that

$$\begin{aligned} & e_{1+1-1}\left(\bigoplus_{l=M+1}^{\infty} |\lambda(N_l)| \theta_{N_l,N_{l+1}}^{1/2-1/q} B_q^{N_l,N_{l+1}}, L^q \cap \overline{\mathcal{T}}_{N_{M+1}}\right) \\ \leq & \sum_{l=M+1}^{\infty} |\lambda(N_l)| \theta_{N_l,N_{l+1}}^{1/2-1/q} e_1\left(B_q^{N_l,N_{l+1}}, L^q \cap \mathcal{T}_{N_l,N_{l+1}}\right) \\ = & \sum_{l=M+1}^{\infty} |\lambda(N_l)| \theta_{N_l,N_{l+1}}^{1/2-1/q}. \end{aligned} \tag{3.17}$$

Thus, we get, by (3.16) and (3.17)

$$\begin{aligned} e_\eta(\Lambda U_2, L^q) &\leq e_k(\Lambda U_2 \cap \mathcal{T}_N, L^q \cap \mathcal{T}_N) + \sum_{l=1}^{M} |\lambda(N_l)| e_{m_l}(B_2^{N_l,N_{l+1}}, L^q \cap \mathcal{T}_{N_l,N_{l+1}}) \\ &+ \sum_{l=M+1}^{\infty} |\lambda(N_l)| \theta_{N_l,N_{l+1}}^{1/2-1/q}. \end{aligned} \tag{3.18}$$

Let us prove first that

$$e_k(\Lambda U_2 \cap \mathcal{T}_N, L^q \cap \mathcal{T}_N) \leq \chi_k. \tag{3.19}$$

It is easy to see that $q \geq 2$

$$|\lambda(N)| B_{(q)}^n \subseteq |\lambda(N)| B_{(2)}^n \subseteq \Lambda_n B_{(2)}^n. \tag{3.20}$$

Let $\Theta = \{z_j\}_{1 \leq j \leq m}$ be a χ_k-separate maximal subset of $\Lambda_n B_{(2)}^n$ in $(\mathbb{R}^n, \|\cdot\|_{(q)})$, in other works, $\|z_i - z_j\|_{(q)} \geq \chi_k$, for all $i \neq j$. It follows from maximality that Θ is a χ_k-net of $\Lambda_n B_{(2)}^n$ in $(\mathbb{R}^n, \|\cdot\|_{(q)})$ and that the balls

$$z_j + \frac{\chi_k}{2} B_{(q)}^n, \quad 1 \leq j \leq m$$

they are mutually disjoint. Applying (3.20), it follows that

$$\bigcup_{j=1}^{m} \left(z_j + \frac{\chi_k}{2} B_{(q)}^n\right) \subseteq \frac{3}{2} \Lambda_n B_{(2)}^n,$$

where the union is disjoint. Hence, we get in terms of volume

$$\left(\frac{\chi_k}{2}\right)^n m \cdot \mathrm{Vol}_n(B_{(q)}^n) \leq \frac{3^n}{2^n} \left(\prod_{j=1}^{N} |\lambda(j)|^{d_j}\right) \mathrm{Vol}_n(B_{(2)}^n),$$

and thus

$$m \leq \left(\frac{3}{\chi_k}\right)^n \frac{\mathrm{Vol}_n(B_{(2)}^n)}{\mathrm{Vol}_n(B_{(q)}^n)} \left(\prod_{j=1}^{N} |\lambda(j)|^{d_j}\right). \tag{3.21}$$

From the definition of χ_k, we obtain

$$\left(\frac{3}{\chi_k}\right)^n \;\le\; 2^{k-1}\left(\frac{\mathrm{Vol}_n(B^n_{(2)})}{\mathrm{Vol}_n(B^n_{(q)})}\prod_{j=1}^{N}|\lambda(j)|^{d_j}\right)^{-1}. \tag{3.22}$$

By (3.21) and (3.22), we conclude that the cardinality m of a $\Theta\ \chi_k$-separate net of $\Lambda_n B^n_{(2)}$ can be estimated by $m\le 2^{k-1}$ and thus (3.19) follows by definition of entropy numbers.

Now, applying the Lemma 3.1 to the Banach Space $E_j = (\mathbb{R}^{\theta_{N_j,N_{j+1}}}, \|\cdot\|_{(q)})$ and m_j, remembering that $m_j = [e^{-\epsilon j}\theta_{N_1,N_2}]+1 \;\ge\; e^{-\epsilon j}\theta_{N_1,N_2}$ and that $M^*(E_j)\le M(E_j)$ we get

$$e_{m_j}(B_2^{N_j,N_{j+1}}, L^q\cap\mathcal{T}_{N_j,N_{j+1}}) \;=\; e_{m_j}(B_{(2)}^{N_j,N_{j+1}}, E_j) \;\ll\; \frac{\theta^{1/2}_{N_j,N_{j+1}}}{m_j^{1/2}}M^*(E_j) \;\ll\; \frac{\theta^{1/2}_{N_j,N_{j+1}}}{\theta^{1/2}_{N_1,N_2}}e^{\epsilon j/2}M(E_j).$$

Thus, applying the Theorem 2.1, we obtain

$$e_{m_j}(B_2^{N_j,N_{j+1}}, L^q\cap\mathcal{T}_{N_j,N_{j+1}}) \;\ll\; \frac{\theta^{1/2}_{N_j,N_{j+1}}}{\theta^{1/2}_{N_1,N_2}}e^{\epsilon j/2}\begin{cases} q^{1/2}, & 2\le q<\infty,\\ (\ln\theta_{N_j,N_{j+1}})^{1/2}, & q=\infty,\end{cases}$$

and thereby, since $|\lambda(N_j)|\le|\lambda(N)|e^{-j+1}$ and $\Lambda\in K_{\epsilon,2}$, we have that

$$\begin{aligned}
&\sum_{j=1}^{M}|\lambda(N_j)|e_{m_j}(B_2^{N_j,N_{j+1}}, L^q\cap\mathcal{T}_{N_j,N_{j+1}})\\
\ll\;& |\lambda(N)|\sum_{j=1}^{M}e^{-j(1-\epsilon/2)}\frac{\theta^{1/2}_{N_j,N_{j+1}}}{\theta^{1/2}_{N_1,N_2}}\begin{cases} q^{1/2}, & 2\le q<\infty,\\ (\ln\theta_{N_j,N_{j+1}})^{1/2}, & q=\infty,\end{cases}\\
\ll\;& \begin{cases} q^{1/2}, & 2\le q<\infty,\\ \sup_{1\le j\le M}(\ln\theta_{N_j,N_{j+1}})^{1/2}, & q=\infty,\end{cases}
\end{aligned} \tag{3.23}$$

and

$$\sum_{j=M+1}^{\infty}|\lambda(N_j)|\theta^{1/2-1/q}_{N_j,N_{j+1}} \;\le\; |\lambda(N)|\sum_{j=M+1}^{\infty}e^{-j+1}\theta^{1/2-1/q}_{N_j,N_{j+1}} \;\ll\; \sum_{j=M+1}^{\infty}e^{-j}\theta^{1/2-1/q}_{N_j,N_{j+1}}. \tag{3.24}$$

Finally, by (3.18), (3.19), (3.23) and (3.24), we obtain (3.15).

□

4 Estimates for sets of finitely differentiable functions

In this section, we apply the results of the preceding section for obtaining estimates for entropy numbers of the multiplier operators $\Lambda^{(1)} = \{\lambda_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}^d}$, $\lambda_{\mathbf{k}} = \lambda(|\mathbf{k}|)$, where $\lambda:[0,\infty)\to\mathbb{R}$ is defined by $\lambda(t) = t^{-\gamma}(\ln t)^{-\xi},\ t>1$ and $\lambda(t)=0$, for $0\le t\le 1$, $\gamma,\xi\in\mathbb{R}$, $\gamma>d/2$, $\xi\ge 0$. We have that $\Lambda^{(1)}U_p$ are sets of finitely differentiable functions on $\mathbb{T}^d$. In particular, if $\xi=0$, $\Lambda^{(1)}U_p$ are Sobolev type classes on $\mathbb{T}^d$.

Lemma 4.1. *For $\gamma,\xi>0$ e $k\in\mathbb{N}$, let $g:[2,\infty)\longrightarrow\mathbb{R}$ be the function defined by*

$$g(x) \;=\; -\frac{k(\ln 2)}{x} - \frac{\gamma}{d}\ln x - \xi\ln(\ln x).$$

Then, there is constants $C_1, C_2, \overline{C}>0$ depending only of the γ,ξ,d, such that the maximum of the function g is assumed in a point x_k satisfying

$$C_1k\le x_k\le C_2k,\quad k\ge 1$$

and

$$g(x_k) \;\le\; \overline{C} - \frac{\gamma}{d}\ln k - \xi\ln(\ln k).$$

Proof. We have that $g^{'}(x) = 0$ if and only if $x = k(\ln 2)d\,(\ln x)/(\gamma \ln x + d\xi)$.

The function $g^{'}$ is zero at a single point x_k, which is a local maximum for the function g. If $h(x) = (\ln x)/(\gamma \ln x + d\xi)$, then $h(2) \le h(x) \le 1/\gamma$ for $x \ge 2$ and therefore

$$\frac{(\ln 2)^2 d}{\gamma \ln 2 + d\xi} k \;\le\; k(\ln 2)d\,\frac{\ln x}{\gamma \ln x + d\xi} \;\le\; \frac{(\ln 2)d}{\gamma} k.$$

Thus, there is constants C_1 e C_2 which depend only d, γ, ξ, satisfying

$$C_1 k \;\le\; x_k \;\le\; C_2 k, \qquad k \ge 1.$$

Let $C_1 \le C \le C_2$ such that $x_k = Ck$, so

$$g(x_k) \;=\; \frac{k \ln 2}{Ck} - \frac{\gamma}{d}\ln(Ck) - \xi \ln(\ln(Ck)) \;\le\; \overline{C} - \frac{\gamma}{d}\ln k - \xi \ln(\ln k).$$

□

Theorem 4.1. *We have that*

$$e_k(\Lambda^{(1)}U_p, L^q) \;\ll\; k^{-\gamma/d}(\ln k)^{-\xi} \begin{cases} 1, & 2 \le p \le \infty,\ q < \infty, \\ (\ln k)^{1/2}, & 2 \le p \le \infty, q = \infty, \end{cases} \tag{4.25}$$

and

$$e_k(\Lambda^{(1)}U_p, L^q) \;\gg\; k^{-\gamma/d}(\ln k)^{-\xi} \begin{cases} 1, & p < \infty,\ q > 1, \\ (\ln k)^{-1/2}, & p < \infty,\ q = 1, \\ (\ln k)^{-1/2}, & p = \infty,\ q > 1, \\ (\ln k)^{-1}, & p = \infty,\ q = 1. \end{cases} \tag{4.26}$$

Proof. Let $n = \dim \mathcal{T}_N$. Since $\lambda(N) = N^{-\gamma}(\ln N)^{-\xi} \asymp n^{-\gamma/d}(\ln n)^{-\xi}$, we obtain by (3.7)

$$e_n(\Lambda^{(1)}U_p, L^q) \;\gg\; n^{-\gamma/d}(\ln n)^{-\xi} \begin{cases} 1, & p < \infty,\ q > 1, \\ (\ln n)^{-1/2}, & p < \infty,\ q = 1, \\ (\ln n)^{-1/2}, & p = \infty,\ q > 1, \\ (\ln n)^{-1}, & p = \infty,\ q = 1. \end{cases}$$

Now, let $k \in \mathbb{N}$ such that $\dim \mathcal{T}_{N-1} \le k \le \dim \mathcal{T}_N = n$. It follows by basic properties of entropy numbers and by the previous inequality that

$$e_k(\Lambda^{(1)}U_p, L^q) \;\ge\; e_n(\Lambda^{(1)}U_p, L^q) \;\gg\; n^{-\gamma/d}(\ln n)^{-\xi} \begin{cases} 1, & p < \infty,\ q > 1, \\ (\ln n)^{-1/2}, & p < \infty,\ q = 1, \\ (\ln n)^{-1/2}, & p = \infty,\ q > 1, \\ (\ln n)^{-1}, & p = \infty,\ q = 1. \end{cases}$$

But, $n \asymp N^d \asymp (N-1)^d \asymp \dim \mathcal{T}_{N-1} \le k \le \dim \mathcal{T}_N = n$, whence $k \asymp n$ and thus, we obtain (4.26) of the above inequality. Now, let us note that

$$\sigma_n \;=\; \left(\prod_{j=2}^{N} |\lambda(j)|^{d_j} \right)^{1/n} \;\ge\; \left(|\lambda(N)|^{\sum_{j=2}^{N} d_j} \right)^{1/n} \;\ge\; |\lambda(N)| \;\asymp\; n^{-\gamma/d}(\ln n)^{-\xi}. \tag{4.27}$$

Moreover

$$\ln \sigma_n \;=\; \ln \left(\prod_{j=2}^{N} |\lambda(j)|^{d_j} \right)^{1/n} \;\le\; -\frac{\gamma}{n}\sum_{j=3}^{N} d_j \ln j - \frac{\xi}{n}\sum_{j=3}^{N} d_j \ln(\ln j) + C_1. \tag{4.28}$$

By (1.3), we have that $d_j \geq Ej^{d-1} - C_2 j^{d-2}$, where $E = 2\pi^{d/2}/\Gamma(d/2)$ and C_2 is a positive constant. Thus, if $f(x) = Ex^{d-1}\ln x - C_2 x^{d-2}\ln x$ and $h(x) = Ex^{d-1}\ln(\ln x) - C_2 x^{d-2}\ln(\ln x)$, then

$$\frac{1}{n}\sum_{j=3}^{N} d_j \ln j \;\geq\; \frac{1}{n}\int_2^N f(x)\,dx \;\geq\; \frac{E}{nd}\,N^d\ln N - \frac{C_2}{n(d-1)}N^{d-1}\ln N + C_3$$

and

$$\frac{1}{n}\sum_{j=3}^{N} d_j \ln(\ln j) \;\geq\; \frac{1}{n}\int_2^N h(x)\,dx \;\geq\; \frac{E}{nd}\,N^d\ln(\ln N) - \frac{C_2}{n(d-1)}N^{d-1}\ln(\ln N) + C_4.$$

Now, by (1.4), we have that $1/n \;\geq\; 1/FN^d - C/F^2N^{d+1}$, where $F = E/d$ and therefore

$$\begin{aligned}\frac{1}{n}\sum_{j=3}^{N} d_j \ln j \;&\geq\; \left(\frac{1}{FN^d} - \frac{C}{F^2N^{d+1}}\right)\left(\frac{E}{d}\,N^d\ln N - \frac{C_2}{d-1}N^{d-1}\ln N + C_3\right)\\ &\geq\; \ln N + C_5. \end{aligned} \tag{4.29}$$

and

$$\begin{aligned}\frac{1}{n}\sum_{j=3}^{N} d_j \ln(\ln j) \;&\geq\; \left(\frac{1}{FN^d} - \frac{C}{F^2N^{d+1}}\right)\left(\frac{E}{d}\,N^d\ln(\ln N) - \frac{C_2}{d-1}N^{d-1}\ln(\ln N) + C_4\right)\\ &\geq\; \ln(\ln N) + C_6. \end{aligned} \tag{4.30}$$

Consequently, we obtain by (4.28), (4.29) and (4.30)

$$\ln\sigma_n \;\leq\; -\gamma\ln N - \xi\ln(\ln N) + C_7$$

and therefore, sice $N \asymp n^d$,

$$\sigma_n \;\ll\; e^{(\ln N)^{-\gamma}} e^{(\ln(\ln N))^{-\xi}} \;\asymp\; n^{-\gamma/d}(\ln n)^{-\xi}. \tag{4.31}$$

By (4.27) and (4.31), it follows that

$$\sigma_n \;\asymp\; n^{-\gamma/d}(\ln n)^{-\xi} \tag{4.32}$$

and thus, we obtain by (3.14) and (4.32), for $2 \leq q < \infty$

$$\chi_k \;\asymp\; \sup_{N\geq 1} 2^{-k/n}\sigma_n \;\asymp\; \sup_{N\geq 1}\; 2^{-k/n} n^{-\gamma/d}(\ln n)^{-\xi}. \tag{4.33}$$

Let

$$g(x) = -\frac{k}{x}\ln 2 - \frac{\gamma}{d}\ln x - \xi\ln(\ln x).$$

Then, follows by (4.33) and by Lemma 4.1, that for $2 \leq q < \infty$,

$$\chi_k \;\asymp\; e^{\sup_N g(n)} \asymp\; e^{g(x_k)} \;\leq\; e^{\overline{C} - (\gamma/d)\ln k - \xi\ln(\ln k)} \;\asymp\; k^{-\gamma/d}(\ln k)^{-\xi}.$$

It follows by ([7], p. 62) for $p = 2$ that $\Lambda^{(1)} \in K_{\epsilon,2}$. Thus, applying the Theorem 3.3 and observing that $\sum_{j=M+1}^{\infty} e^{-j}\theta_{N_j,N_{j+1}}^{1/2-1/q} \;\ll\; 1$ (ver [7], p. 62), we obtain

$$e_\eta(\Lambda^{(1)}U_p, L^q) \;\ll\; \chi_k \;\ll\; k^{-\gamma/d}(\ln k)^{-\xi}, \qquad 2 \leq p \leq \infty,\; 2 \leq q < \infty. \tag{4.34}$$

Let $n_k \in \mathbb{N}$ such that $x_k \in [n_k, n_k{+}1)$. Then $e^{g(n_k)} \asymp e^{g(x_k)} \asymp \chi_k$ and $n_k \asymp x_k \asymp k$. We apply the Theorem 3.3 for k and $N = [(n_k)^{1/d}] \asymp k^{1/d}$ and thus $n = \dim\mathcal{T}_N \asymp n_k \asymp k$. Now, since for $k \geq 1$, $\theta_{N_k,N_{k+1}} \asymp N_{k+1}^d$, $N_{k+1} \ll e^{k/\gamma}N$ (ver [7], p. 61) and $n \asymp N^d$, we get by Remark 3.1

$$\eta \;=\; k + \sum_{j=1}^{M} m_j \;\asymp\; k + \sum_{j=1}^{M} e^{-\epsilon j}\theta_{N_1,N_2} \;\asymp\; k + N^d \;\asymp\; k + n \;\asymp\; k$$

and thus, by (4.34) it follows that

$$e_k(\Lambda^{(1)}U_p, L^q) \ \ll \ k^{-\gamma/d}(\ln k)^{-\xi}, \qquad 2 \le p \le \infty,\ 1 \le q < \infty. \tag{4.35}$$

For $2 \le p \le \infty$ e $q = \infty$, using (3.14), we obtain with analogous reasoning

$$e_k(\Lambda^{(1)}U_p, L^q) \ \ll \ k^{-\gamma/d}(\ln k)^{-\xi}(\ln k)^{1/2}. \tag{4.36}$$

By (4.35) and (4.36), finally we get (4.25), concluding the demonstration.

□

5 Estimates for sets of infinitely differentiable functions

In this section, we apply the results of Section 3 in obtaining estimates for entropy numbers of multipliers operators $\Lambda^{(2)} = \{\lambda_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}^d}$, $\lambda_{\mathbf{k}} = \lambda(|\mathbf{k}|)$, where $\lambda : [0,\infty) \to \mathbb{R}$ is defined by $\lambda(t) = e^{-\gamma t^r}$, $\gamma, r > 0$. We have that $\Lambda^{(2)}U_p$ are sets of infinitely differentiable functions if $0 < r < 1$ or analitics if $r = 1$ on the torus $\mathbb{T}^d$.

Remark 5.1. *Let $\gamma, r \in \mathbb{R}$, $\gamma, r > 0$. For $N, k \in \mathbb{N}$ and $n = \dim \mathcal{T}_N$, let*

$$A_{N,k} \ = \ -\frac{1}{n}\left(k\ln 2 + \gamma\sum_{l=1}^{N} l^r\ d_l\right).$$

We know by (1.3) that $d_l \asymp l^{d-1}$ and $n \asymp N^d$, and thus

$$\sum_{l=1}^{N} l^r\ d_l \ \asymp \ \sum_{l=1}^{N} l^{d+r-1} \ \asymp \ \int_0^N x^{d+r-1}\,dx \ \asymp \ N^{d+r},$$

and therefore

$$A_{N,k} \ \asymp \ g(N), \tag{5.37}$$

where $g(x) = -kx^{-d} - x^r$. The function g assumes absolute maximum value in $x_k = (d/r)^{1/(d+r)}k^{1/(d+r)}$. Since for all $r > 0$ e $x > 1$, we have that $-dkx^{-d-1} \ \le\ k[(x+1)^{-d} - x^{-d}] \ \le\ 0$ and $0 \ \le\ (x+1)^r - x^r \ \le\ rx^{r-1}$, then

$$g(x+1) \ \ge\ g(x) - rx^{r-1}.$$

In this manner, since g is decreasing for $x > x_k$, it follows that

$$g(x) - rx^{r-1} \ \le\ g(x+1) \ \le\ g(x+t) \ \le\ g(x), \qquad x \ge x_k,\ 0 \le t \le 1.$$

But, there is $\overline{N} \in \mathbb{N}$ such that $x_k \ \le\ \overline{N} \ \le\ x_k + 1$, and therefore

$$g(x_k) - rx_k^{r-1} \ \le\ g(\overline{N}) \ \le\ \sup_N g(N) \ \le\ g(x_k).$$

Hence

$$-C_1k^{r/(d+r)} - C_2k^{(r-1)/(d+r)} \ \le\ \sup_N g(N) \ \le\ -C_1k^{r/(d+r)},$$

with $C_1 = (d+r)d^{-d/(d+r)}r^{-r/(d+r)}$ and $C_2 = r^{(d+1)/(d+r)}d^{(r-1)/(d+r)}$. Thus, we obtain by (5.37),

$$\sup_N A_{N,k} \ \asymp\ \sup_N g(N) \ \asymp\ k^{r/(d+r)}.$$

Theorem 5.1. *For $0 < r \leq 1$, we have that*

$$e_k(\Lambda^{(2)}U_p, L^q) \gg e^{-\mathcal{C}k^{r/(d+r)}} \begin{cases} 1, & p < \infty,\ q > 1, \\ (\ln k)^{-1/2}, & p < \infty,\ q = 1, \\ (\ln k)^{-1/2}, & p = \infty,\ q > 1, \\ (\ln k)^{-1}, & p = \infty,\ q = 1, \end{cases} \tag{5.38}$$

where

$$\mathcal{C} = \gamma^{d/(d+r)} \left(\frac{(d+r)d\Gamma(d/2)(\ln 2)}{2r\pi^{d/2}} \right)^{r/(d+r)}.$$

Proof. Since

$$2^{-k/n} \left(\prod_{l=1}^{N} |\lambda(l)|^{d_l} \right)^{1/n} = e^{\ln 2^{-k/n}} \left(e^{-\gamma/n} \sum_{l=1}^{N} l^r d_l \right) = e^{A_{N,k}},$$

we obtain by the Theorem 3.2

$$e_k(\Lambda^{(2)}U_p, L^q) \gg e^{A_{N,k}} \begin{cases} 1, & p < \infty,\ q > 1, \\ (\ln n)^{-1/2}, & p < \infty,\ q = 1, \\ (\ln n)^{-1/2}, & p = \infty,\ q > 1, \\ (\ln n)^{-1}, & p = \infty,\ q = 1. \end{cases} \tag{5.39}$$

Let $f(x) = \left(2\pi^{d/2}/\Gamma(d/2)\right) x^{d+r-1} + C_1 x^{d+r-2}$. By (1.3), it follows that

$$\begin{aligned} \sum_{l=1}^{N} l^r d_l \ &\leq\ \sum_{l=1}^{N} f(l) \ \leq\ \int_1^{N+1} f(x)\, dx \\ &\leq\ \frac{2\pi^{d/2}}{\Gamma(d/2)(d+r)} (N+1)^{d+r} + C_1 (N+1)^{d+r-1}. \end{aligned}$$

By (1.4) we have that $n \geq \left(2\pi^{d/2}/d\Gamma(d/2)\right) N^d$ and thus

$$\begin{aligned} \frac{\gamma}{n} \sum_{l=1}^{N} l^r d_l \ &\leq\ \frac{d\Gamma(d/2)\gamma}{2\pi^{d/2} N^d} \left(\frac{2\pi^{d/2}}{\Gamma(d/2)(d+r)} (N+1)^{d+r} + C_1 (N+1)^{d+r-1} \right) \\ &\leq\ \frac{d\gamma}{(d+r)} (N+1)^r + C_2 \left(1 + \frac{1}{N}\right)^r N^{r-1} + C_3 (N+1)^{r-1} + C_4 \left(1 + \frac{1}{N}\right)^{r-1} N^{r-2} \\ &\leq\ \frac{d\gamma}{(d+r)} (N+1)^r + C_5. \end{aligned}$$

Now, since $0 < r \leq 1$, we have that $(N+1)^r = N^r \left(1 + N^{-1}\right)^r \leq N^r + C_r N^{r-1} \leq N^r + C_6$ and thus

$$\frac{\gamma}{n} \sum_{l=1}^{N} l^r d_l \leq \frac{d\gamma}{(d+r)} N^r + C_7. \tag{5.40}$$

Then, we get by (5.40) and (1.3)

$$A_{N,k} \geq \frac{-d\Gamma(d/2)(\ln 2)k}{2\pi^{d/2}} N^{-d} - \frac{d\gamma}{d+r} N^r - C_7. \tag{5.41}$$

Let

$$g(x) = \frac{-d\Gamma(d/2)(\ln 2)k}{2\pi^{d/2}}\, x^{-d} - \frac{d\gamma}{d+r}\, x^r - C_7.$$

It is easy to see that the absolute maximum of the function g is attained at

$$x_k = \left(\frac{(d+r)d\Gamma(d/2)(\ln 2)}{2r\gamma\pi^{d/2}}\right)^{1/(d+r)} k^{1/(d+r)}.$$

We can show, as in the Remark 5.1, that there is a constant C_8 such that

$$g(x) - C_8 \;\leq\; g(x+t) \;\leq\; g(x), \;\; x \geq x_k, \; 0 \leq t \leq 1,$$

and thus we obtain

$$g(x_k) - C_8 \;\leq\; \sup_N g(N+1) \;\leq\; g(x_k).$$

Therefore, it follows by (5.41),

$$\sup_N A_{N,k} \;\geq\; \sup_N g(N+1) \;\geq\; g(x_k) - C_8.$$

But

$$g(x_k) \;\;=\;\; -\gamma^{d/(d+r)} \left(\frac{(d+r)d\Gamma(d/2)(\ln 2)}{2r\pi^{d/2}}\right)^{r/(d+r)} k^{r/(d+r)} - C_7 \;=\; -\mathcal{C}k^{r/(d+r)} - C_7$$

and thus

$$\sup_N A_{N,k} \;\geq\; -\mathcal{C}k^{r/(d+r)} - C_9.$$

Hence, by (5.39) we get

$$e_k(\Lambda^{(2)}U_p, L^q) \;\gg\; e^{-\mathcal{C}k^{r/(d+r)}} \begin{cases} 1, & p < \infty, \; q > 1, \\ (\ln n)^{-1/2}, & p < \infty, \; q = 1, \\ (\ln n)^{-1/2}, & p = \infty, \; q > 1, \\ (\ln n)^{-1}, & p = \infty, \; q = 1. \end{cases}$$

By the Remark 5.1, we have $N \asymp k^{1/(d+r)}$ and thus $n \asymp N^d \asymp k^{d/(d+r)}$, whence $\ln n \asymp \ln k^{d/(d+r)} \asymp \ln k$ and therefore

$$e_k(\Lambda^{(2)}U_p, L^q) \;\gg\; e^{-\mathcal{C}k^{r/(d+r)}} \begin{cases} 1, & p < \infty, \; q > 1, \\ (\ln k)^{-1/2}, & p < \infty, \; q = 1, \\ (\ln k)^{-1/2}, & p = \infty, \; q > 1, \\ (\ln k)^{-1}, & p = \infty, \; q = 1. \end{cases}$$

□

Theorem 5.2. *For $0 < r \leq 1$, we have that*

$$e_k(\Lambda^{(2)}U_p, L^q) \;\ll\; e^{-\mathcal{C}k^{r/(d+r)}} \begin{cases} 1, & 2 \leq p \leq \infty, \; 1 \leq q < \infty, \\ \ln k, & 2 \leq p \leq \infty, \; q = \infty, \end{cases}$$

where $\mathcal{C}$ is the constant in the statement of the theorem 5.1.

Proof. By ([7], p. 67), we know that there is a constant δ' such that $\theta_{N_k,N_{k+1}} \;\leq\; C_1 e^{\delta' k} N^{d-r}$ for all $k \in \mathbb{N}$ and we have that $M \leq \epsilon^{-1} \ln \theta_{N_1,N_2} \leq C_2 \ln N$. Thus, for $1 \leq k \leq M$, $\ln \theta_{N_k,N_{k+1}} \;\leq\; C_3 \ln N \;\leq\; C_4 \ln n$ and hence

$$\sup_{1 \leq k \leq M} \left(\ln \theta_{N_k,N_{k+1}}\right)^{1/2} \;\leq\; C_5 \left(\ln n\right)^{1/2}. \tag{5.42}$$

For $2 \le q \le \infty$, we obtain by (3.14),

$$\begin{aligned}
\chi_k &\le \sup_{n\ge 1} C_6 2^{-k/n} \left(\prod_{j=1}^{N} |\lambda(j)|^{d_j}\right)^{1/n} \begin{cases} 1, & 2\le q<\infty, \\ (\ln n)^{1/2}, & q=\infty, \end{cases} \\
&= C_6 \sup_N e^{-\left(k\ln 2+\gamma\sum_{j=1}^{N} j^r d_j\right)/n} \begin{cases} 1, & 2\le q<\infty, \\ (\ln n)^{1/2}, & q=\infty, \end{cases} \\
&= C_6 \sup_N e^{A_{N,k}} \begin{cases} 1, & 2\le q<\infty, \\ (\ln n)^{1/2}, & q=\infty. \end{cases}
\end{aligned}$$

By (3.14) we get too

$$\chi_k \ \ge\ C_7 \sup_N e^{A_{N,k}}$$

and thus

$$e^{\sup_N\ A_{N,k}} \ll \chi_k \ll e^{\sup_N\ A_{N,k}} \begin{cases} 1, & 2\le q<\infty, \\ (\ln n)^{1/2}, & q=\infty. \end{cases} \tag{5.43}$$

If $f(x) = \left(2\pi^{d/2}/\Gamma(d/2)\right) x^{d+r-1} - Cx^{d+r-2}$, then by (1.3)

$$\sum_{l=1}^{N} l^r d_l \ \ge\ \sum_{l=1}^{N} f(l) \ \ge\ \int_0^N f(x)\,dx \ =\ \frac{2\pi^{d/2}}{\Gamma(d/2)(d+r)} N^{d+r} - \frac{C}{d+r-1} N^{d+r-1}. \tag{5.44}$$

By (1.4), we have that

$$\frac{k}{n} \ \ge\ \frac{k}{FN^d} - \frac{\overline{C}k}{F^2N^{d+1}}, \quad F = \frac{2\pi^{d/2}}{d\Gamma(d/2)}.$$

But, by Remark 5.1, we have that $N \asymp k^{1/(d+r)}$ and then, since $0<r\le 1$, we can guarantee the existence of an absolute constant C_8 satisfying

$$\frac{k}{n} \ \ge\ \frac{k}{FN^d} - C_8 \ =\ \frac{d\Gamma(d/2)k}{2\pi^{d/2}} N^{-d} - C_8. \tag{5.45}$$

Once again using (1.4), it follows by (5.44) for $0<r\le 1$,

$$\begin{aligned}
\frac{\gamma}{n}\sum_{l=1}^{N} l^r d_l &\ge \frac{2\pi^{d/2}\gamma}{F(d+r)\Gamma(d/2)} N^r - \frac{2\pi^{d/2}\gamma\overline{C}}{F^2(d+r)\Gamma(d/2)} N^{r-1} \\
&- \frac{\gamma C}{F(d+r-1)} N^{r-1} + \frac{\gamma C\overline{C}}{F^2(d+r-1)} N^{r-2} \\
&\ge \frac{d\gamma}{d+r} N^r - C_9.
\end{aligned} \tag{5.46}$$

Thus, by (5.45) and (5.46), we get

$$A_{N,k} \ =\ -\frac{k}{n}\ln 2 - \frac{\gamma}{n}\sum_{l=1}^{N} l^r d_l \ \le\ -\frac{d\Gamma(d/2)(\ln 2)\,k}{2\pi^{d/2}} N^{-d} - \frac{d\gamma}{(d+r)} N^r + C_{10}. \tag{5.47}$$

Let

$$g_1(x) \ =\ -\frac{d\Gamma(d/2)(\ln 2)\,k}{2\pi^{d/2}} x^{-d} - \frac{d\gamma}{d+r} x^r + C_{10}.$$

The absolute maximum value of g_1 is attained at the point

$$x_k \ =\ \left(\frac{(d+r)d\Gamma(d/2)(\ln 2)}{2r\gamma\pi^{d/2}}\right)^{1/(d+r)} k^{1/(d+r)}$$

and thus

$$\begin{aligned}\sup_N A_{N,k} &\le \sup_N g(N) \le g_1(x_k)\\ &= -\gamma^{d/(d+r)}\left(\frac{(d+r)d\Gamma(d/2)(\ln 2)}{2r\pi^{d/2}}\right)^{r/(d+r)} k^{r/(d+r)} + C_{10}\\ &= -\mathcal{C}k^{r/(d+r)} + C_{10}.\end{aligned}$$

Then, It follows by (5.43) that

$$\chi_k \ll \begin{cases} e^{-\mathcal{C}k^{r/(d+r)}}, & 2 \le q < \infty, \\ e^{\sup_N A_{N,k}}(\ln n)^{1/2}, & q = \infty. \end{cases} \tag{5.48}$$

In particular, for $q = \infty$, using (5.47) and remembering that $n \asymp N^d$, we obtain

$$\begin{aligned}\chi_k &\ll \sup_N e^{A_{N,k}}(\ln N)^{1/2} \ll \sup_N e^{A_{N,k}+\frac{1}{2}\ln(\ln N)}\\ &\le \sup_N e^{-\frac{d\Gamma(d/2)(\ln 2)\, k}{2\pi^{d/2}}N^{-d}-\frac{d\gamma}{d+r}N^r+\frac{1}{2}\ln(\ln N)+C_{10}}.\end{aligned}$$

Let now

$$g_2(x) = -\frac{d\Gamma(d/2)(\ln 2)\, k}{2\pi^{d/2}}x^{-d} - \frac{d\gamma}{d+r}x^r + \frac{1}{2}\ln(\ln x) + C_{10}.$$

Analogously to has been done in the Lemma 4.1, we observe that the maximum value of the function g_2 is obtained at a point $\overline{x}_k$ satisfying $\overline{C}_1 x_k \le \overline{x}_k \le \overline{C}_2 x_k$, whence $\sup_N g_2(N)$ is obtained when $N \asymp \overline{x}_k \asymp x_k \asymp k^{1/(d+r)}$. Therefore for $q = \infty$, we have that

$$\chi_k \ll \sup_N e^{A_{N,k}}(\ln k)^{1/2} \ll e^{-\mathcal{C}k^{r/(d+r)}}(\ln k)^{1/2}.$$

and by (5.48)

$$\chi_k \ll e^{-\mathcal{C}k^{r/(d+r)}} \begin{cases} 1, & 2 \le q < \infty, \\ (\ln k)^{1/2}, & q = \infty. \end{cases} \tag{5.49}$$

Since $\Lambda^{(2)} \in K_{\epsilon,2}$, for any $\epsilon > 0$ (See [7] p. 67), applying the Theorem 3.3, we get

$$\begin{aligned}&e_\eta(\Lambda^{(2)}U_p, L^q)\\ &\ll \chi_k \left(\left\{ \begin{array}{cc} 1, & 2 \le p \le \infty,\ 1 \le q < \infty, \\ \sup\limits_{1\le j\le M} \left(\ln \theta_{N_j,N_{j+1}}\right)^{1/2}, & 2 \le p \le \infty,\ q = \infty, \end{array} \right\} + \sum_{j=M+1}^{\infty} e^{-j}\theta_{N_j,N_{j+1}}^{1/2-1/q} \right).\end{aligned}$$

But $\sum_{j=M+1}^{\infty} e^{-j}\theta_{N_j,N_{j+1}}^{1/2-1/q} \ll 1$ (See [7], p. 62), $\ln n \asymp \ln N^d \asymp \ln k^{d/(d+r)} \asymp \ln k$, and by (5.42)

$$e_\eta(\Lambda^{(2)}U_p, L^q) \ll \chi_k \begin{cases} 1, & 2 \le p \le \infty,\ 1 \le q < \infty, \\ (\ln k)^{1/2}, & 2 \le p \le \infty,\ q = \infty. \end{cases}$$

Then, it follows by (5.49)

$$e_\eta(\Lambda^{(2)}U_p, L^q) \ll \begin{cases} \chi_k, & 2 \le p \le \infty,\ 1 \le q < 2, \\ e^{-\mathcal{C}k^{r/(d+r)}}, & 2 \le p \le \infty,\ 2 \le q < \infty, \\ e^{-\mathcal{C}k^{r/(d+r)}}\ln k, & 2 \le p \le \infty,\ q = \infty. \end{cases}$$

For $1 \le q < 2$, by (5.49) we have that $\chi_k^{(q)} \le \chi_k^{(2)} \ll e^{-\mathcal{C}k^{r/(d+r)}}$ and thus

$$e_\eta(\Lambda^{(2)}U_p, L^q) \ll e^{-\mathcal{C}k^{r/(d+r)}} \begin{cases} 1, & 2 \le p \le \infty,\ 1 \le q < \infty, \\ \ln k, & 2 \le p \le \infty,\ q = \infty. \end{cases} \tag{5.50}$$

We have that $\sum_{j=1}^{M} m_j \le C_\epsilon \theta_{N_1,N_2}$. Thus using the fact that $\theta_{N_1,N_2} \asymp N^{d-r}$ and remembering that $N \asymp k^{1/(d+r)}$, we obtain

$$\eta = k + \sum_{j=1}^{M} m_j \le k + C_{11} k^{(d-r)/(d+r)},$$

hence, it follows by basic properties of the entropy numbers and by (5.50), that

$$e_{[k+C_{11}k^{(d-r)/(d+r)}]}(\Lambda^{(2)}U_p, L^q) \ll e^{-\mathcal{C}k^{r/(d+r)}} \begin{cases} 1, & 2 \le p \le \infty,\, 1 \le q < \infty, \\ \ln k, & 2 \le p \le \infty,\, q = \infty. \end{cases} \tag{5.51}$$

Let $\varphi_k = k + C_{11}k^{(d-r)/(d+r)}$. Then

$$\begin{aligned} \mathcal{C}\left(-k^{r/(d+r)} + \varphi_k^{r/(d+r)}\right) &= \mathcal{C}k^{r/(d+r)}\left(-1 + \left(1 + C_{11}k^{-2r/(d+r)}\right)^{r/(d+r)}\right) \\ &= \mathcal{C}\left(\frac{rC_{11}}{d+r}k^{-r/(d+r)} - \frac{rdC_{11}^2}{2(d+r)^2}k^{-3r/(d+r)} + \cdots\right) \\ &\le C_{12} \end{aligned}$$

and hence

$$e^{-\mathcal{C}k^{r/(d+r)}} \le e^{C_{12}} e^{-\mathcal{C}\varphi_k^{r/(d+r)}} \ll e^{-\mathcal{C}\varphi_k^{r/(d+r)}}.$$

Then, by (5.51) we get

$$e_{[\varphi_k]}(\Lambda^{(2)}U_p, L^q) \ll e^{-\mathcal{C}\varphi_k^{r/(d+r)}} \begin{cases} 1, & 2 \le p \le \infty,\, 1 \le q < \infty, \\ \ln k, & 2 \le p \le \infty,\, q = \infty. \end{cases} \tag{5.52}$$

We observe now that

$$\begin{aligned} \varphi_k^{r/(d+r)} &= k^{r/(d+r)}\left(1 + C_{11}k^{-2r/(d+r)}\right)^{r/(d+r)} \\ &= k^{r/(d+r)}\left(1 + \frac{rC_{11}}{d+r}k^{-2r/(d+r)} - \frac{rdC_{11}^2}{2(d+r)^2}k^{-4r/(d+r)} + \cdots\right), \end{aligned}$$

and thus

$$\varphi_k^{r/(d+r)} \ge k^{r/(d+r)}\left(1 + \frac{rC_{11}}{d+r}k^{-2r/(d+r)} - \frac{rdC_{11}^2}{2(d+r)^2}k^{-4r/(d+r)}\right)$$

and

$$\varphi_{k+1}^{r/(d+r)} \le (k+1)^{r/(d+r)}\left(1 + \frac{rC_{11}}{d+r}(k+1)^{-2r/(d+r)}\right).$$

Since $0 < (k+1)^{r/(d+r)} - k^{r/(d+r)} \le r/(d+r)$, it follows that

$$0 \le \varphi_{k+1}^{r/(d+r)} - \varphi_k^{r/(d+r)} \le (k+1)^{r/(d+r)} - k^{r/(d+r)} + C_{13} \le r/(d+r) + C_{13} = C_{14}$$

and therefore

$$1 \le \frac{e^{-\mathcal{C}\varphi_k^{r/(d+r)}}}{e^{-\mathcal{C}\varphi_{k+1}^{r/(d+r)}}} \le C_{15}. \tag{5.53}$$

Finally, since $0 < r \le 1$, if $[\varphi_k] \le l \le [\varphi_{k+1}]$, we obtain by basic properties of entropy numbers, by (5.52) and (5.53)

$$\begin{aligned} e_l(\Lambda^{(2)}U_p, L^q) &\le e_{[\varphi_k]}(\Lambda^{(2)}U_p, L^q) \\ &\ll e^{-\mathcal{C}\varphi_k^{r/(d+r)}} \begin{cases} 1, & 2 \le p \le \infty,\, 1 \le q < \infty, \\ \ln k, & 2 \le p \le \infty,\, q = \infty, \end{cases} \\ &\ll e^{-\mathcal{C}\varphi_{k+1}^{r/(d+r)}} \begin{cases} 1, & 2 \le p \le \infty,\, 1 \le q < \infty, \\ \ln \varphi_k, & 2 \le p \le \infty,\, q = \infty, \end{cases} \\ &\ll e^{-\mathcal{C}l^{r/(d+r)}} \begin{cases} 1, & 2 \le p \le \infty,\, 1 \le q < \infty, \\ \ln l, & 2 \le p \le \infty,\, q = \infty. \end{cases} \end{aligned}$$

□

Remark 5.2. *Now, consider the multiplier operators*

$$\Lambda_*^{(1)} = \left\{|\mathbf{k}|_*^{-\gamma}(\ln|\mathbf{k}|_*)^{-\xi}\right\}_{\mathbf{k}\in\mathbb{Z}^d} \quad \textit{and} \quad \Lambda_*^{(2)} = \left\{e^{-\gamma|\mathbf{k}|_*^r}\right\}_{\mathbf{k}\in\mathbb{Z}^d}, \ \ \gamma,\xi,r\in\mathbb{R}, \ \ \gamma,r>0, \ \xi\geq 0.$$

Making simple adjustments in the proofs in this paper, we can show that the Theorems 3.2 *and* 3.3 *also hold if we change* $\mathcal{H}_l$, $\mathcal{T}_N$, d_l *and* $\lambda_{\mathbf{k}}$ *by* $\mathcal{H}_l^*$, $\mathcal{T}_N^*$, d_l^* *and* $\lambda_{\mathbf{k}}^*$, *the Theorem* 4.1, *if we change* $\Lambda^{(1)}$ *by* $\Lambda_*^{(1)}$ *and the Theorems* 5.1 *and* 5.2 , *also hold if we change* $\Lambda^{(2)}$ *by* $\Lambda_*^{(2)}$ *and the constant* $\mathcal{C}$ *by the constant*

$$\mathcal{C}_* = \gamma^{d/(d+r)}\left(\frac{(d+r)(\ln 2)}{2^d r}\right)^{r/(d+r)}.$$

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