AN ALPHABETICAL APPROACH TO NIVAT'S CONJECTURE ☆

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Abstract

Since techniques used to address the Nivat's conjecture usually relies on Morse-Hedlund Theorem, an improved version of this classical result may mean a new step towards a proof for the conjecture. In this paper, we consider an alphabetical version of the Morse-Hedlund Theorem. Following methods highlighted by Cyr and Kra [1], we show that, for a configuration $\eta \in \mathcal{A}^{\mathbb{Z}^2}$ that contains all letters of a given finite alphabet $\mathcal{A}$, if its complexity with respect to a quasi-regular set $\mathcal{U} \subset \mathbb{Z}^2$ (a finite set whose convex hull on $\mathbb{R}^2$ is described by pairs of edges with identical size) is bounded from above by $\frac{1}{2}|\mathcal{U}| + |\mathcal{A}| - 1$, then η is periodic.

Keywords:
Combinatorics on words, Formal languages, Symbolic dynamics.

1. Introduction

Fixed a finite alphabet $\mathcal{A}$ (with at least two elements), for each $n \in \mathbb{N}$, the n-complexity of an infinite sequence $\xi = (\xi_i)_{i\in\mathbb{Z}} \in \mathcal{A}^{\mathbb{Z}}$, denoted by $P_\xi(n)$, is defined to be the number of distinct words of the form $\xi_j\xi_{j+1}\cdots\xi_{j+n-1}$ appearing in ξ. In 1938, Morse and Hedlund [2] proved one of the most famous results in symbolic dynamics which establishes a connection between periodic sequences (sequences for which there is an integer $m \geq 1$ such that $\xi_{i+m} = \xi_i$ for all $i \in \mathbb{Z}$) and complexity. More specifically, they proved that $\xi \in \mathcal{A}^{\mathbb{Z}}$ is periodic if, and only if, there exists $n \in \mathbb{N}$ such that $P_\xi(n) \leq n$.

A natural extension of the complexity function to higher dimensions is obtained when we consider, instead of words, blocks of symbols. More precisely, the $n_1 \times \cdots \times n_d$-complexity of a configuration $\eta = (\eta_g)_{g\in\mathbb{Z}^d} \in \mathcal{A}^{\mathbb{Z}^d}$, denoted by $P_\eta(n_1, \ldots, n_d)$, is defined to be the number of distinct $n_1 \times \cdots \times n_d$ blocks of symbols appearing in η. Of course periodicity also has a natural higher dimensional generalization: $\eta \in \mathcal{A}^{\mathbb{Z}^d}$ is periodic if there exists a vector $h \in (\mathbb{Z}^d)^*$, called period of η, such that $\eta_{g+h} = \eta_g$ for all $g \in \mathbb{Z}^d$. A configuration that is not periodic is said to be aperiodic.

The Nivat's Conjecture [3] is the natural generalization of the Morse-Hedlund Theorem for the two-dimensional case.

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